Equivariant Simple Poincaré Duality

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Introduction

In this paper we describe, for a finite transformation group G, the simple homotopy theory of cell complexes whose cells are discs of representations. Our primary goal is to define equivariant simple Poincaré complexes and to prove a π–π theorem for equivariant simple Poincaré pairs.

There is in the literature on equivariant surgery much about the question: When is a degree-1 normal map between G-manifolds normally cobordant to a G-equivalence or to a simple G-equivalence? (See [DP], [DR] and [LM]; for a comprehensive overview see [DS].) In this paper we begin to address the question: When is a G–CW complex simply G-homotopy equivalent to a smooth G-manifold? If we drop the requirement that the equivalence be simple, some results are given in [CW1] and [CW2].

The first obstruction to discussing the case of simple equivalence is the question of what is meant by a simple G-Poincaré complex, and behind this is the question of what is meant by G-Poincaré duality. It has been customary [DR; Lü] to consider a nonequivariant triangulation in which G permutes the simplices; the nonequivariant chains then have an induced $\mathbf{Z}G$-action and nonequivariant Poincaré duality gives an equivalence of $\mathbf{Z}G$ chains. This equivalence fails in general to be a simple $\mathbf{Z}G$ equivalence, and in fact Lück defines the Poincaré torsion of a smooth G-manifold which measures this failure [Lü, 18.G]. Thus, under this interpretation, smooth G-manifolds are usually not simple Poincaré complexes, and so a theory of simple G-Poincaré complexes would seem pointless.

From the point of view of equivariant stable homotopy theory there is another problem. From this point of view "ordinary homology" means Bredon's ordinary equivariant homology theory [Br]. This uses the same CW decomposition as above, but the Bredon chains incorporate more information about the fixed sets of the G-complex in question. In this theory we do not even have Poincaré duality itself for smooth G-manifolds, except in very special cases such as free actions or trivial actions.

We can resolve both these difficulties if we redefine what we mean by equivariant Poincaré duality. It is well known that it is useful to extend the

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grading of equivariant homology theories from $\mathbf{Z}$ to $RO(G)$; in [LMM] it is shown how this can be done for ordinary equivariant homology. It then turns out that Poincaré duality holds for a significantly wider class of G-manifolds, namely manifolds locally modeled on a single representation of G, as introduced by [Pu]. If M is modeled on the representation V, then the Poincaré duality isomorphism has the form $H_k^G(M) \cong H_G^{V-k}(M)$. This isomorphism cannot be described on the chain level using only the cell structures discussed above. It is based instead on the natural one-to-one correspondence between these cells and the dual cells. The dual cells are not permuted by G, but instead are copies of the unit disc of V (minus some trivial summands). This cell structure, and not any subdivision of it into ordinary cells, is the appropriate one to use in the construction of H_G^{V-k}. This Poincaré duality isomorphism is in fact a simple isomorphism on the chain level, because of the underlying geometry. Thus the mathematics forces us to take these dual cells seriously.

The next logical step is to extend the grading further in order to obtain simple Poincaré duality for general G-manifolds. The extended grading is needed to account for the changing local representations of subgroups of G occurring in the tangent bundle, these being the representations that show up in the action of G on the dual cell structure. We have already used cell structures such as these in [CW1] to obtain a theory of equivariant Poincaré duality, which we then used in [CW2] to study the equivariant Spivak normal fibration.

In this paper we refine this further in order to define equivariant simple Poincaré complexes and to obtain the following main results. The first of these is a "one out of three" result that is easy to prove nonequivariantly from the simplicity of Poincaré duality. Dovermann and Rothenberg [DR, 5.c.0] prove an equivariant version for manifolds. We give the following similar result for equivariant simple Poincaré complexes.

PROPOSITION. *Let $(X, \partial X)$ and $(Y, \partial Y)$ be simple Poincaré G-pairs such that the inclusions $\partial X \to X$ and $\partial Y \to Y$ are π-equivalences. Suppose that $f: (X, \partial X) \to (Y, \partial Y)$ is a degree-1 G-homotopy equivalence covered by a simple equivalence of local representation data. If $f: X \to Y$ is a simple G-homotopy equivalence, then so is the map $\partial f: \partial X \to \partial Y$ and the map of pairs $f: (X, \partial X) \to (Y, \partial Y)$.*

This will be proved as Proposition 7.7. It is a key technical result in showing the following generalization to equivariant simple Poincaré complexes of the π–π theorem of [DR] for manifolds.

π–π THEOREM. *Let $(X, \partial X)$ be a simple Poincaré G-pair such that the inclusion $\partial X \to X$ is a π-equivalence; we also assume that the dimensions of the fixed sets satisfy certain hypotheses (to be specified in Section 7). If M is a smooth compact G-manifold and $f: (M, \partial M) \to (X, \partial X)$ is a degree-1 map covered by a simple equivalence of local representation data and a bundle* *map $b: \nu_M \to \xi$ for some bundle ξ over X, then (f, b) is normally G-cobordant to a simple G-homotopy equivalence of pairs.*

This appears as Theorem 7.10.

The bulk of this paper is taken up with developing the simple homotopy theory of G-cell complexes whose cells are the discs of representations. The associated Whitehead torsion lives in Whitehead groups containing those used by various authors [I1; I3; Lü; Ro]. We show along the way that the dual cell structure of a smooth G-manifold is unique up to simple G-homotopy equivalence.

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1. Background

We shall be discussing a theory of G-cell complexes constructed from discs of representations, where G is a finite group. Such a theory was constructed in [CW1], and here we shall give a slightly more restrictive version more suitable for discussing simple homotopy. As is becoming standard practice, we shall use the equivariant fundamental groupoid in order to keep track of the representations (cf. [Lü] or [CW1]). If X is a G-space, the *fundamental groupoid* $\pi(X; G)$ (or just πX, if G is understood) *of* X is the category whose objects are the G-maps $x: G/H \to X$, where H ranges over the subgroups of G; equivalently, x is a point in X^H. A morphism $x \to y$, $y: G/K \to X$, is the equivalence class of a pair (σ, ω), where $\sigma: G/H \to G/K$ is a G-map, and where $\omega: G/H \times I \to X$ is a G-homotopy from x to $y \circ \sigma$. Two such maps are equivalent if there is a G-homotopy $k: \omega \simeq \omega'$ such that $k(\alpha, 0, t) = x(\alpha)$ and $k(\alpha, 1, t) = y \circ \sigma(\alpha)$ for $\alpha \in G/H$ and $t \in I$.

Let $\mathcal{G}$ be the category of G-orbits and G-maps between them. There is a functor $\phi: \pi X \to \mathcal{G}$, given by $\phi(x: G/H \to X) = G/H$ on objects and by $\phi(\sigma, \omega) = \sigma$ on morphisms. This turns πX into a groupoid over $\mathcal{G}$ in the sense of [CKMW]. If $f: X \to Y$ is a G-map, then there is an induced map $f_*: \pi X \to \pi Y$ over $\mathcal{G}$.

Let $h\mathcal{O}_n$ be the category of n-dimensional orthogonal G-bundles over G-orbits and G-homotopy classes of linear maps, so there is again a functor $\phi: h\mathcal{O}_n \to \mathcal{G}$, giving the base space. An *n-dimensionial linear representation of* πX is a functor $\rho: \pi X \to h\mathcal{O}_n$ such that $\phi\rho = \phi$; that is, it is a functor over $\mathcal{G}$. If $\phi(x) = G/H$, then we shall often write $\rho(x) = G \times_H V(x)$, so that $V(x)$ is an orthogonal representation of H. A *linear map* of representations of πX is a natural transformation over the identity. More generally, if $f: X \to Y$ is a G-map, ρ is a representation of πX, and ρ' is a representation of πY, then a map $\rho \to \rho'$ covering f is given by a natural transformation $\eta: \rho \to \rho' \circ f_*$ over the identity. If ξ is an n-dimensional G-bundle over the G-space X, then ξ determines a representation $\rho(\xi)$ of πX given by $\rho(\xi)(x: G/H \to X) = x^*(\xi)$ on objects. $\rho(\xi)$ is defined on maps using the covering homotopy property for G-bundles. Similarly, a map of G-bundles gives rise to a map of induced representations.

If M is any smooth G-manifold, then its *tangent representation* τ is defined to be the representation of πM associated with the tangent bundle of M.

We can define *spherical representations* by replacing $h\mathcal{O}_n$ with the category $h\mathfrak{F}_n$ of spherical bundles over G-orbits and G-fiber homotopy equivalences between them. *Spherical maps* of representations are again natural transformations. We can consider any linear representation as a spherical representation via the functor $h\mathcal{O}_n \to h\mathfrak{F}_n$.

2. Equivariant Simple Homotopy Theory

In this section we introduce a generalization of the usual notion of equivariant simple homotopy theory [I1; I3; Lü; Ro]. Let X be a G-space and let γ be an l-dimensional linear representation of the fundamental groupoid $\pi(X;G)$.

DEFINITION 2.1. *A* G–CW(γ) structure on X is a decomposition of X as $\operatorname{colim} X^n$, where:

(a) X^0 is a disjoint union of orbits $G/H \to X$ such that $\gamma(G/H \to X)$ is a product $G/H \times \mathbf{R}^l$.

(b) $X^n = X^{n-1} \cup_{\phi_n} (\bigcup_m e^n_m)$ where for each m there is a specified G-orbit $x\colon G/H \to e^n_m$ and a specified G-homeomorphism

$$e^n_m \cong D(\gamma(x) + \mathbf{R}^{n-l})$$

that identifies x with the zero section. Here, if $n<l$ then $\gamma(x)$ must contain an $(l-n)$-dimensional summand, and $\gamma(x)+\mathbf{R}^{n-l}$ denotes the complement of this summand. We call the resulting map

$$D(\gamma(x) + \mathbf{R}^{n-l}) \to X$$

the *characteristic map* for the cell e^n.

We identify two such structures on X if they use the same skeleta X^n and if, for each cell, the two given characteristic maps give rise to a G-homeomorphism $S^{\gamma(x)+n-l} \to S^{\gamma(y)+n-l}$ that is G-homotopic to a linear isomorphism.

Note that if γ is trivial then this coincides with the usual notion of G–CW structure, since any homeomorphism of a trivial sphere is homotopic to a linear isomorphism. In the general case the choice of a class of characteristic maps is mainly a technical device that allows us to deal simultaneously with several kinds of cell structure on the same space (see Definition 4.1, which leads eventually to Proposition 5.2). In this section the main effect of this choice is on the definition of "trivial unit" given below. (See also Lemma 7.6, which effectively allows one to vary γ up to weak equivalence.)

A cellular G-map $f\colon (X,\gamma)\to(Y,\mu)$ consists of a G-map f that is cellular in the obvious sense together with a linear map $\gamma\to\mu$. If A is a sub-G-complex of X, then the inclusion $\iota\colon (A,\gamma|A)\to(X,\gamma)$ is then cellular with the associated natural transformation $\gamma|A\to\iota^*\gamma$ being the identity. In [CW1] we prove cellular approximation theorems: any G-map $f\colon (X,\gamma)\to(Y,\mu)$ covered by a linear map $\gamma\to\mu$ is G-homotopic to a cellular G-map; and if γ is a representation of πX for a G-space X then there is a G–CW(γ) complex ΓX and a weak G-homotopy equivalence $(\Gamma X,\gamma)\to(X,\gamma)$.

Armed with such a cellular theory, we may now describe geometric simple homotopy theory along the lines of [Co], [I1] and [Lü].

DEFINITION 2.2. A cellular inclusion $\iota\colon (X,\gamma|X)\to(Y,\gamma)$ is an *elementary expansion* if $Y=X\cup e^n\cup e^{n+1}$, with a choice of characteristic maps such that

$$e^{n+1}=G\times_H D(W\oplus\mathbf{R})\cong G\times_H D(W)\times I$$

and

$$e^n=G\times_H[D(W)\times 1\cup S(W)\times I]\cong G\times_H D(W).$$

We also say that X is an *elementary collapse* of Y.

Definition 2.2 leads to the standard definition of a simple homotopy equivalence as a G-map G-homotopic to a finite sequence of elementary collapses and expansions. If X is a G–CW(γ) complex, we can now define the associated Whitehead group, $\mathrm{Wh}_G(X,\gamma)$, as the group of equivalence classes of relatively finite cellular pairs (Z,X) in which X is a G-deformation retract of Z, under the relation $\simeq_s$ of equivariant simple homotopy equivalence rel X. The usual lemmas now apply to show that Wh_G is a homotopy functor on the category of G–CW(γ) complexes and G-maps. If $f\colon X\to Y$ is a cellular G-homotopy equivalence, then we take $\tau(f)=[Mf\cup_X Y, Y]\in\mathrm{Wh}_G(Y,\gamma)$. $\tau(f)$ is the obstruction to f being a simple G-homotopy equivalence. Notice that if $\gamma=0$ or any integer then $\mathrm{Wh}_G(X,\gamma)$ is the group $\mathrm{Wh}_G(X)$ defined in [I2] if X is finite, or the group $\mathrm{Wh}^G(X)$ defined in [Lü].

The following "simplified form" result follows from the arguments of Cohen and Illman [Co; I2].

PROPOSITION 2.3. *Let (Z,X) be a relatively finite G-cellular pair with X a strong deformation retract of Z. Then, for sufficiently large r, $Z\simeq_s W$ rel X, where $W=X\cup\bigcup e_i^r\cup\bigcup e_i^{r+1}$.*

In [CW1] we construct the cellular chain complex $C_{\gamma+*}(X)$ of a G–CW(γ) complex X. This is a complex of $\hat{\pi}X$-groups—that is, contravariant additive functors from $\hat{\pi}X$ to the category of abelian groups, where $\hat{\pi}X$ is the stable fundamental groupoid constructed in [CW1] (and see below). In addition, there is a canonical isomorphism

$$C_{\gamma+*}(X)\cong\textstyle\sum_i \hat{\pi}X(-,x_i)$$

where the $\hat{\pi}X(-,x_i)$ are free functors on the centers of the cells (see [CW1, Lemma 4.2]). Although this $\hat{\pi}X$-structure suffices for stable theories (such as ordinary G-homology), we need to refine it in order to deal with simple G-homotopy theory.

First recall from [CW1] that $\hat{\pi}X$ is the category whose objects are those of πX and whose morphisms $x \to y$ are formal sums of equivalence classes of diagrams $x \leftarrow z \to y$ in πX. This is a generalization of the construction of the stable orbit category due to Lindner [Li] and Lewis [Le].

DEFINITION 2.4. If γ is a representation of πX then let $\hat{\pi}^{\gamma}X$ be the subcategory of $\hat{\pi}X$ whose morphisms $x \to y$ are formal sums of equivalence classes of diagrams $x \leftarrow z \to y$ such that, if $z: G/K \to X$ and $x: G/H \to X$, then G/K embeds in $\gamma(x)$ over G/H; that is, we have a commutative diagram

$$\begin{array}{ccc} G/K & \to & \gamma(x) \\ & \searrow \quad \swarrow & \\ & G/H, & \end{array}$$

where the top map is an embedding and where the remaining maps come from γ and the morphism $z \to x$. It is not hard to see that $\hat{\pi}^{\gamma}X$ is indeed a subcategory.

In [CKMW] we give an alternate definition of $\hat{\pi}X$ in which $\hat{\pi}X(x,y)$ is the set of stable H-maps from $S^{V(x)}$ to a certain space. $\hat{\pi}^{\gamma}X$ can be described similarly, except that we permit stabilization only by trivial representations.

The cellular chain complex $C_{\gamma+*}(X)$ of a G–CW(γ) complex X can now be redefined to be a complex of $\hat{\pi}^{\gamma}X$-groups, the point being that the attaching maps define morphisms in the subcategory $\hat{\pi}^{\gamma}X$ of $\hat{\pi}X$ because they are given by $\mathbf{R}$-stable maps between γ-spheres. $C_{\gamma+*}(X)$ is *based* in the sense that there is an isomorphism $C_{\gamma+*}(X) \cong \sum_i \hat{\pi}^{\gamma}X(-,x_i)$, where the $\hat{\pi}^{\gamma}X(-,x_i)$ are free $\hat{\pi}^{\gamma}X$-functors on the centers of the cells (this isomorphism depends only on the choice of characteristic maps for the cells). In general, we shall say that the $\hat{\pi}^{\gamma}X$-group C is *free* if there exists a natural isomorphism Φ: $C \cong \sum_i \hat{\pi}^{\gamma}X(-,x_i)$, and *based* if such an isomorphism has been chosen. In either event, we shall refer to C as finite-dimensional if the sequence of objects (x_j) can be taken to be finite.

We now construct the algebraic Whitehead group $\mathrm{Wh}_G(\pi X, \gamma)$. When $\gamma = 0$, this will coincide with the group $\mathrm{Wh}_G(\pi X)$ of [Lü]. Following the standard procedure (as in [Lü]) we consider the category $\mathfrak{F}$ of finite-dimensional free $\hat{\pi}^{\gamma}X$-groups. Define $K_1(\mathfrak{F})$ to be the quotient of the free abelian group generated by the automorphisms of objects in $\mathfrak{F}$ by the subgroup generated by the elements $[f \circ g] - [f] - [g]$ for any two automorphisms f and g of the same module, and the elements $[f] - [f_0] - [f_1]$ for which there is a commutative diagram

$$\begin{array}{ccccc} A_0 & \to & A_0 \oplus A_1 & \to & A_1 \\ \downarrow f_0 & & \downarrow f & & \downarrow f_1 \\ A_0 & \to & A_0 \oplus A_1 & \to & A_1 \end{array}$$

in which the horizontal arrows are the obvious inclusions and projections. A *trivial unit* in $K_1(\mathfrak{F})$ is an element of the form $[\epsilon\alpha_*]$, where α is an automorphism of an object x in πX, ϵ is the homotopy class of an $\mathbf{R}$-stable G-homeomorphism $S^{\gamma(x)} \to S^{\gamma(x)}$ induced by a G-linear isomorphism, and the automorphism $\epsilon\alpha_*: \hat{\pi}^\gamma X(-,x) \to \hat{\pi}^\gamma X(-,x)$ is given by $\beta \mapsto \epsilon\alpha\circ\beta$. We let $\mathrm{Wh}_G(\pi X,\gamma)$ be the quotient of $K_1(\mathfrak{F})$ by the subgroup generated by the trivial units.

In order to relate this to the geometry, we shall find it convenient to use the following equivalent description. Fix a skeleton σX of πX and let $\hat{\sigma}^\gamma X$ be the corresponding skeleton of $\hat{\pi}^\gamma X$. (Here, by σX we mean a skeleton of πX as a category rather than as a groupoid over the orbit category.) Let $\mathrm{M}(\hat{\pi}^\gamma X)$ be the monoid of matrices $A = [a_{\alpha\beta}]$, where $\alpha = (x_\alpha, n)$ and $\beta = (x_\beta, m) \in (\mathrm{obj}\,\hat{\sigma}^\gamma X) \times \mathbf{N}$ and where $a_{\alpha\beta} \in \hat{\pi}^\gamma X(x_\beta, x_\alpha)$. We also require that all but finitely many entries satisfy $a_{\alpha\beta} = 0$ if $\alpha \neq \beta$ and $a_{\alpha\alpha} = 1_{x_\alpha}$. Let $\mathrm{GL}(\hat{\pi}^\gamma X) \subset \mathrm{M}(\hat{\pi}^\gamma X)$ be the group of invertible matrices. As usual, we take $\mathrm{E}(\hat{\pi}^\gamma X)$ to be the subgroup generated by the elementary matrices; that is, matrices $I + a_{\alpha\beta}$ obtained from the identity by adding a single off-diagonal entry $a_{\alpha\beta}$. It now follows as in [Co] that $\mathrm{E}(\hat{\pi}^\gamma X)$ is the commutator subgroup of $\mathrm{GL}(\hat{\pi}^\gamma X)$. Let $\mathrm{E}_{\pi X}$ be the subgroup of $\mathrm{GL}(\hat{\pi}^\gamma X)$ generated by $\mathrm{E}(\hat{\pi}^\gamma X)$ and matrices $J(a_{\alpha\alpha})$ obtained from the identity by replacing the αth diagonal entry by the trivial unit $a_{\alpha\alpha} \in \hat{\pi}^\gamma X(x_\alpha, x_\alpha)$. It follows as in the non-equivariant case that $\mathrm{Wh}_G(\pi X,\gamma) \cong \mathrm{GL}(\hat{\pi}^\gamma X)/\mathrm{E}_{\pi X}$.

LEMMA 2.5. *If $\Phi: \sum_i \hat{\pi}^\gamma X(-,x_i) \to \sum_j \hat{\pi}^\gamma X(-,y_j)$ is an isomorphism of finite sums, then the sequences (x_i) and (y_j) agree up to order.*

Proof. This is Lemma 10.40 of [Lü]; $\hat{\pi}^\gamma X$ is not an EI-category as used in [Lü], but this result does not require that assumption (see Lück's comments after 9.42). □

It follows that to any isomorphism ϕ between finite-dimensional based $\hat{\pi}^\gamma X$-groups we can associate a matrix $[\phi]$ and hence an element of $\mathrm{Wh}_G(\pi X,\gamma)$. This allows us to define the map $\Phi: \mathrm{Wh}_G(X,\gamma) \to \mathrm{Wh}_G(\pi X,\gamma)$ connecting the geometric and the algebraic Whitehead groups. If $[Z,X]$ is an element of $\mathrm{Wh}_G(X,\gamma)$, we consider the relative chain complex $C(Z,X) = C_{\gamma+*}(Z,X)$. This is an acyclic based complex of $\hat{\pi}^\gamma Z$-groups; we choose a retraction $\pi Z \to \pi X$ giving a retraction $\hat{\pi}^\gamma Z \to \hat{\pi}^\gamma X$ so that we can consider $C(Z,X)$ as a based complex of $\hat{\pi}^\gamma X$-groups. Choose a contracting homotopy s, and consider the map

$$\partial + s: C(Z,X)_{\mathrm{odd}} \to C(Z,X)_{\mathrm{even}}$$

between two based $\hat{\pi}^\gamma X$-groups. By [Lü, 11.5], this map is an isomorphism and so the matrix $[\partial + s]$ determines an element of $\mathrm{Wh}_G(\pi X,\gamma)$. This element is independent of the choice of s, again by [Lü, 11.5]. It is easy to see that it is also unchanged by elementary expansions and contractions. Finally, choosing different characteristic maps for the cells or a different retraction $\pi Z \to \pi X$ would only change the matrix by trivial units, and so not change its image in the Whitehead group. We let $\Phi[Z,X]=[\partial+s]$. Similarly, if f is a chain homotopy equivalence between two $\hat{\pi}^\gamma X$-chain complexes, then we can define its torsion $\tau(f)\in \mathrm{Wh}_G(\pi X,\gamma)$ by applying the above construction to the acyclic complex given by the algebraic mapping cone of f.

Our next goal is to prove the following theorem.

THEOREM 2.6. *$\Phi: \mathrm{Wh}_G(X,\gamma)\to \mathrm{Wh}_G(\pi X,\gamma)$ is an isomorphism for any G-CW(γ) complex X.*

[Ro], [I3], and [Lü] rely on a splitting to reduce this to the nonequivariant case. We cannot expect such a splitting in our context, but we can generalize the nonequivariant proof directly.

Proof. We first show that Φ is injective. Suppose that $\Phi[Z,X]=0$. We can assume that (Z,X) is in simplified form by Proposition 2.3. Thus $Z= X\cup\bigcup e_i^r\cup\bigcup e_i^{r+1}$, where we may assume that r is even and at least $|\gamma|+2$. The relative chain complex $C(Z,X)$ is then concentrated in dimensions r and $r+1$, and $\Phi[Z,X]$ is given by the matrix of attaching maps of the $(r+1)$-cells to the r-cells with respect to some choice of characteristic maps. Since $\Phi[Z,X]=0$ it follows that this matrix can be reduced to the identity by a series of elementary column operations, multiplications of columns by trivial units, and expansion of the matrix by the identity. If the matrix actually were the identity it would be clear that $[Z,X]=0$, so it suffices to show how to realize each such matrix operation by a simple homotopy equivalence.

Let C be a column of the matrix corresponding to the cell e_i^{r+1} with attaching map ϕ_i and center $z_i: G/H\to Z$. Let $\rho: \pi Z\to\pi X$ be the chosen retraction with $\rho(z_i)=x_i$. Let $\epsilon\alpha\in\hat{\pi}^\gamma X(x_i,x_i)$ be a trivial unit as above and consider the operation $C\mapsto C(\epsilon\alpha)$. The operation $C\mapsto C\epsilon$ is realized by changing the attaching map of e_i^{r+1} by precomposing with the (linear) homeomorphism ϵ (assuming that we have made r large enough). The operation $C\mapsto C\alpha$ is realized by replacing the retraction ρ with the retraction ρ' defined by $\rho'(z)=\rho(z)$ on objects, while

$$\rho'(\beta: u\to v)=\eta(v)^{-1}\circ\rho(\beta)\circ\eta(u),$$

where $\eta(z)$ is α if $z=z_i$ and is the identity on $\rho(z)$ otherwise.

Now let C_i and C_j be two columns, corresponding to the cells e_i^{r+1} and e_j^{r+1} with attaching maps ϕ_i and ϕ_j. Consider the operation $C_i\mapsto C_i+C_j\beta$, where $\beta\in\hat{\pi}^\gamma X(\rho(z_i),\rho(z_j))$, $\rho:\pi Z\to\pi X$ being again the chosen retraction. We want to realize this by replacing ϕ_i by $\phi_i+\phi_j\beta$; what this means is the following. Let η be the natural isomorphism from the identity to ρ. Write the morphism $\eta(z_j)^{-1}\beta\eta(z_i)$ as a sum of terms of the form $z_i\leftarrow u\to z_j$ and consider one of these terms. This is equivalent to a diagram $G/H \xleftarrow{\sigma} G/L \xrightarrow{\tau} G/K$ and a G-homotopy h from the composite $z_i\circ\sigma$ to $z_j\circ\tau$. We may assume that $L\subset H$ and that this homotopy has the following form: the radial path outward from $z_i\circ\sigma$ to an L-fixed point w_i in ∂e_i^{r+1}, followed by an L-homotopy $\bar{h}$ from $\phi_i(w_i)$ to $\phi_j(w_j)$, where w_j is a K-fixed point on ∂e_j^{r+1}, followed by the radial path inward from w_j to z_j. We then add $\phi_j \bar{h}$ to ϕ_i along the orbit of w_i, and this extends to define $\phi_i' = \phi_i + \phi_j \beta$ in general. Call the resulting complex Z'. We must also define a retraction $\rho' : \pi Z' \to \pi X$. To do this, let $J: \phi_i \to \phi_i'$ be a (free) G-homotopy, and consider the complex T formed by attaching to Z two cells: one, e_i', attached by ϕ_i'; and the other, f, one dimension higher and attached by J in the obvious way to e_i^{r+1}, e_i', and to X. T contains Z' as a subcomplex. Since Z is a deformation retract of T, we obtain a retraction $\pi T \to \pi Z$. We let ρ' be the composite $\pi Z' \to \pi T \to \pi Z \to \pi X$. With this retraction one can check that the matrix of (Z', X) is the original matrix with the operation $C_i \mapsto C_i + C_j \beta$ performed. T shows that (Z', X) is simply homotopy equivalent to $(Z, X) \operatorname{rel} X$.

Expansion of the matrix by the identity is achieved in the obvious way by an elementary expansion.

We now show that Φ is surjective. Suppose that A is a finite invertible matrix in $\mathrm{GL}(\hat{\pi}^\gamma X)$. Construct a complex Z from X as follows. Let r be an even number larger than $\dim X$ and at least $|\gamma| + 2$. Let $x_1, \ldots, x_k$ be the objects in πX associated with the columns of A, with $x_i : G/H_i \to X$. Attach to X one r-cell for each x_i via the attaching map $G \times_{H_i} S(V(x_i)) \to G/H_i \to X$. Each column in A then describes how to attach an $(r+1)$-cell to the resulting complex, giving us the cell complex Z. The retraction $\rho : \pi Z \to \pi X$ is defined on the r-cell associated with x_i by sending every point of that cell to x_i; on the $(r+1)$-cell associated with the ith column we send all interior points again to x_i. Now the relative chain complex $C(Z, X)$ is contractible and the associated matrix is equivalent to A in $\mathrm{Wh}_G(\pi X, \gamma)$. We need to show that Z is G-homotopy equivalent to X. For each H the constructions of [CW1, §5] give a functor from $\hat{\pi}^\gamma X$-groups to $\mathbf{Z}[\pi X^H]$-groups taking $C(Z, X)$ to $C(\tilde{Z}^H, \tilde{X}^H)$. The latter is the sum over the components of X^H of the non-equivariant relative chains of the universal covers of the components, possibly with some shifts in dimension. The contractibility of $C(Z, X)$ then implies the contractibility of each summand in $C(\tilde{Z}^H, \tilde{X}^H)$, whence $\tilde{X}^H \to \tilde{Z}^H$ is a homology equivalence. Since $X^H \to Z^H$ is a π_1-equivalence, it follows that $X^H \to Z^H$ is a homotopy equivalence. Since this is true for each H, it follows that $X \to Z$ is a G-homotopy equivalence. Thus $[Z, X] \in \mathrm{Wh}_G(X, \gamma)$ and $\Phi[Z, X] = [A]$ as desired. □

As in [Lü, 4.27] we can define the Whitehead group $\mathrm{Wh}_G(X, \gamma)$ for any G-space X and representation γ of πX by setting $\mathrm{Wh}_G(X, \gamma) = \mathrm{Wh}_G(Z, \gamma)$ for any G–CW(γ)-approximation Z of X. This is well-defined up to canonical isomorphism.

We shall also need the following definitions.

Definition 2.7. Let X be a G–CW(γ) complex and let A be a subcomplex. Similarly, let Y be a G–CW(δ) complex and let B be a subcomplex. A G-map $f: (X, A) \to (Y, B)$ covered by a map of representations $\gamma \to \delta$ is said to be a *G-simple homotopy equivalence of pairs* if the two maps $X \to Y$ and $A \to B$ are both G-simple homotopy equivalences. If $C(X, A) \to C(Y, B)$ is a chain homotopy equivalence with zero torsion, then we say that f is a *relative* G-simple homotopy equivalence.

As in Lemma 2.5 of [Wal] we have the following.

PROPOSITION 2.8. *Let $(X, A) \to (Y, B)$ be a map of pairs of complexes as above. If two out of the three chain maps $C(A) \to C(B)$, $C(X) \to C(Y)$, and $C(X, A) \to C(Y, B)$ are simple equivalences, then so is the third, and then the map $(X, A) \to (Y, B)$ is a G-simple homotopy equivalence of pairs.*

Finally, we need in various places the following result, which is proved as in [Lü, 14.21].

PROPOSITION 2.9. *If $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ are G-homotopy equivalences of complexes, then there are invariants $\chi(Y_1)$ and $\chi(Y_2)$ such that*

$$\tau(f_1 \times f_2) = \tau(f_1)\chi(Y_2) + \chi(Y_1)\tau(f_2).$$

In particular, if f_1 and f_2 are both G-simple homotopy equivalences, then so is the product $f_1 \times f_2$.

3. Applications to Smooth Manifolds

Let M be a compact G-manifold and let τ be its tangent representation. In this section we shall describe the duality between G–CW structures and G–CW(τ) structures on M, and show that M carries an essentially unique smooth G–CW(τ) structure.

DEFINITION 3.1. Let M be a G-manifold with tangent representation τ. A *smooth simplicial* G–CW(τ)*-structure* on M consists of a smooth G-triangulation with skeleta $\{M^k\}$ and a filtration of M by G-subcomplexes $M^{\tau-k}$ such that $\{M^{\tau-k}\}$ is the collection of skeleta of a G–CW(τ) structure on M. We also require that the closed cells of the G–CW(τ) structure be smoothly embedded in M. We shall use the word *simplex* to mean a simplex of the G-triangulation, and the word *cell* to mean a cell of the G–CW(τ) structure.

LEMMA 3.2. *Any smooth G-manifold has a smooth simplicial G*–CW(τ) *structure.*

Proof. If M is a smooth G-manifold, then it has a smooth G-triangulation by [I2]. The dual G–CW(τ) structure is constructed as the usual dual complex; for example, the τ-dimensional cells are the orbits of the stars in the first barycentric subdivision of the original vertices. The smooth simplicial G–CW(τ) structure is then given by the first barycentric subdivision of the triangulation together with the G–CW(τ) structure dual to the original triangulation. □

Let M be a smooth G-manifold with boundary ∂M. A smooth G-triangulation of M is algebraically related to its dual G-CW(τ) structure as follows.

Let $C_*(M)$ be the chain complex of the triangulation; thus each $C_k(M)$ is a contravariant functor on πM. Let $C_{\tau+*}(M)$ be the chain complex of the G–CW(τ)-structure; since the cells $G\times_H D(V(x)-k)$ are embedded in M, we can consider $C_{\tau-k}(M)$ to be a contravariant functor on πX^{op}. We define cochain complexes in the following way.

Definition 3.3. If C is a contravariant functor from a category $\mathcal{B}$ to the category of abelian groups, we define its dual C^T to be the covariant functor given by

$$C^T(a) = \mathbf{Z}\operatorname{Hom}_{\mathcal{B}}(C, \mathbf{Z}\mathcal{B}(-, a)).$$

(Notice that if $C = \mathbf{Z}\mathcal{B}(-, b)$ then $C^T = \mathbf{Z}\mathcal{B}(b, -)$ by Yoneda's lemma.) If C_* is a chain complex of contravariant functors on $\mathcal{B}$, we let C^* be the cochain complex of covariant functors on $\mathcal{B}$ defined by $C^k = (C_k)^T$. In particular, we let $C^*(M)$ be the cochain complex of the triangulation of M, and we let $C^{\tau+*}(M)$ be the cochain complex of the G–CW(τ) structure. We define the chain and cochain complexes of the pair $(M, \partial M)$ similarly.

Proposition 3.4. *Let M be a compact smooth G-manifold, let $C_*(M)$ be the chains of a smooth triangulation, and let $C_{\tau-*}(M)$ be the chains of the dual G–CW(τ) structure. Then there are canonical based isomorphisms of complexes*

$$\begin{aligned} C_*(M) &\cong C^{\tau-*}(M, \partial M), \\ C_*(M, \partial M) &\cong C^{\tau-*}(M), \\ C_{\tau-*}(M) &\cong C^*(M, \partial M), \quad \textit{and} \\ C_{\tau-*}(M, \partial M) &\cong C^*(M). \end{aligned}$$

Proof. By construction there is a one-to-one correspondence between the simplices of M and the relative τ-cells of $(M, \partial M)$ that associates to each k-simplex a $(\tau-k)$-cell. This establishes the isomorphism

$$C_k(M) \cong C^{\tau-k}(M, \partial M),$$

and one can check that this preserves the respective boundaries. The other three isomorphisms are similar. □

We now wish to show how to construct the dual G–CW complex to any smooth simplicial G–CW(τ) structure on a G-manifold.

Construction 3.5. (a) Let M be a G-manifold with a smooth simplicial G–CW(τ) structure. We construct a new smooth triangulation of M having exactly one vertex in the interior of each cell of M. The construction is by induction on the skeleta of the G–CW(τ) structure. The beginning of the induction is forced. Assume that we have triangulated $M^{\tau-k-1}$ as desired. We extend this to a triangulation of $M^{\tau-k}$ as follows. Using a diffeomorphism of a $(\tau-k)$-cell with $G\times_H D(V(x)-k)$, we take the vertex in this cell to be the point corresponding to the origin, and the triangulation of the cell to be the cone on the triangulation of its boundary, as in barycentric subdivision. We call this simplicial structure the *barycentric triangulation* of M associated to the G–CW(τ) structure.

(b) Let M be as above. The G–CW structure dual to the G-CW(τ) structure is constructed in the following way. Let $n=|\tau|$. The n-cells of the desired structure are the orbits of the stars of the vertices of the G–CW(τ) structure taken in the barycentric triangulation. The lower-dimensional cells are the orbits of the intersections of these stars.

This construction is inverse to the construction used in Lemma 3.2. We could generalize Lemma 3.2 by using the same procedure as in Construction 3.5 to make these precisely inverse. Both constructions are illustrated by Figure 1, which shows part of a G-manifold. The shaded regions indicate some of the simplices of a G-triangulation, while the solid bold figures are cells in the dual G–CW(τ) structure. The remaining lines give the barycentric subdivision or barycentric triangulation.

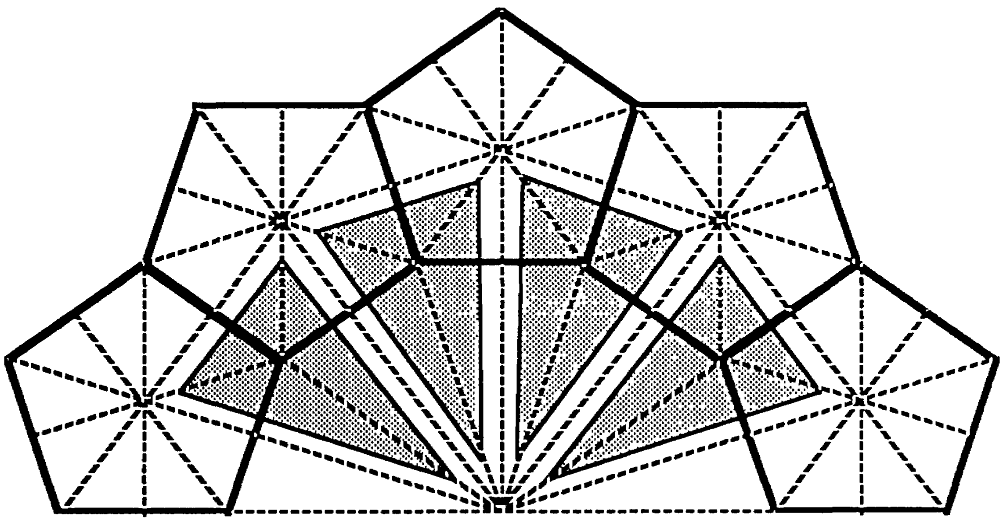

Figure 1

We now have the following proposition.

PROPOSITION 3.6. *Let M be a compact smooth G-manifold with a smooth G–CW(τ) structure. If $C_*(M)$ denotes the chain complex of the dual G–CW structure, then there are canonical isomorphisms of complexes*

$$C_*(M) \cong C^{\tau-*}(M, \partial M),$$
$$C_*(M, \partial M) \cong C^{\tau-*}(M),$$
$$C_{\tau-*}(M) \cong C^*(M, \partial M), \quad \textit{and}$$
$$C_{\tau-*}(M, \partial M) \cong C^*(M).$$

DEFINITION 3.7. If M is a compact smooth G-manifold with a smooth G–CW(τ) structure $\mathcal{S}$, then a *cosubdivision* of $\mathcal{S}$ is the G–CW(τ) structure

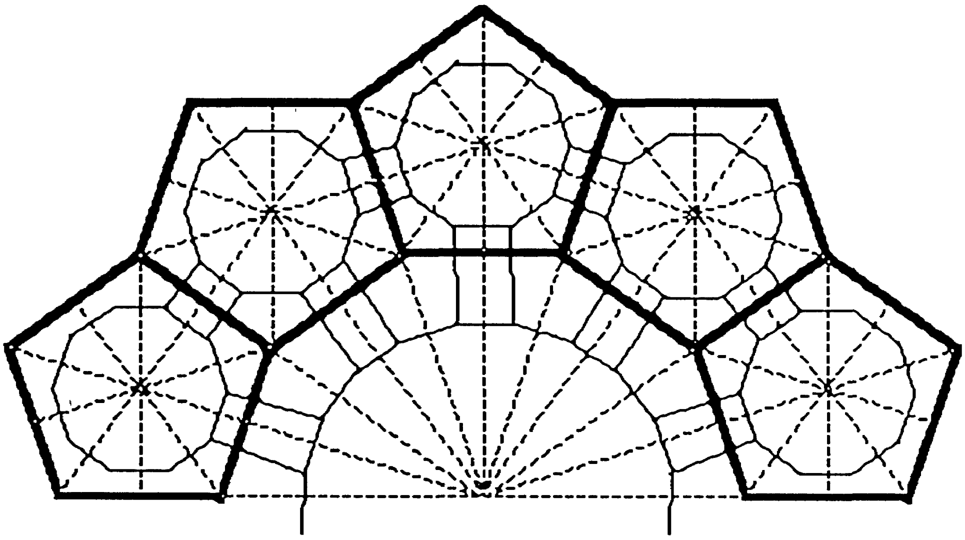

Figure 2

dual to any smooth subdivision of the G–CW structure dual to $\mathcal{S}$. This is illustrated in Figure 2. The original G–CW(τ) structure is given by the solid bold lines. A cosubdivision is given by the solid thin lines; this structure is dual to the one given by the dotted lines.

Now let $\mathcal{S}$ be a smooth G–CW(τ) structure on M, and let $\mathcal{D}$ be its dual G–CW structure. Let $\mathcal{D}'$ be a subdivision of $\mathcal{D}$ and let $\mathcal{S}'$ be the dual to $\mathcal{D}'$, so that $\mathcal{S}'$ is a cosubdivision of $\mathcal{S}$. The map $i\colon (M;\mathcal{D}) \to (M;\mathcal{D}')$ given by the identity is cellular, and so defines maps of chains:

$$j_*\colon C_{\tau-*}(M;\mathcal{S}') \cong C^*(M,\partial M;\mathcal{D}') \to C^*(M,\partial M;\mathcal{D}) \cong C_{\tau-*}(M;\mathcal{S})$$

and

$$j_*\colon C_{\tau-*}(M,\partial M;\mathcal{S}') \cong C^*(M;\mathcal{D}') \to C^*(M;\mathcal{D}) \cong C_{\tau-*}(M,\partial M;\mathcal{S}).$$

LEMMA 3.8. *There is a cellular approximation to the identity $j\colon (M;\mathcal{S}') \to (M;\mathcal{S})$ inducing the maps j_* displayed above.*

Proof. The algebraic map j_* tells us where to send vertices; in fact, any vertex is in the interior of some top-dimensional simplex of $\mathcal{D}$, and will be sent to the center of this simplex. We construct j inductively on the skeleta of $\mathcal{S}'$ to have the following property: If s' is a cell of $\mathcal{S}'$ dual to the cell d' in $\mathcal{D}'$, then d' is contained in a cell d of minimal dimension in $\mathcal{D}$; we require that $j(s')$ be contained in the cell $s = \phi(s')$ dual to d. Assume then that j has been constructed on the $(\tau - k - 1)$-skeleton of $\mathcal{S}'$ with this property. We now want to extend j over a $(\tau - k)$-G-cell s'. If we know that $j(\partial s') \subset \phi(s')$ then we can extend j over s' by coning. But this inclusion follows easily from the following two facts and the inductive hypothesis: (1) $a \subset \partial b$ if and only if $\beta \subset \partial\alpha$, where α is dual to the cell a and β is dual to b; and (2) if $a' \subset \partial b'$ then $a \subset \partial b$, where a is the minimal cell of $\mathcal{D}$ containing the cell a' of $\mathcal{D}'$ and similarly for b and b'.

It is not difficult to check that j is homotopic to the identity since it takes each cell of $\mathcal{S}$ to itself by a degree-1 map. It is also easy to see that j induces the algebraic map j_*. □

The map j and the homotopy can be seen in Figure 2. Each decagon inside a pentagon expands to fill the pentagon, while each hexagon surrounding a vertex of a pentagon contracts to that vertex.

THEOREM 3.9. *Let $(M, \partial M)$ be a smooth compact G-manifold. Then any two smooth G-*CW(τ) *structures on $(M, \partial M)$ are G-simply homotopy equivalent as pairs.*

Proof. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two smooth G–CW(τ) structures on M, with dual G–CW structures $\mathfrak{D}_1$ and $\mathfrak{D}_2$. By [I2], $\mathfrak{D}_1$ and $\mathfrak{D}_2$ have a common subdivision $\mathfrak{D}$. Let $\mathcal{S}$ be the dual G–CW(τ) structure, so that $\mathcal{S}$ is "common co-subdivision" of $\mathcal{S}_1$ and $\mathcal{S}_2$. By [I1] the map $i^*: C^*(M;\mathfrak{D}) \to C^*(M;\mathfrak{D}_1)$ is a simple equivalence and similarly for $(M, \partial M)$, so the map $j_*: C_{\tau-*}(M;\mathcal{S}) \to C_{\tau-*}(M;\mathcal{S}_1)$ is a simple equivalence, as is the map on pairs. By Theorem 2.6 we then get that the identity map $(M;\mathcal{S}) \to (M;\mathcal{S}_1)$ is a G-simple homotopy equivalence. Similarly, the identity map $(M;\mathcal{S}) \to (M;\mathcal{S}_2)$ is a G-simple homotopy equivalence, and so the identity map $(M;\mathcal{S}_1) \to (M;\mathcal{S}_2)$ is a G-simple homotopy equivalence, and similarly for the pair $(M, \partial M)$. By Proposition 2.8, the identity map is a G-simple homotopy equivalence of pairs. □

If $\tau = \gamma + \delta$ then we could similarly define what one means by a smooth simplicial G–CW(γ) structure on M. We conjecture that such structures exist and are unique up to simple equivalence. Such a result would simplify some of the constructions of the next section.

4. Spaces with Two Cell Structures

For use in later sections, we need some results about spaces having simultaneously two cellular structures. These generalize the simplicial G–CW(τ) structures used in the last section.

DEFINITION 4.1. Let X be a G-space and let γ and ρ be representations of πX. A G–CW$(\gamma, \gamma+\rho)$ *structure* on X consists of a G–CW(γ) structure with skeleta $\{X^{\gamma+*}\}$ and a filtration of X by G-subcomplexes $X^{\gamma+\rho+*}$ such that $\{X^{\gamma+\rho+*}\}$ is the collection of skeleta of a G–CW$(\gamma+\rho)$ structure on X. We also make the following restriction on the G-cells of the G–CW$(\gamma+\rho)$ structure: Such a cell is to be the image of a copy of $G\times_H D(V(x)\oplus W(x)+k)$ with a G–CW$(\gamma(x))$ structure, and we require that this structure on the pair $(G\times_H D(V(x)\oplus W(x)+k),\ G\times_H S(V(x)\oplus W(x)+k))$ be G-simply homotopy equivalent to the product (over G/H) of a smooth simplicial G–CW$(\gamma(x))$ structure on $G\times_H D(V(x)+i)$ and a smooth G–CW structure on $G\times_H D(W(x)+j)$ (with $i+j=k$ possibly negative).

Note that a change in the characteristic map of a cell as allowed by Definition 2.1 will take such a G–CW($\gamma(x)$) structure to another one, so the requirement above that the G–CW($\gamma(x)$) structure on a cell be equivalent to a product structure does not depend on the choice of characteristic map.

LEMMA 4.2. *Any two product structures on*

$$(G\times_H D(V\oplus W+k), G\times_H S(V\oplus W+k))$$

as above are G-simply homotopy equivalent as pairs.

Proof. Using Theorem 3.9, we have the following H-simple homotopy equivalences of H–CW(V) complexes:

$$\begin{aligned} D(V+W+k) &= D(V+i)\times D(W+j) \\ &\simeq_s D(V-V^H)\times D(V^H+i)\times D(W^H+j)\times D(W-W^H) \\ &\simeq_s D(V-V^H)\times D(V^H+W^H+k)\times D(W-W^H), \end{aligned}$$

and similarly for these complexes relative to their bounding spheres. Since the smooth structures on each factor of the last space are unique up to equivalence of pairs (being respectively an H–CW(V), an H–CW, and an H–CW structure), the result follows from Propositions 2.8 and 2.9. □

LEMMA 4.3. *Let X be a G–CW(γ) complex, let F: $(Y_1, B_1)\to(Y_2, B_2)$ be a cellular G-map that is a relative G-simple homotopy equivalence, and let $f_1: B_1\to X$ and $f_2: B_2\to X$ be cellular maps such that $f_2\simeq f_1\circ F$. Then*

$$X\cup_{f_1}Y_1\simeq_s X\cup_{f_2}Y_2.$$

Proof. This follows by considering the diagram

$$\begin{array}{ccccccccc} 0 \to & C(X) & \to & C(X\cup_{f_1}Y_1) & \to & C(X\cup_{f_1}Y_1, X) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & C(X) & \to & C(X\cup_{f_2}Y_2) & \to & C(X\cup_{f_2}Y_2, X) & \to 0. \end{array}$$

Since $C(X\cup_{f_1}Y_1, X)\cong C(Y_1, B_1)$ and $C(X\cup_{f_2}Y_2, X)\cong C(Y_2, B_2)$, the outer two vertical arrows are G-simple equivalences, and so the middle one is as well. □

These two lemmas imply that the G–CW(γ) simple homotopy type of a G–CW($\gamma, \gamma+\rho$) complex is essentially independent of the particular G–CW(γ) structure, but depends only on the G–CW($\gamma+\rho$) structure. In other words, a given G–CW($\gamma+\rho$) complex has, up to G–CW(γ) simple homotopy equivalence, a unique G–CW($\gamma, \gamma+\rho$) structure.

We can now develop the theory of G–CW($\gamma, \gamma+\rho$) complexes along the lines of more familiar complexes. In particular, we have the following definitions and results.

DEFINITION 4.4. If X is a G–CW($\gamma, \gamma+\rho$) complex and Y is a G–CW($\delta, \delta+\sigma$) complex, then a *cellular map* $f: X\to Y$ is a map that is cellular with respect to both the G–CW(γ) and the G–CW($\gamma+\rho$) structures on X. In particular, f must really be a triple of maps: one, the G-map $f: X \to Y$; and the others, maps of representations $\gamma \to \delta$ and $\rho \to \sigma$.

PROPOSITION 4.5. *Let X be a G–CW($\gamma, \gamma+\rho$) complex and let Y be a G–CW($\delta, \delta+\sigma$) complex. Let $f: X \to Y$ be a G-map covered by maps of representation $\gamma \to \delta$ and $\rho \to \sigma$, and assume that f is cellular on a subcomplex $A \subset X$. Then $f \simeq f'$ rel A, where f' is cellular on all of X with the homotopy covered by maps of representations extending the given ones.*

Proof. The proof is by induction on the subcomplexes $A \cup X^{\gamma+\rho+k}$. Thus, assume that $f \mid A \cup X^{\gamma+\rho+k-1} \simeq f'$ rel A, where $f': A \cup X^{\gamma+\rho+k-1} \to Y$ is cellular. Using G–CW($\gamma+\rho$) approximation, we can extend f' to a map $f'': A \cup X^{\gamma+\rho+k} \to Y$ that is cellular with respect to the G–CW($\gamma+\rho$) structure, and we can extend the homotopy to a homotopy $f \mid A \cup X^{\gamma+\rho+k} \simeq f''$ rel A. Considering f'' as a map into $Y^{\delta+\sigma+k}$ that is already cellular on $A \cup X^{\gamma+\rho+k-1}$ with respect to the G–CW(γ) structure, we now homotope f'' rel $A \cup X^{\gamma+\rho+k-1}$ to a map that is cellular with respect to the G–CW(γ) structure. This map we take as f' for the next step, as it clearly respects both cellular structures. We leave to the reader the job of fitting these homotopies together in the case where X is an infinite complex. □

DEFINITION 4.6. A cellular inclusion $X \to Y$ of G–CW($\gamma, \gamma+\rho$) complexes is an *elementary expansion* if Y is an elementary expansion of X as a G–CW($\gamma+\rho$) complex.

This leads to the notion of G–CW($\gamma, \gamma+\rho$) simple homotopy equivalence, and to the definition of the geometric Whitehead group $\mathrm{Wh}_G(X, \gamma, \gamma+\rho)$. By definition, a G–CW($\gamma, \gamma+\rho$) simple homotopy equivalence is a G–CW($\gamma+\rho$) simple homotopy equivalence. We also have the following.

PROPOSITION 4.7. *If $f: X \to Y$ is a G–CW($\gamma, \gamma+\rho$) simple homotopy equivalence, then f is also a G–CW(γ) simple homotopy equivalence.*

Proof. It suffices to consider the case where f is an elementary expansion. Let $Y = X \cup e^n \cup e^{n+1}$, where

$$e^{n+1} = G \times_H D(V+W+k+1) \cong G \times_H D(V+W+k) \times I$$

and

$$e^n = G \times_H [D(V+W+k) \times 1 \cup S(V+W+k) \times I] \cong G \times_H D(V+W+k),$$

with some H–CW(V) structures on these discs. Let E^{n+1} be

$$G \times_H D(V+W+k) \times I$$

with the cell structure given by the G–CW($G \times_H V$) structure on

$$G \times_H D(V+W+k) \times 0$$

crossed with an ordinary structure on I. By Lemma 4.2, this structure is G–CW($G\times_H V$) equivalent to the structure on $(e^n, \partial e^n)$. Let

$$E^n = G\times_H[D(V+W+k)\times 1\cup S(V+W+k)\times I]$$

with the structure it inherits as a subcomplex of E^{n+1}. Again, E^n is G-simply homotopy equivalent to e^n. We now want to say that $X\cup E^n\cup E^{n+1}$ is G–CW(γ) equivalent to $Y \operatorname{rel} X$, where we use exactly the same attaching maps as in Y. We do this in two steps: $X\cup E^n \simeq_s X\cup e^n$, by Lemma 4.3. Then $(X\cup E^n)\cup E^{n+1}\simeq_s (X\cup e^n)\cup e^{n+1}$, by the same argument modified to account for the fact that the spaces to which we are attaching cells are not identical but are simply homotopy equivalent. It is obvious that

$$X\to X\cup E^n\cup E^{n+1}$$

is a G–CW(γ) simple homotopy equivalence, because it is a mapping cylinder, and the result follows. □

COROLLARY 4.8. *There is a homomorphism*

$$\epsilon: \mathrm{Wh}_G(X,\gamma,\gamma+\rho)\to \mathrm{Wh}_G(X,\gamma)$$

given by forgetting the G–CW($\gamma+\rho$) structures.

5. Stable Whitehead Groups

In order to simplify our theory somewhat, we show in this section that all of the groups $\mathrm{Wh}_G(X,\gamma)$ for varying γ inject naturally into a group $\underline{\mathrm{Wh}}_G(\pi X)$, which we call the *stable Whitehead group* of X. Our argument uses the spaces introduced in the last section.

PROPOSITION 5.1. *Let (Z,X) be a relatively finite G–CW($\gamma,\gamma+\rho$) pair, with X a strong deformation retract of Z. Then $Z\simeq_s W \operatorname{rel} X$, where $W = X\cup\bigcup e_i^r\cup\bigcup e_i^{r+1}$ and r can be taken as large as desired.*

As in Proposition 2.3, this follows by the arguments of [Co] or [I1], using γ-cellular approximation of maps in order to make the constructed complexes be G–CW($\gamma,\gamma+\rho$).

PROPOSITION 5.2. $\Phi: \mathrm{Wh}_G(X,\gamma,\gamma+\rho)\to \mathrm{Wh}_G(\pi X,\gamma+\rho)$ *is an isomorphism.*

Proof. Simply repeat the proof of Theorem 2.6, but use cellular approximation to make all attaching maps cellular. □

COROLLARY 5.3. $\mathrm{Wh}_G(X,\gamma,\gamma+\rho)\to \mathrm{Wh}_G(X,\gamma+\rho)$ *is an isomorphism.*

Let X be a G–CW(γ)-complex and let V be a representation of G. If we give $D(V)$ its smooth G–CW($0,V$) structure, then multiplication by $D(V)$ defines a map

$$\sigma_V: \mathrm{Wh}_G(X,\gamma)\to \mathrm{Wh}_G(X\times D(V),\gamma,\gamma+V).$$

LEMMA 5.4. *$\sigma_V: \mathrm{Wh}_G(X,\gamma) \to \mathrm{Wh}_G(X \times D(V), \gamma, \gamma+V)$ is split injective.*

Proof. The splitting is given by the map ϵ of Lemma 4.8 and the fact that $X \times D(V) \simeq_s X$ rel X as G–CW(γ) complexes. □

Since the projection $X \times D(V) \to X$ is a G–CW$(\gamma, \gamma+V)$ approximation to X, this defines an injection

$$\sigma_V: \mathrm{Wh}_G(X,\gamma) \to \mathrm{Wh}_G(X,\gamma,\gamma+V) \cong \mathrm{Wh}_G(\pi X, \gamma+V).$$

On the other hand, there is an algebraic stabilization map

$$\sigma_V: \mathrm{Wh}_G(\pi X,\gamma) \to \mathrm{Wh}_G(\pi X, \gamma+V)$$

defined by the inclusion of $\hat{\pi}^\gamma X$ in $\hat{\pi}^{\gamma+V} X$.

LEMMA 5.5. *If V has the property that $|V^H|$ is even for all subgroups H, then the diagram*

$$\begin{array}{ccc} \mathrm{Wh}_G(X,\gamma) & \xrightarrow{\sigma} & \mathrm{Wh}_G(X,\gamma+V) \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi}\downarrow \\ \mathrm{Wh}_G(\pi X,\gamma) & \xrightarrow{\sigma} & \mathrm{Wh}_G(\pi X,\gamma+V) \end{array}$$

commutes.

Proof. $\Phi\sigma[Z,X]$ is the torsion of

$$C_{\gamma+V+*}(Z \times D(V), X \times D(V)) \cong C_{\gamma+*}(Z,X) \otimes C_{V+*}(D(V)).$$

As in [Lü] this is equal to the torsion of $C_{\gamma+*}(Z,X)$ multiplied by the Euler characteristic $\chi_V(D(V))$ of $C_{V+*}(D(V))$. In our context, $\chi_V(D(V))$ is the alternating sum of the orbits of the centers of the cells of $D(V)$ and lives in $A_V(G)$, the subgroup of the Burnside ring generated by those orbits of G that embed in V. Since the $(V-k)$-cells in the G–CW(V) structure on $D(V)$ are in one-to-one correspondence with the relative k-cells in an ordinary G–CW structure on $(D(V), S(V))$, it follows that $\chi_V(D(V)) = \chi(D(V), S(V))$, the ordinary Euler characteristic given in [CWW] or [LMS]. Any element of the Burnside ring is determined by its fixed sets, and as in [CWW] or [LMS] these are given by

$$\chi(D(V), S(V))^H = \chi(D(V)^H, S(V)^H) = 1.$$

Thus $\chi_V(D(V)) = 1$, and the diagram commutes. □

Now define $\underline{\mathrm{Wh}}_G(\pi X)$ in the same way that we defined $\mathrm{Wh}_G(\pi X,\gamma)$, except that we use $\hat{\pi} X$ in place of $\hat{\pi}^\gamma X$. Since the inclusions induce an isomorphism $\mathrm{colim}_V\, \hat{\pi}^{\gamma+V} X \cong \hat{\pi} X$ for any γ, it follows that $\underline{\mathrm{Wh}}_G(\pi X) \cong \mathrm{colim}_V\, \mathrm{Wh}_G(\pi X, \gamma+V)$ for any γ.

THEOREM 5.6. *For any γ, the homomorphisms $\mathrm{Wh}_G(\pi X,\gamma) \to \underline{\mathrm{Wh}}_G(\pi X)$ and $\mathrm{Wh}_G(X,\gamma) \to \mathrm{Wh}_G(\pi X,\gamma) \to \underline{\mathrm{Wh}}_G(\pi X)$ are injective. Further, the* *induced map* $\operatorname{colim}_V \mathrm{Wh}_G(X \times D(V), \gamma + V) \to \underline{\mathrm{Wh}}_G(\pi X)$ *is an isomorphism for any* γ.

6. Algebra

In preparation for studying Poincaré duality in the next section, we recall some algebra from [CW1]. As there, it is easiest to allow arbitrary G-sets in places where we have heretofore only allowed G-orbits (i.e., in $\mathcal{G}$ and πX), and we will do so without further comment.

Recall that *$\hat{\pi}X$-groups* are contravariant additive functors $\hat{\pi}X \to \mathcal{Ab}$ that take disjoint unions to direct sums. At times it will be useful to consider a covariant functor to be a contravariant functor via the isomorphism of $\hat{\pi}X$ with its opposite category. This isomorphism is given by taking a morphism $x \leftarrow u \to y$ to its dual $y \leftarrow u \to x$. In particular, if T is a contravariant functor, $f: x \to y$ is a map, and $\bar{f}: y \to x$ is its dual, we write f^* for $T(f)$ and f_* for $T(\bar{f})$.

Definition 6.1. If S and T are $\hat{\pi}X$-groups, we define the group

$$S \otimes_{\hat{\pi}X} T = \textstyle\sum_z S(z) \otimes T(z)/\approx,$$

where $f^*s \otimes t \approx s \otimes f_* t$; we define $\mathrm{Hom}_{\hat{\pi}X}(S, T)$ to be the group of natural transformations from S to T. When using these constructions, one should keep in mind the isomorphisms

$$\mathrm{Hom}_{\hat{\pi}X}(\hat{\pi}X(-,x), S) \cong S(x) \cong \hat{\pi}X(-,x) \otimes_{\hat{\pi}X} S.$$

The homology and cohomology groups of X with coefficients in S were defined in [CW1] to be the homology groups of the chain complexes $C_*(X) \otimes_{\hat{\pi}X} S$ and $\mathrm{Hom}_{\hat{\pi}X}(C_* X, S)$, respectively. Here, $C_*(X)$ denotes the chains with respect to a G–CW structure, giving the integer-graded homology, or the chains with respect to a G–CW(γ) structure, giving the $(\gamma + *)$-homology. We can also repeat the definition of the cochain complex given in Definition 3.3:

$$C^*(X) = \mathrm{Hom}_{\hat{\pi}X}(C_*(X), \hat{\pi}X(-,-)).$$

Here we consider $\hat{\pi}X(-,-)$ as a functor of two variables, contravariant in the first and covariant in the second. "$\mathrm{Hom}_{\hat{\pi}X}$" is then taken with respect to the contravariant variable, making the result a covariant functor of the remaining variable. As mentioned at the beginning of the section, we can then use the self-duality of $\hat{\pi}X$ to consider $C^k(X)$ to be a $\hat{\pi}X$-group. If X is a G-cellular complex of finite type (i.e., has finitely many cells in each dimension), then

$$C^*(X) \otimes_{\hat{\pi}X} S \cong \mathrm{Hom}_{\hat{\pi}X}(C_* X, S),$$

and so we can compute the cohomology using either chain complex.

We need the following notation: If $x: A \to X$ and $y: B \to Y$ are objects in $\hat{\pi}X$ and $\hat{\pi}Y$ respectively, then $x \times y$ will denote the object in $\hat{\pi}(X \times Y)$ given by the map $x\times y: A\times B\to X\times Y$. Notice that not all objects of $\hat{\pi}(X\times Y)$ have this form.

DEFINITION 6.2. If S is a $\hat{\pi}X$-group and T is a $\hat{\pi}Y$-group, then the $\hat{\pi}(X\times Y)$-group $S\,\Box\, T$ is defined by

$$(S\,\Box\, T)(z)=\Sigma_{z\to x\times y}\, S(x)\otimes T(y)/\approx,$$

where $(s\otimes t)_{z\to x'\times y'\xrightarrow{f\times g} x\times y}\approx (f^*s\otimes g^*t)_{z\to x'\times y'}$.

The importance to us of this "box product" comes from the easily checked observation that

$$\hat{\pi}X(-,x_0)\,\Box\,\hat{\pi}Y(-,y_0)\cong \hat{\pi}(X\times Y)(-,x_0\times y_0),$$

from which it follows that $C_*(X\times Y)\cong C_*(X)\,\Box\, C_*(Y)$. (Note that, if (X,γ) and (Y,δ) are G-cellular spaces, then $X\times Y$ has an obvious G–CW$(\gamma\times\delta)$ structure, and this is the one we use; this is simplified somewhat if we allow cells to have the form of disc bundles over finite G-sets, rather than orbits, so that a cell in X and one in Y give a product cell in $X\times Y$.) The analogous statement holds for products of pairs.

Suppose now that U is a $\hat{\pi}(X\times Y)$-group and that T is a $\hat{\pi}Y$-group. Then we define the $\hat{\pi}X$-group $T\otimes_{\hat{\pi}Y}U$ by

$$(T\otimes_{\hat{\pi}Y}U)(x)=T\otimes_{\hat{\pi}Y}U(x\times -),$$

where, on the right, we regard $U(x\times -)$ as a $\hat{\pi}Y$-group and use $\otimes_{\hat{\pi}Y}$ in its previous sense. One of the properties of this tensor is that

$$(S\,\Box\, T)\otimes_{\hat{\pi}X\times Y}U\cong S\otimes_{\hat{\pi}X}(T\otimes_{\hat{\pi}Y}U);$$

this follows by playing with the definitions.

Another piece of algebra we need is this: Suppose that $f: X\to Y$ is a G-map. Given a $\hat{\pi}Y$-group T, we can form the $\hat{\pi}X$-group f^*T by composing with the induced map $\hat{\pi}X\to\hat{\pi}Y$. There is a left adjoint to this construction. Given a $\hat{\pi}X$-group S, we can form a $\hat{\pi}Y$-group f_*S by letting

$$(f_*S)(y)=\Sigma_{y\to fx}\, S(x)/\approx,$$

where $s_{y\to fx'\xrightarrow{fh} fx}\approx (h^*s)_{y\to fx'}$. The homomorphisms of $\hat{\pi}X$-groups $S\to f^*T$ are in one-to-one correspondence with the homomorphisms of $\hat{\pi}Y$-groups $f_*S\to T$. It is also not difficult to see that $f^*T\otimes_{\hat{\pi}X}S\cong T\otimes_{\hat{\pi}Y}f_*S$.

Finally, we introduce the coefficient system that, in this theory, plays the role that $\mathbf{Z}$ plays nonequivariantly. Let $\mathcal{A}=\phi^*\hat{\mathcal{G}}(-,G/G)$, so that $\mathcal{A}(x)=\hat{\mathcal{G}}(\phi(x),G/G)$; if $\phi(x)=G/H$ than $\mathcal{A}(x)$ is the Burnside ring of H. We call $\mathcal{A}$ the *Burnside ring coefficient system*. An important property of this system is the following.

LEMMA 6.3. $(\Delta_*\mathcal{A})(x\times y)\cong\hat{\pi}X(x,y)$ *naturally in x and y.*

Proof. There is a natural map $\hat{\pi}X(x,y)\to(\Delta_*\mathcal{A})(x\times y)$ taking a morphism $x\leftarrow p\to y$ to the element $1_{\Delta p\to x\times y}$, where $1\in\mathcal{A}(p)$ is the unit in the Burnside ring, given by the projection $\phi(p)\to G/G$.

In the other direction, suppose that we have an element $a_{\Delta z \to x \times y}$. Let the map $\Delta z \to x \times y$ be represented by the diagram $\Delta z \leftarrow d \to x \times y$, and let $a \in \mathfrak{A}(z)$ be given by the diagram $\phi(z) \leftarrow s \to G/G$. We can form the following diagram in $\hat{\pi}(X \times X)$:

$$\begin{array}{ccccccc} & & \Delta p & & & & \\ & \swarrow & & \searrow & & & \\ \Delta(z \circ \sigma) & & & & d & & \\ & \searrow & & \swarrow & & \searrow & \\ & & \Delta z & & & & x \times y. \end{array}$$

Here $\sigma: s \to \phi(z)$ is the map representing a, and the square is a pullback square. This square is constructed by first forming the pullback of the underlying G-sets, and then using composition of maps to get p and the left side of the square; there is then a unique morphism in $\hat{\pi}(X \times X)$ from $\Delta p \to d$ completing the diagram. We now take a to the map $[x \leftarrow p \to y] \in \hat{\pi}X(x, y)$. One now checks that this is well-defined, and is inverse to the map first described. □

COROLLARY 6.4. *If S is any $\hat{\pi}X$-group, then* $S \otimes_{\hat{\pi}X} \Delta_* \mathfrak{A} \cong S$.

Here, the tensor product of a $\hat{\pi}X$-group and a $\hat{\pi}(X \times X)$-group is the one defined after Definition 6.2.

Proof. $(S \otimes_{\hat{\pi}X} \Delta_* \mathfrak{A})(x) = S \otimes_{\hat{\pi}X} \Delta_* \mathfrak{A}(x \times -) \cong S \otimes_{\hat{\pi}X} \hat{\pi}X(x, -) \cong S(x)$. □

We put these pieces together to define a chain-level cap product. Let X be a G–CW$(0, \tau)$ complex with chains C_k and $C_{\tau-k}$, and let $\xi \in C_\tau \otimes_{\hat{\pi}X} \mathfrak{A}$ be a cycle. Give $X \times X$ the G–CW(τ) structure that is the product of the G–CW(τ) structure on the first factor and the ordinary G–CW structure on the second. We can then approximate the diagonal $\Delta: X \to X \times X$ by a map that is cellular with respect to the G–CW(τ) structure on X. This induces a map $\Delta_*: C_\tau \to \Delta^* C_\tau(X \times X) \cong \sum_k \Delta^*(C_{\tau-k} \Box C_k)$. We then define *capping with* ξ to be the following chain map, natural in y:

$$\begin{aligned} C^k(y) \xrightarrow{\otimes \xi} C^k(y) \otimes (C_\tau \otimes_{\hat{\pi}X} \mathfrak{A}) &\xrightarrow{\Delta_*} C^k(y) \otimes (\Delta^*(C_{\tau-k} \Box C_k) \otimes_{\hat{\pi}X} \mathfrak{A}) \\ &\to \Delta^*(C_{\tau-k} \Box \hat{\pi}X(-, y)) \otimes_{\hat{\pi}X} \mathfrak{A} \\ &\cong C_{\tau-k}(y). \end{aligned}$$

The third map is evaluation, while the last isomorphism is given by

$$\begin{aligned} \Delta^*(C_{\tau-k} \Box \hat{\pi}X(-, y)) \otimes_{\hat{\pi}X} \mathfrak{A} &\cong (C_{\tau-k} \Box \hat{\pi}X(-, y)) \otimes_{\hat{\pi}X \times X} \Delta_* \mathfrak{A} \\ &\cong C_{\tau-k} \otimes_{\hat{\pi}X} (\hat{\pi}X(-, y) \otimes_{\hat{\pi}X} \Delta_* \mathfrak{A}) \\ &\cong C_{\tau-k} \otimes_{\hat{\pi}X} \hat{\pi}X(-, y) \\ &\cong C_{\tau-k}(y). \end{aligned}$$

The first two isomorphisms were described above, the third is Corollary 6.4, and the last was pointed out after Definition 6.1. The naturality in y and the fact that this is a chain map are straightforward to check. There is another chain map $-\cap\xi: C^{\tau-k}\to C_k$ that is defined similarly.

If $f: X\to Y$ is a cellular π-equivalence, then there is a map $f^*: C^k(Y)\to C^k(X)$, and the following diagram commutes up to chain homotopy:

$$\begin{array}{ccc} C^k(X) & \to & C_{\tau-k}(X) \\ {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} \\ C^k(Y) & \to & C_{\tau-k}(Y). \end{array}$$

Here the upper arrow is $-\cap\xi$ while the lower arrow is $-\cap f_*(\xi)$. This is easiest to check if we can assume that the diagonal approximations agree, and this can be arranged by mapping both X and Y into the mapping cylinder and extending their diagonal approximations to a cellular approximation to the diagonal map on the mapping cylinder.

We can introduce coefficients by tensoring with a coefficient system S. The resulting map $\mathrm{Hom}_{\hat{\pi}X}(C_k, S)\cong C^k\otimes_{\hat{\pi}X}S\to C_{\tau-k}\otimes_{\hat{\pi}X}S$ is easily seen, using Corollary 6.4, to be the same cap product defined in [CW1].

We can define a cap product in the relative case as well. If A and B are subcomplexes of X, and if $\xi\in C_\tau(X, A\cup B)\otimes\mathfrak{A}$ is a cycle, then cap product with ξ defines chain maps

$$-\cap\xi: C^k(X,A)\to C_{\tau-k}(X,B)$$

and

$$-\cap\xi: C^{\tau-k}(X,A)\to C_k(X,B).$$

Finally, we comment on restriction to fixed sets. Let K be a subgroup of G and let $WK=NK/K$. In [CW1, §5] we constructed a functor from $\hat{\pi}(X;G)$-groups to $\hat{\pi}(X^K;WK)$-groups, denoted $S\mapsto S^K$, and we showed that there is a natural isomorphism

$$C_\tau^G(X)^K\cong C_{\tau^K}^{WK}(X^K)$$

that preserves the cellular bases. Further, this functor preserves cap products and simple equivalences. We also constructed in [CW1, §5] a functor from $\hat{\pi}(X;G)$-groups to $\hat{\pi}(X;K)$-groups, denoted by $S\mapsto S|K$, and a natural isomorphism

$$C_\tau^G(X)\,|\,K\cong C_\tau^K(X)$$

preserving cellular bases. Again, this functor preserves cap products and simple equivalences.

7. Simple Poincaré *G*-Complexes

Generalizing the behavior of smooth G-manifolds seen in Propositions 3.4 and 3.6, we make the following definitions.

DEFINITION 7.1. A *simple Poincaré G-complex of dimension* τ is a finite G-CW$(0,\tau)$ complex X with a *fundamental cycle* $[X]\in C_\tau(X)\otimes_{\hat{\pi}X}\mathfrak{A}$ such that the chain level cap products with $[X]$,

$$-\cap[X]: C^k(X) \to C_{\tau-k}(X)$$

and

$$-\cap[X]: C^{\tau-k}(X) \to C_k(X),$$

are simple equivalences of based $\hat{\pi}X$-chain complexes. Similarly, a *simple Poincaré G-pair of dimension* τ is a pair $(X, \partial X)$ of finite G–CW$(0, \tau)$ complexes with a fundamental cycle $[X, \partial X] \in C_\tau(X, \partial X) \otimes_{\hat{\pi}X} \mathfrak{A}$ such that the following chain level cap products are simple equivalences:

$$-\cap[X, \partial X]: C^{\tau-k}(X, \partial X) \to C_k(X),$$

$$-\cap[X, \partial X]: C^{\tau-k}(X) \to C_k(X, \partial X),$$

$$-\cap[X, \partial X]: C^k(X, \partial X) \to C_{\tau-k}(X),$$

$$-\cap[X, \partial X]: C^k(X) \to C_{\tau-k}(X, \partial X),$$

and such that ∂X is a simple Poincaré G-complex with fundamental cycle $\partial[X, \partial X]$.

REMARKS 7.2. (a) It follows from the remarks at the end of Section 6 that if X is a simple Poincaré G-complex then X^K is a simple Poincaré WK-complex for any subgroup K of G, and that X is a simple Poincaré K-complex for any K. Similar statements hold for pairs.

(b) In the definition of simple Poincaré G-complex, it is really only necessary to assume that the map $C^k(X) \to C_{\tau-k}(X)$ is a simple equivalence, for this is a simple equivalence if and only if $C^{\tau-k}(X) \to C_k(X)$ is (the latter being the transpose of the former). Similarly, in the case of pairs, it suffices to assume that $C^k(X) \to C_{\tau-k}(X, \partial X)$ and $C^{\tau-k}(X) \to C_k(X, \partial X)$ are simple isomorphisms, together with the fact that ∂X is a simple Poincaré G-complex. These assumptions then generalize those made by Wall [Wal].

PROPOSITION 7.3. *If M is a compact smooth G-manifold, then $(M, \partial M)$ is a simple Poincaré G-pair.*

Proof. As in Section 3, let τ be the tangent representation of M, choose a smooth triangulation of M, and let the G–CW$(0, \tau)$ structure on M be given by the dual G–CW(τ) cell structure. We then have that

$$C_\tau(M, \partial M) \cong \sum \hat{\pi}M(-, x),$$

where the sum runs over the centers of the τ-dimensional cells of M. We claim that, as the fundamental cycle for M, we can take the sum

$$[M, \partial M] = \sum 1_x \otimes u \in C_\tau(M, \partial M) \otimes_{\hat{\pi}M} \mathfrak{A},$$

where $u \in \mathfrak{A}(\phi(x))$ is the projection $\phi(x) \to G/G$. That this is a cycle follows from the geometry as in the nonequivariant case. As an approximation to the diagonal $D: M \to M \times M$, we take the following map. The first component $M \to M$ is the map j constructed in Lemma 3.8. The second component is a map k constructed in much the same way that j was, except that we start by sending the vertices to different places: Using the notation of 3.8, every vertex of $\mathcal{S}'$ is contained in the interior of a top-dimensional cell of $\mathcal{S}$, and we send such a vertex to the center of that cell. The resulting map $D=(j,k)$: $M\to M\times M$ is cellular with respect to the G–CW(τ) structure on M and the product of the τ and ordinary structures ($\mathcal{S}$ and $\mathfrak{D}'$) on $M\times M$. It can now be checked that with this diagonal approximation and the cycle $[M,\partial M]$ described above, the map $-\cap[M,\partial M]\colon C^{\tau-k}\to C_k$ takes a cellular basis element to the sum of the cells in the subdivision that make up its dual cell. In other words, this is the composite of the isomorphism of Proposition 3.4 and the simple equivalence given by the inclusion of the triangulation in its subdivision. Similar comments apply to the case $C^k\to C_{\tau-k}$. Finally, by similar arguments it is easy to see that $\partial[M,\partial M]$ is a fundamental cycle for ∂M. □

By the naturality of the cap product, it is also easy to show that any finite G–CW$(0,\tau)$ complex τ-simply equivalent to a τ-dimensional smooth closed G-manifold is itself a simple Poincaré G-complex, and similarly for pairs (see the Corollary to 2.1 in [Wal]).

We now consider degree-1 maps between Poincaré complexes.

DEFINITION 7.4. Let $(Y,\partial Y)$ be a (simple) Poincaré G-pair of dimension δ, and let $(X,\partial X)$ be a (simple) Poincaré G-pair of dimension γ. A *degree-1 map* from Y to X is a G-map $f\colon(Y,\partial Y)\to(X,\partial X)$ covered by a spherical map of representations $\tilde f\colon\delta\to\gamma$ such that $f_*[Y,\partial Y]$ is homologous to $[X,\partial X]$.

This depends on the observation that a spherical map like $\tilde f$ is sufficient to determine a map of chains $f_*\colon C_\delta(Y,\partial Y)\to C_\gamma(X,\partial X)$ (this follows from the definition of the chains given in [CW1]).

DEFINITION 7.5. Let γ and γ' be representations of πX. A *simple equivalence* $F\colon\gamma\mapsto\gamma'$ is a continuous G-map of pairs $(D(\gamma),S(\gamma))\to(D(\gamma'),S(\gamma'))$ that is a simple equivalence of G–CW pairs when each disc is given its canonical (up to simple equivalence) smooth triangulation. Further, we assume for each x that if $\phi(x)=G/H$ then the map $D(V(x))\to D(V'(x))$ is induced by a map $D(V(x)_H)\to D(V'(x)_H)$, where V_H is the orthogonal complement to the H-fixed sets.

LEMMA 7.6. *Let $(X,\partial X)$ be a simple Poincaré G-pair of dimension γ, and let $F\colon\gamma'\to\gamma$ be a simple equivalence. Then there is a simple Poincaré G-pair $(X',\partial X')$ of dimension γ', and a degree-1 map $f\colon(X,\partial X)\to(X',\partial X')$ covered by F such that f is a simple equivalence of G–CW pairs.*

Proof. We construct X' by induction on the skeleta of X. Suppose by induction that we have constructed $((X')^n,(\partial X')^n)$ and a simple equivalence f^n: $(X^n,\partial X^n)\to((X')^n,(\partial X')^n)$. Suppose that $S(\gamma(x)+k)\to X^n$ is the attaching map for the $(n+1)$-cell e^{n+1}, and consider the following diagram:

$$\begin{array}{ccccc} S(\gamma(x)+k) & \to & X^n & \to & X^n \cup e^{n+1} \\ \downarrow & & \downarrow & & \vdots \\ S(\gamma'(x)+k) & \to & X'^n & \to & X'^n \cup (e')^{n+1}. \end{array}$$

We choose the attaching map $S(\gamma'(x)+k) \to X'^n$ to be cellular with respect to the G–CW(0) structures, and to make the left square homotopy commute. If the original attaching map $S(\gamma(x)+k) \to X^n$ lands in ∂X, we require the same of the new attaching map. Continuing in this way, we construct a map $f: (X, \partial X) \to (X', \partial X')$.

That f is a simple equivalence with respect to the G–CW(0) structures follows by induction from the diagram above and the assumption that the map $S(\gamma(x)+k) \to S(\gamma'(x)+k)$ is a simple equivalence. It is also clear from the construction that f induces a basis-preserving isomorphism of $C_\gamma(X)$ with $C_{\gamma'}(X')$ and similarly for the pairs. If we then take $[X', \partial X'] = f_*[X, \partial X]$, the following diagram commutes:

$$\begin{array}{ccc} C^k(X) & \longleftarrow & C^k(X') \\ \downarrow & & \downarrow \\ C_{\gamma-k}(X, \partial X) & \to & C_{\gamma'-k}(X', \partial X'). \end{array}$$

Since the top, left, and bottom arrows are simple equivalences, the rightmost arrow is as well, and $(X', \partial X')$ is a simple Poincaré G-pair. □

The following result generalizes the key Lemma 5.c.0 of [DR], but requires different assumptions.

PROPOSITION 7.7. *Let $(Y, \partial Y)$ be a simple Poincaré G-pair of dimension δ and $(X, \partial X)$ a simple Poincaré G-pair of dimension γ, such that the maps $\pi(\partial Y) \to \pi(Y)$ and $\pi(\partial X) \to \pi(X)$ are equivalences of categories. Suppose that $f: (Y, \partial Y) \to (X, \partial X)$ is a degree-*1 *G-homotopy equivalence, and suppose further that f is covered by a simple equivalence $\tilde{f}: \delta \to \gamma$. If $f: Y \to X$ is a simple G-homotopy equivalence with respect to the ordinary G–CW structures, then $f: (Y, \partial Y) \to (X, \partial X)$ is a simple G-homotopy equivalence of pairs.*

We shall need the following lemma.

LEMMA 7.8. (a) *If $f: Y \to X$ is a G-homotopy equivalence of finite G–CW$(0, \gamma)$ complexes that is a γ-simple homotopy equivalence, then it is a* 0*-simple homotopy equivalence.*

(b) *If $f: (Y, B) \to (X, A)$ is a G-homotopy equivalence of G–CW$(0, \gamma)$ pairs, with $\pi B \to \pi Y$ and $\pi A \to \pi X$ equivalences of categories, and if f is a relative γ-simple equivalence, then f is a relative* 0*-simple equivalence.*

Proof. By Corollary 4.8 and Proposition 5.2, there is a well-defined homomorphism $\mathrm{Wh}_G(\pi X, \gamma) \to \mathrm{Wh}_G(\pi X, 0)$. For part (a), it is easy to see that this map takes $\tau(f; \gamma)$, the torsion of f with respect to the G–CW(γ) structures, to $\tau(f;0)$, the torsion with respect to the ordinary structures. Since $\tau(f;\gamma)=0$ by assumption, $\tau(f;0)=0$ also. For part (b), consider the diagram

$$\begin{array}{ccccc} C_{\gamma+k}(B) & \to & C_{\gamma+k}(Y) & \to & C_{\gamma+k}(Y,B) \\ \downarrow f' & & \downarrow f & & \downarrow f'' \\ C_{\gamma+k}(A) & \to & C_{\gamma+k}(X) & \to & C_{\gamma+k}(X,A). \end{array}$$

From this diagram we get

$$\tau(f'';\gamma)=\tau(f;\gamma)-\tau(f';\gamma),$$

and similarly

$$\tau(f'';0)=\tau(f;0)-\tau(f';0).$$

Since $\tau(f;\gamma)$ and $\tau(f';\gamma)$ are taken to $\tau(f;0)$ and $\tau(f';0)$ as above, it follows that $\tau(f'';\gamma)$ is taken to $\tau(f'';0)$, and so we are done. □

Proof of Proposition 7.7. Applying Lemma 7.6 to the simple equivalence $\delta\to\gamma$, we can replace $(Y,\partial Y)$ in the hypothesis with a γ-dimensional Poincaré G-pair; that is, we may assume that $\delta=\gamma$ and that the map of representations is the identity. We have then the following diagram:

$$\begin{array}{ccc} C^k(Y) & \longleftarrow & C^k(X) \\ \downarrow & & \downarrow \\ C_{\gamma-k}(Y,\partial Y) & \to & C_{\gamma-k}(X,\partial X). \end{array}$$

Since the top and sides are simple equivalences of $\hat{\pi}X$-chain complexes, so is the bottom map. By Theorem 5.6 we can conclude that it is also a simple equivalence of $\hat{\pi}^{\gamma}X$-chain complexes, so that the map $f\colon(Y,\partial Y)\to(X,\partial X)$ is a relative simple equivalence of G-CW(γ) complexes. By Lemma 7.8, $f\colon(Y,\partial Y)\to(X,\partial X)$ is a relative simple equivalence of G-CW complexes. It follows that $\partial f\colon\partial Y\to\partial X$ is also a simple equivalence of G-CW complexes. □

We now write down some dimensional assumptions used in [CW2] to obtain a π-π theorem.

DEFINITION 7.9. (a) A representation V of G is *ideal* if, for each subgroup K of G, V decomposes into a sum of irreducible representations of K as

$$V=\mathbf{R}^{n_0}\oplus\sum Z_i^{n_i},$$

where $n_0<d_i(n_i+1)-1$ for each i; here $d_i=1$, 2, or 4 if Z_i is (respectively) real, complex, or quaternionic. A representation γ of πX is said to be ideal if, for every x, the representation $V(x)$ is an ideal representation of H.

(b) γ satisfies the *gap hypothesis* if, for every x, if $\gamma(x)=G\times_H V$ then for every $K\subset H$ we have either $V^K=V^H$ or $\dim V^K\geq 2\dim V^H$.

(c) γ has fixed sets of dimension at least n if, for every x, if $\gamma(x)=G\times_H V$ then $\dim V^H\geq n$.

THEOREM 7.10 (π–π theorem). *Let $(X, \partial X)$ be a simple Poincaré G-pair of dimension γ; suppose that γ is ideal, satisfies the gap hypothesis, and has fixed sets of dimension of least 5. Suppose further that $\pi(\partial X) \to \pi(X)$ is an equivalence of categories. If M is a smooth compact G-manifold of dimension τ, and $f: (M, \partial M) \to (X, \partial X)$ is a degree-1 map covered by a simple equivalence $\tau \to \gamma$ and a bundle map $b: \nu_M \to \xi$ for some bundle ξ over X, then (f, b) is normally cobordant to a simple G-homotopy equivalence of pairs.*

The proof follows the proof of Theorem 6.5 of [DR], using Proposition 7.7 in place of their Lemma 5.c.0.

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