A Remark on Quasiconformal Mappings on Carnot Groups

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1. Introduction

In a letter of May 1991, A. Koranyi and M. Reimann informed me that a result of theirs on the theory of quasiconformal mappings on the Heisenberg groups contradicted inequality (20.17) in my monograph [2] which is asserted there without proof and used in the proof of Proposition 21.3. The latter proposition deals with the extension of a certain mapping φ between hyperbolic space over the division algebra $\mathbf{K}(\mathbf{K} := \mathbf{R}, \mathbf{C}, \mathbf{H} :=$ quaternions, or $\mathbf{O} :=$ octonions) to their boundaries at infinity. The boundary map φ_0 had previously been proved to be a quasiconformal mapping over $\mathbf{K}$. Proposition 21.3 asserts that φ_0 is absolutely continuous on almost all curves of a specified type.

The boundary minus one point is the free action orbit of any maximal unipotent subgroup of the isometry group of the hyperbolic space. In case $\mathbf{K} = \mathbf{C}$, the unipotent group is the Heisenberg group; for a general $\mathbf{K}$, it is a two-step unipotent Carnot group.

Proposition 21.3 is an essential step in proving strong rigidity for locally hyperbolic spaces over $\mathbf{K}$. This paper offers a correction of the proof of Proposition 21.3 via bypassing the faulty inequality (20.17). The method used can be generalized directly to simplify the definition of quasiconformal mappings on two-step Carnot groups. In [2] the notion of quasiconformal mapping on the boundary of hyperbolic space is defined in terms of the boundary "semimetric". Subsequently, Pansu (in [3]), and Koranyi and Reimann (in an earlier version of [1]) studied a similar notion of quasiconformal mapping with respect to a "Carnot–Carathéodory metric", which required an extra "doubling hypothesis". In Section 4 it is pointed out that, as a result of the method used here, the extra doubling hypothesis is superfluous. This method was subsequently adopted by Koranyi and Reimann in [1].

2. Setting the Stage

Occurring in the proof of Proposition 21.3 are two commuting fibrations $\pi^{\mathbf{K}}$ and $\pi_{\mathbf{R}}$ by Hopf fibers and by quarter great $\mathbf{R}$-circles respectively. The point

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of the Koranyi–Reimann observation is that the flow along the $\pi_{\mathbf{R}}$ fibers can distort the boundary semimetric d_0 unboundedly and thus contradicts (20.17), an inequality on the base space of $\pi_{\mathbf{R}}$. The proof can be corrected by two small modifications which we explain below. Most of the argument remains the same.

For the convenience of the reader, we list the relevant definitions, continuing the notation of [2].

$X = H^n_{\mathbf{K}}$, hyperbolic n-space over $\mathbf{K}$;
$k = \dim \mathbf{K}$;
$X_0 =$ boundary of $X \approx S^m$, $m = nk-1$;
$G = \operatorname{Aut} X$;
$0 =$ origin of X in the model of §20 as a ball in $\mathbf{K}^n$;
$G_0 =$ stabilizer of 0. G_0 preserves the standard Euclidean metric of $\mathbf{K}^n$ as well as the function $d^{\mathbf{K}}$ on $\mathbf{K}^n \times \mathbf{K}^n$ and its restriction d_0 to $X_0 \times X_0$ (the boundary semimetric; cf. (20.14), (20.15)). The boundary ball $\mathbf{K}(p,s) := \{q \in X_0;\ d_0(p,q) < s\}$.

Fix a point $p_0 \in X_0$, let L denote the $\mathbf{K}$-line through p_0, and let $L^\perp$ denote the orthogonal complement to L with respect to the standard Euclidean metric on $\mathbf{K}^n$. Fix a point $q_0 \in L^\perp \cap X_0$, and let L_2 denote the $\mathbf{K}$-line through q_0. Let K denote the subgroup of G_0 which stabilizes each of the $\mathbf{K}$-lines that contain a point of the great $\mathbf{R}$-circle through p_0, q_0. Set $H = G_{0,L}$, the stabilizer of L in G_0, and H_{p_0,q_0}, the fixer of p_0, q_0 in H. Set ${}^1\mathbf{K} = \{x \in \mathbf{K};\ \bar{x}x = 1\}$.

If $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, then $K = {}^1\mathbf{K} \cdot H_{p_0,q_0}$ (direct product). If $\mathbf{K} = \mathbf{O}$, the description is more intricate: $H = \operatorname{Spin} 8$ acting on L via even $\frac{1}{2}$-spinors. Let $\operatorname{Spin} 7_1 = H_{p_0}$, $\operatorname{Spin} 7_2 = H_{q_0}$, and let $\operatorname{Spin} 7_3$ denote the subgroup of Spin 8 acting on $(\mathbf{R}p_0 + \mathbf{R}q_0) \otimes \mathbf{O}$ via identity $\otimes$ spinor representation of Spin7. $\operatorname{Spin} 7_3$ is the image of $\operatorname{Spin} 7_1$ via the triality automorphism of order 3. $H_{p_0,q_0} = \operatorname{Spin} 7_1 \cap \operatorname{Spin} 7_2 = G_2 \subset \operatorname{Spin} 7_3$, the exceptional group G_2 being the group of algebra automorphisms of the octonions. Here $K = \operatorname{Spin} 7_3$ and $K/H_{p_0,q_0} \simeq Kp_0 = L \cap X_0$.

Let T denote the one-parameter subgroup of G_0 which stabilizes $\mathbf{R}p_0 + \mathbf{R}q_0$, is the identity on $(L+L_2)^\perp$, and centralizes K; T is a circle group parameterized by angle of rotation $0 \le t < 2\pi$. Set $T_* = \{t \in T;\ 0 < t < \pi/2\}$, $T_1 =$ a (nonempty) compact subset of T_*. Tp_0 is the great $\mathbf{R}$-circle through p_0, q_0, and H_{p_0,q_0} fixes each point of Tp_0. The fibration $\pi_{\mathbf{R}}: X_* \to Y$ of (20.15) has fibers hT_*p_0, $h \in H$. The group H permutes the great $\mathbf{R}$-circles and $htp_0 = tp_0$ with $t \in T_*$ implies that $h \in H_{p_0,q_0} \subset K$ and hence $th = ht$ for all $h \in H_{p_0,q_0}$ and $t \in T_*$. Thus $Y := \pi_{\mathbf{R}}(X_*) \cong H/H_{p_0,q_0} \simeq S^{k-1} \times S^{(n-1)k-1}$. Set $X_1 = HT_1p_0$. Then $X_1 \cong H/H_{p_0,q_0} \times T_1$ and $\pi_{\mathbf{R}}(X_1) = Y$.

The erroneous inequality (20.17) is used only once in the proof of Proposition 21.3, on page 164 line 13, to justify (21.20):

$$\limsup_{s \to 0} \frac{\mu(\phi_0(E(y,s)))}{s^{m+k-2}} < \infty \quad \text{a.e. } y \in Y,$$

where μ denotes the standard S^m measure on X_0.

3. The Modifications

Our first modification in the proof of Proposition (21.3) is to change lim sup to lim inf in the definition of $\tau(y)$ in (21.19); hereafter,

$$\tau(y) := \liminf_{s\to 0} \frac{\mu(\varphi_0(E(y,s)))}{s^{m+k-2}}. \tag{21.19'}$$

An examination of the proof of Proposition 21.3 reveals that (21.20) was used only once: in the justification of inequality (21.26). Inasmuch as the inference there that $(Nt)^{m+k'-2} \le \mu_1(E)+a$ is valid for any sequence of t converging to zero, inequality (21.26) follows equally well from the new (and weaker than in [2]) inequality

$$\tau(y) < \infty \text{ a.e. } y \in Y. \tag{21.20'}$$

It remains only to justify this assertion.

Recall from page 154 the inequality relating the boundary ball with the polydisc (with respect to the boundary semimetric d_0) for small s:

$$D_0(p, s/2) \subset \mathbf{K}(p,s) \subset D_0(p,s).$$

For any $y \in \pi_{\mathbf{R}}(X_1)$ and $s>0$ suitably small, on page 164, line 4, we defined

$$E(y,s) := \{p \in X_0;\ d_0(p,q) < s \text{ for some } q \in hT_1 p_0\};$$

here $y = hT_1 p_0$, $h \in H$; by abuse of notation, we write $y = hT_1 p_0$ for $y = \pi_{\mathbf{R}}(hT_1 p_0)$. Thus

$$E(y,s) = hT_1 \mathbf{K}(p_0, s),$$

so that

$$hT_1 D_0(p_0, s/2) \subset E(y,s) \subset hT_1 D_0(p_0, s).$$

The second modification in the proof of Proposition 21.3 is to exploit the composite fibering $\pi_{\mathbf{R}} \circ \pi^{\mathbf{K}}$. Set

$$X_0^{\mathbf{K}} = \pi^{\mathbf{K}}(X_0) \cong P_{\mathbf{K}}^{n-1};$$

$\pi^{\mathbf{K}}: X_0 \to X_0^{\mathbf{K}}$ is a G_0-map. Set

$$\begin{aligned} X_1^{\mathbf{K}} = \pi^{\mathbf{K}}(X_1) &= HT_1 p_0^{\mathbf{K}},\ p_0^{\mathbf{K}} = \pi^{\mathbf{K}}(p_0) \\ &\cong (H/K) \times T_1. \end{aligned}$$

The map $\pi^{\mathbf{K}}: X_1 \to X_1^{\mathbf{K}}$ is an H-map. The Hopf fibers ($\cong {}^1\mathbf{K}$) which meet $T_1 p_0$ are orbits of the group K, each orbit being isomorphic to $K/H_{p_0,q_0}$. The fibration $\pi^{\mathbf{K}}$ of X_1 descends to $\pi_{\mathbf{R}}(X_1) := Y$, and we denote the induced fiber map $Y \to Y^{\mathbf{K}}$ by $\pi^{\mathbf{K}}$ also. The map $Y \to Y^{\mathbf{K}}$ is equivalent to $H/H_{p_0,q_0} \to H/K$. Set $\xi = \pi_{\mathbf{R}} \circ \pi^{\mathbf{K}} = \pi^{\mathbf{K}} \circ \pi_{\mathbf{R}}$.

For any $h \in H$ and $t \in T_1$, set

$$\begin{aligned} y &= \pi_{\mathbf{R}}(htp_0), & D(y,s) &= hT_1 D_0(p_0,s), \\ y^{\mathbf{K}} &= \pi^{\mathbf{K}}(y), & D^{\mathbf{K}}(y^{\mathbf{K}},s) &= \xi D(y,s). \end{aligned}$$

For any $y^{\mathbf{K}} \in Y^{\mathbf{K}}$, set

$$\tau^{\mathbf{K}}(y^{\mathbf{K}}) = \limsup_{s \to 0} \frac{\mu(\phi_0(\xi^{-1}D_{\mathbf{K}}(y^{\mathbf{K}}, s)))}{\mu(\xi^{-1}D^{\mathbf{K}}(y^{\mathbf{K}}, s))}.$$

By the usual theorem on differentiability of completely additive finite-valued set functions defined on all closed subsets of the metric space $Y^{\mathbf{K}}$, $\tau^{\mathbf{K}}(y^{\mathbf{K}}) < \infty$ a.e. on $Y^{\mathbf{K}}$.

For two families of sets $f(s)$ and $g(s)$, we write $f(s) \sim g(s)$ if there exist positive constants b_1 and b_2 with $f(b_1 s) \leq g(s) \leq f(b_1 s)$ for all small s. For convenience, define

$$^{\mathbf{K}}S := (\pi^{\mathbf{K}})^{-1}\pi^{\mathbf{K}}S \quad \text{for } S \subset X_0,$$

$$^{\xi}S := \xi^{-1}\xi S \qquad \text{for } S \subset X_1,$$

and

$$B_r(S) := \text{tubular neighborhood of radius } r$$

around the subset S with respect to the standard Euclidean metric on X_0. Thus, for $y = hT_1 p_0$ with $h \in H$,

$$\tau^{\mathbf{K}}(y^{\mathbf{K}}) = \limsup_{s \to 0} \frac{\mu(\varphi_0(^{\xi}hT_1D_0(p_0, s)))}{\mu(^{\xi}hT_1D_0(p_0, s))}.$$

Set $t_1 = \inf\{t; t \in T_1\}$. We note that, given $\lambda > 1$,

$$^{\mathbf{K}}[B_s(tp_0)] \subset KB_{\lambda s}(tp_0) \subset {}^{\mathbf{K}}[B_{\lambda^2 s}(tp_0)]$$

and

$$^{\xi}[B_s(tp_0)] \subset {}^{\mathbf{K}}[B_{cs}(T_1 p_0)], \quad c = 1/\sin t_1,$$

for all $t \in T_1$ and all sufficiently small $s > 0$. This yields, for any $t \in T_1$,

$$^{\xi}D_0(tp_0, s) \subset {}^{\xi}\mathbf{K}(tp_0, 2s) \subset {}^{\xi}[B_{2s}(tp_0)] \subset {}^{\mathbf{K}}[B_{2cs}(T_1 p_0)] \subset KB_{3cs}(T_1 p_0)$$

for all sufficiently small $s > 0$.

On the other hand, for any $t \in T_1$,

$$^{\xi}[D_0(tp_0, s)] \supset \bigcup_{t \in T_1} {}^{\mathbf{K}}[D_0(tp_0, s)] \supset \bigcup_{t \in T_1} {}^{\mathbf{K}}[B_{s/2}(tp_0)] \supset K \cdot B_{s/2\lambda}(T_1 p_0).$$

Consequently,

$$\begin{aligned} ^{\xi}[D_0(tp_0, s)] &\sim KB_s(T_1 p_0) = KT_1 B_s(p_0) = T_1 KB_s(p_0) \\ &\sim T_1 KD_0(p_0, s) \\ &= KT_1 D_0(p_0, s). \end{aligned}$$

Furthermore, for any $c > 1$, given $g, g' \in K$ and $t, t' \in T_1$ with

$$D_0(gtp_0, s) \cap D_0(g't'p_0, s)$$

nonempty and with s sufficiently small, we have $g'p_0 \in B_{cs^2}(gp_0)$; for locally, the system of subspaces (cf. [2, p. 152]) which define the polydiscs D_0 is approximated asymptotically by a product structure whose polydiscs have radius $s^2/2$ along the **K**-fibers of X_0. Thus, given $c>1$, for all sufficiently small s (i.e., $s<\epsilon_c$) we have that

$$gT_1D_0(p_0,s)\cap g'T_1D_0(p_0,s) \text{ is nonempty implies } g'p_0\in B_{cs^2}(gp_0). \qquad (*)$$

We have seen above that for $y=hT_1p_0$ with $h\in H$,

$$E(y,s):=hT_1\mathbf{K}(p_0,s)\sim hT_1D_0(p_0,s).$$

We now prove assertion (21.20′) by contradiction. Suppose (21.20′) were false. Then there would exist a subset H_0 of H with positive measure such that for all $h\in H_0$, there is a subset C_h in $K/H_{p_0,q_0}$ of measure equal to c_1 measure $(K/H_{p_0,q_0})$, $c_1>0$, satisfying $\tau(hgT_1p_0)=\infty$ for all $g\in C_h$ (note that gtp_0 is well-defined for all $g\in K/H_{p_0,q_0}$, $t\in T$) and $\tau^{\mathbf{K}}(hKT_1p_0)$ finite. For any $s>0$, let U_s denote the ball of radius $s^2/2$ about the identity coset H_{p_0,q_0} in $K/H_{p_0,q_0}$ with respect to the K-invariant metric induced from the **K**-fibers.

Choose any $h\in H_0$. By a standard result, it is possible to extract from the family of translates $\{gU_s;\ g\in C_h\}$ a disjoint subfamily $\mathcal{C}_s$ such that $\{gU_{\sqrt{3}s};\ gU_s\in\mathcal{C}_s\}$ covers C_h. By (*), if $gU_sT_1D_0(p,s)$ meets $g'U_sT_1D_0(p_0,s)$ with $g,g'\in K$ then $g^{-1}g'\in U_{\sqrt{2c}s}H_{p_0,q_0}$. Hence we can extract a subfamily $\mathcal{C}'_s$ of translates of U_s such that

(a) $\{gU_sT_1D_0(p_0,s);\ gU_s\in\mathcal{C}'_s\}$ is a disjoint family, and
(b) $\{gU_{\sqrt{6c}s}T_1D_0(p_0,s);\ gU_s\in\mathcal{C}'_s\}$ covers $C_hT_1D_0(p_0,s)$.

Given any $A>0$ however large and any $g\in C_h$, there is an H_{p_0,q_0} invariant neighborhood U_s of radius $s^2/2$ of the identity coset of $K/H_{p_0,q_0}$ such that

$$\mu(\varphi_0(hgU_sT_1D_0(p_0,s)))>A\mu(hgU_sT_1D_0(p_0,s))$$

for all $s<\epsilon_{g,h}$ where $\epsilon_{g,h}>0$. Let $C_{n,h}=\{g\in C_h;\ \epsilon_g\geq 1/n\}$. Then $C_{n,h}$ expands to C_h as $n\to\infty$, so no generality is lost in assuming that $C_h=C_{n,h}$: that is, $\epsilon_{g,h}\geq 1/n$ for all $g\in C_h$.

Now for any s, choose a family $\mathcal{C}'_s$ of translates of U_s centered at points of C_h satisfying the disjointness property (a) and the covering property (b). Then

$$\begin{aligned}\mu\left(\varphi_0\left(h\bigcup_{gU_s\in\mathcal{C}'_s} gU_sT_1D_0(p_0,s)\right)\right)&=\sum\mu(\varphi_0(hgU_sT_1D_0(p_0,s)))\\ &>A\sum_{gU_s\in\mathcal{C}'_s}\mu(gU_sT_1D_0(p_0,s))\\ &>Ac_2\sum_{gU_s\in\mathcal{C}'_s}\mu(gU_{\sqrt{6c}s}T_1D_0(p_0,s)),\end{aligned}$$

where c_2 is a positive constant independent of s. This last term is greater than $Ac_2\mu(hCT_1D_0(p_0,s))=Ac_2c_1\mu(hKT_1D_0(p_0,s))$. Thus we obtain

$$\begin{aligned}\mu(\varphi_0({}^{\xi}[hT_1D_0(p_0,3s)]))>\mu(\varphi_0(hKT_1D_0(p_0,s)))&>Ac_2c_1\mu(hKT_1D_0(p_0,s))\\ &>Ac_3c_2c_1\mu({}^{\xi}[hT_1D_0(p_0,s)])\end{aligned}$$

for all small s, where c_3 is a constant independent of s. This implies that, for $y = hT_1 p_0$ with $h \in H_0$, $\tau^{\mathbf{K}}(y^{\mathbf{K}}) > A$ for arbitrary A. This contradiction establishes (21.20′).

Deleting (20.17) and the paragraph following (21.20) (which appeals to (20.17) of [2], the proof of Proposition 21.3 remains the same. In this modified proof, Lemma 20.3 becomes superfluous, for we needed Lebesgue's theorem on differentiation of set functions on the space $\pi_{\mathbf{R}} \circ \pi^{\mathbf{K}}(X_1)$ in which the balls are standard, in contrast with $\pi_{\mathbf{R}}(X_1)$.

4. Implication for the Theory of Quasiconformal Mappings on Carnot Groups

Quasiconformal mappings between Carnot groups have been studied by Pansu in [3]. Special examples of Carnot groups are the maximal unipotent subgroups of simple Lie groups of **R**-rank 1. Before Pansu's paper, the usual definition of a quasiconformal mapping $f: X \to X'$ between metric spaces was taken to be a homeomorphism f such that

$$H(x) := \limsup_{r \to 0} \frac{\max_{d(x,y)=r} d(f(x), f(y))}{\min_{d(x,y)=r} d(f(x), f(y))}$$

is uniformly bounded on X.

For Carnot–Carathéodory metrics on Carnot groups, Pansu has imposed an additional condition, which Koranyi–Reimann (in their study of Heisenberg groups) have formulated as the doubling hypothesis:

$$D(x) := \limsup_{r \to 0} \frac{\mu(f\{y; d(x,y) \leq 2r\})}{\mu(f\{y; d(x,y) \leq r\})}$$

is uniformly bounded; here μ is a bi-invariant measure on Carnot groups.

Hitherto, the only correct proofs of the "absolute continuity on lines" property for quasiconformal mappings of Carnot groups made use of the doubling hypothesis. The method used in [2] applies as indicated there to the maximal unipotent subgroups of **R**-rank 1 simple groups. That method, modified as in Section 1, yields the desired absolute continuity property of quasiconformal mappings between Carnot groups which are two-step nilpotent.

It is natural to conjecture that it is possible to drop the doubling hypothesis for quasiconformal homeomorphisms $f: N \to N'$ between Carnot–Carathéodory spaces, retaining only the hypothesis on H, and still deduce the desired absolute continuity properties.

References

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