Convexity and the Schwarz–Christoffel Mapping

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1. Introduction

In 1952 the author wrote an article [2] on close-to-convex functions. The present paper shows how additional conclusions about univalent functions can be obtained from the results and methods in [2]. We also provide a new proof of the main result of [2] with the aid of the support angle function of Study.

For a discussion of close-to-convex functions one is referred to [1], especially pages 46–51. We recall that a function $f(z)$ analytic in the unit disc Δ is called *close-to-convex* if $\operatorname{Re}(f'(z)/\phi'(z))>0$ for some convex function $\phi(z)$ in Δ. Every close-to-convex function is necessarily univalent.

In [2], the following theorem is proved:

THEOREM A. *Let $f(z)$ be locally univalent in Δ. Then f is close-to-convex in Δ if and only if*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} d\theta > -\pi, \quad z=re^{i\theta}, \tag{1}$$

for each r, $0<r<1$, and each pair of real numbers θ_1, θ_2 with $\theta_1<\theta_2$.

From Theorem 3 of [2], one deduces the following theorem.

THEOREM B. *If $f(z) \neq$ constant is analytic in Δ and continuous on $\bar{\Delta}$ and $u=\operatorname{Re}[f(z)]$ is monotone nondecreasing as z moves around $\partial\Delta$ from z_0 to $z_1 \neq z_0$ in the positive direction and monotone nonincreasing as z moves around $\partial\Delta$ from z_1 to z_0 in the positive direction, then f is close-to-convex in Δ.*

2. The Support Angle Function

A basic tool will be the support angle function (Stützwinkelfunktion) introduced by Study [4, p. 89]. If f is analytic in Δ and locally univalent, then for the disc $|z| \le \rho < 1$ $(0<\rho<1)$, a support angle function is

$$S_{f,\rho}(\theta) = p_f(\rho,\theta)+\theta, \quad -\infty<\theta<\infty, \tag{2}$$

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where $p_f(r,\theta)=\arg f'(re^{i\theta})$ is chosen to be continuous in Δ; hence $p_f(r,\theta)$ has period 2π in θ. If the branch of $\arg f'$ is changed, then $S_{f,\rho}(\theta)$ is changed by addition of a constant. In general, we allow any function differing from $S_{f,\rho}(\theta)$ by a constant as a support angle function for f on the disc $|z|\le\rho$.

If $p_f(r,\theta)$ is bounded on Δ, then it has radial limits as $r\to 1^-$ for almost all θ:

$$\lim_{r\to 1^-} p_f(r,\theta)=p_f(\theta) \quad \text{for } \theta\in E_f, \tag{3}$$

where $p_f(\theta)$ has period 2π and $E_f\cap[0,2\pi]$ has linear measure 2π. We define each function

$$S_f(\theta)=p_f(\theta)+\theta+\text{const}, \quad \theta\in E_f, \tag{4}$$

as a support angle function for f on Δ (denoted $\Theta(\theta)$ by Study). (If f is analytic at $e^{i\theta}$, then $S_f(\theta)=p_f(\theta)+\theta+\pi/2$ gives the angle of inclination to the u-axis in the w-plane, $w=u+iv$, of the tangent to $f(\partial\Delta)$ at $f(e^{i\theta})$.) By the periodicity of $p_f(\theta)$,

$$S_f(\theta+2\pi)-S_f(\theta)=2\pi. \tag{5}$$

If $S_f(\theta)$ is defined for all θ and is of bounded variation on $[0,2\pi]$, then f has an integral representation:

$$f(z)=A\int_0^z \exp\left[-\frac{1}{\pi}\int_0^{2\pi}\log\left(1-\frac{z}{e^{i\theta}}\right)dS_f(\theta)\right]dz+B,$$

where A and B are constants [4, p. 103]. This is effectively the same as the representation found by Paatero for the functions of bounded boundary rotation [1, p. 270].

In general for f close-to-convex, $S_f(\theta)$ need not be of bounded variation on $[0,2\pi]$. For, if v is an arbitrary bounded harmonic function in Δ with $|v(z)|<\pi/2$ in Δ, and φ is a convex function in Δ, then f can be chosen so that $\arg f'=v+\arg\varphi'$ in Δ and $S_f=v(e^{i\theta})+S_\varphi(\theta)$ a.e. ($v(e^{i\theta})=\lim v(re^{i\theta})$ as $r\to 1^-$). Hence f is close-to-convex, but S_f is not generally of bounded variation.

As pointed out in [2], the criterion (1) for close-to-convexity is equivalent to the condition

$$S_{f,\rho}(\theta_2)-S_{f,\rho}(\theta_1)>-\pi, \quad 0<\rho<1, \ \theta_1<\theta_2. \tag{6}$$

A passage to the limit, $\rho\to 1^-$, leads to a boundary form of the criterion. We formulate this as part of a general theorem which includes Theorem 2 of [2] [1, p. 48].

THEOREM 1. *Let f be locally univalent in Δ and let a branch of* $\arg f'(z)$ *be chosen in Δ. Then the following conditions are equivalent:*

(a) *f is close-to-convex in Δ;*
(b) *condition* (6) *holds;*
(c) $\arg f'$ *is bounded in Δ and*

$$S_f(\theta_2)-S_f(\theta_1)\ge-\pi, \quad \theta_1<\theta_2, \ \theta_1,\theta_2\in E_f. \tag{7}$$

Proof. The implication (a) $\Rightarrow$ (b) is proved as in [1, pp. 48–49]. For the implication (b) $\Rightarrow$ (c) we need a lemma.

LEMMA 1. *Let $u(z)$ be harmonic in Δ and let $u(z_1)-u(z_2)$ be bounded for z_1, z_2 in Δ and $|z_1|=|z_2|$. Then $u(z)$ is bounded in Δ.*

Proof. Let $w(z)=u(z)+iv(z)$ be analytic in Δ and let $F(z)=\exp(w(z))$, so that $|F|=e^u$. By the hypothesis, $|F(z_1)/F(z_2)|$ is bounded for z_1, z_2 in Δ, $|z_1|=|z_2|$. Now the maximum modulus function $M(r)$ for F is nondecreasing. If F is not bounded, then $M(r)\to\infty$ as $r\to 1^-$ and hence, by the boundedness of $|F(z_1)/F(z_2)|$, $|F(z)|\to\infty$ uniformly as $r\to 1^-$, so that $1/F(z)\to 0$ uniformly as $r\to 1^-$. This is impossible. Hence F is bounded and hence u is bounded above; similarly, $-u$ is bounded above and therefore u is bounded. □

We now prove (b) $\Rightarrow$ (c). From (6) we deduce that $|u(z_1)-u(z_2)|<3\pi$ for $|z_1|=|z_2|$, so that, by Lemma 1, $u=\arg f'$ is bounded in Δ. Hence $S_f(\theta)$ exists as above. Passage to the limit gives (7).

For the final step (c) $\Rightarrow$ (a) we follow the ideas in [1, pp. 48–51] with some modifications. For completeness we give the main steps.

LEMMA 2. *Let E be a nonempty subset of the real line such that $\theta\in E\Rightarrow \theta\pm 2\pi\in E$. Let $t(\theta)$ be a real function defined on E such that*

$$t(\theta+2\pi)-t(\theta)=2\pi$$

and

$$t(\theta_2)-t(\theta_1)>-\pi,\quad \theta_1<\theta_2,\ \theta_1,\theta_2\in E.$$

Then there exists a nondecreasing function $s(\theta)$, $\theta\in E$, such that

$$s(\theta+2\pi)-s(\theta)=2\pi \quad \text{and} \quad |s(\theta)-t(\theta)|\le \pi/2,\quad \theta\in E.$$

Here E is arbitrary except for its invariance under translation, and no continuity is involved (cf. [2, p. 174]). The proof is the same as in [1, p. 48], with

$$s(\theta)=\sup_{\theta'\le\theta} t(\theta')-\pi/2 \quad (\theta,\theta'\in E).$$

We apply Lemma 2 to $t(\theta)=S_f(\theta)=p_f(\theta)+\theta$ ($\theta\in E_f$) and obtain $s(\theta)$, nondecreasing, on E_f. Since E_f is dense in the reals, $s(\theta)$ can be extended to all θ, $-\infty<\theta<\infty$, to remain nondecreasing. For $z\in\Delta$ we let

$$h(z)=\frac{i}{2\pi}\int_0^{2\pi}\frac{e^{i\alpha}+z}{e^{i\alpha}-z}(s(\alpha)-\alpha)\,d\alpha,$$

$$\phi(z)=\int_0^z \exp(h(t))\,dt,$$

and verify that $S_\phi(\theta)=s(\theta)$ a.e. Hence

$$|S_f(\theta)-S_\phi(\theta)|\le\pi/2 \text{ a.e.},$$

so that

$$|p_f(\theta)-p_\phi(\theta)|\le \pi/2 \text{ a.e.}$$

By the maximum principle, this implies that $|\arg f'(z)-\arg\varphi'(z)|\le\pi/2$ in Δ, so that f is close-to-convex. (Equality for one z in the last inequality implies that f is itself convex.) □

3. Schwarz–Christoffel Mappings

These form a class of mappings in Δ. All have the form

$$w=f(z)=A\int_0^z\prod_{j=1}^n(t-z_j)^{-\mu_j}\,dt+B \text{ in } \Delta, \tag{8}$$

where $n\ge 3$, $A\ne 0$, and B and the z_j are complex constants. The μ_j are real constants, and definite analytic branches of the functions $(t-z_j)^{-\mu_j}$ in Δ are chosen. Further we assume that for $j=1,\dots,n$, $z_j=e^{i\theta_j}$ and $0<|\mu_j|<1$, $(\theta_1<\theta_2<\cdots<\theta_n<\theta_1+2\pi,\ \mu_1+\cdots+\mu_n=2)$. It is convenient to extend the definitions of θ_j, z_j, and μ_j to all $j\in\mathbf{Z}$ by requiring

$$\theta_{j+n}=\theta_j+2\pi,\quad \mu_{j+1}=\mu_j,\quad z_{j+n}=z_j \tag{9}$$

for all $j\in\mathbf{Z}$. Under these hypotheses it is known that f has a continuous extension to $\bar\Delta$ (also denoted by f), so that $w_j=f(z_j)$ is defined for all j, with $w_{j+n}=w_j$ for all j; further, it is known that f maps each arc $z=e^{i\theta}$, $\theta_{j-1}\le\theta\le\theta_j$ on a line segment $w_{j-1}w_j$, with $\arg(w_j-w_{j-1})=\alpha_j\pi$. Here α_j can be chosen for all j to satisfy

$$\alpha_{j+1}-\alpha_j=\mu_j,\quad \alpha_{j+n}-\alpha_j=2, \tag{10}$$

and α_j is uniquely determined by these conditions, once α_1 has been chosen. The mapping f thus takes the circular path $z=e^{i\tau}$, $0\le\tau\le 2\pi$, onto a polygonal path Γ, with "exterior angles" $\mu_j\pi$.

REMARK. If we allow $0\le|\mu_j|<1$, then the above conditions still yield a Schwarz–Christoffel mapping. At least three of the μ_j must be nonzero, since $\mu_1+\cdots+\mu_n=2$, so that Γ has at least three normal vertices. The vertices with $\mu_j=0$ are illusory vertices.

Each mapping (8) has a support angle function $S_f(\theta)$ defined for all θ, $-\infty<\theta<\infty$. We can choose $p_f(\theta)=\arg f'(e^{i\theta})$ for $\theta\ne\theta_j$, all j, so that

$$S_f(\theta)=p_f(\theta)+\theta+\pi/2=\alpha_j\pi \quad \text{for } \theta_{j-1}<\theta<\theta_j$$

and hence find, for θ so restricted, that

$$\alpha_j\pi=\arg A-\sum\mu_j\arg(e^{i\theta}-z_j)+\theta+\pi/2. \tag{11}$$

Thus $S_f(\theta)$ is a step function with jump μ_j at θ_j. If $0<\mu_j<1$ for all j, then $S_f(\theta)$ is nondecreasing and f is a convex function in Δ, as is well known.

THEOREM 2. *Under the above hypotheses, f is close-to-convex in Δ if and only if for $0\le p\le m\le 2n$ and $m-p<n$,*

$$\mu_p + \cdots + \mu_m \geq -1. \tag{12}$$

Furthermore, when f is close-to-convex in Δ, there is a convex Schwarz-Christoffel mapping φ on Δ such that $\operatorname{Re}(f'/\varphi') > 0$.

Proof. Condition (12) is equivalent to

$$\alpha_{m+1}\pi - \alpha_p \pi \geq -\pi. \tag{13}$$

If this condition holds for $0 \leq p \leq m \leq 2n$, $m - p < n$, then it holds for $-\infty < p \leq m < \infty$, as follows from the second condition in (10). Further, the condition is equivalent to the condition

$$S_f(\theta'') - S_f(\theta') \geq -\pi \quad \text{for } \theta'' > \theta',$$

provided $\theta' \neq \theta_j$ and $\theta'' \neq \theta_j$ for all j. It now follows from Theorem 1 that (12) holds if and only if f is close-to-convex.

The construction of the convex function φ in the proof of Theorem 1 uses

$$s(\theta) = \sup_{\theta' \leq \theta} [p_f(\theta') + \theta'] - \pi/2. \tag{14}$$

From (14) it follows that $s(\theta)$ is also a step function, with jumps $\gamma_j \geq 0$ at the θ_j. These facts alone show that φ is also a Schwarz-Christoffel mapping. Pursuing the construction further, one finds that $\gamma_j = 0$ when $\mu_j < 0$ and $0 \leq \gamma_j \leq \mu_j < 1$ for $\mu_j > 0$, with $\gamma_1 + \cdots + \gamma_n = 2$, and that

$$S_\varphi(\theta) = p_\varphi(\theta) + \theta = s(\theta). \tag{15}$$

□

4. Discussion

The theorem just proved in Section 3 gives rise to a number of questions and comments.

(a) The conclusion is that f is close to a convex Schwarz-Christoffel mapping. Accordingly one can introduce a new class "C-C-S-C" of univalent functions f in Δ such that $\operatorname{Re}(f'/\varphi') > 0$ for a convex Schwarz-Christoffel mapping φ. Are these *all* the close-to-convex functions in Δ? If not, how can one further describe the class? How are these questions affected if one includes degenerate Schwarz-Christoffel mappings (8) with $\mu_j = \pm 1$ allowed?

(b) Theorem 2 describes a subclass of the class of univalent Schwarz-Christoffel mappings. This subclass is described solely by a condition on the exterior angles $\mu_j \pi$. Is this the largest subclass which can be so described? For each set $\mu = \{\mu_1, \ldots, \mu_n\}$ not satisfying (12), is there a Schwarz-Christoffel mapping which is univalent? To study this question, it may be helpful to consider the area functional

$$A_f = \int_\Delta \int |f'|^2 \, r \, dr \, d\theta \tag{16}$$

in the class of functions (8) with given μ (and, say, $A = 1$, $B = 0$). Here A_f is finite and thus $f' \in L_2(\Delta)$ with $\|f'\| = A_f^{1/2}$. Thus the extreme points for A_f

have a simple geometric meaning. The question can also be considered as one of pure Euclidean geometry: Given a closed polygonal path in the plane with exterior angles $\mu_1\pi, \ldots, \mu_n\pi$, with all $|\mu_j|<1$ and $\sum \mu_j=2$, can one always modify the path without changing the angles so that it becomes a simple closed path? Perhaps the modification could be done by a homotopy within the class of polygonal paths.

(c) Equations (14) and (15) allow one to find φ explicitly from a given Schwarz–Christoffel mapping f: (14) gives $s(\theta)$ in detail as a step function, and hence gives the $\gamma_j \geq 0$. The z_j are unchanged, so that φ can be taken as

$$\varphi(z)=A_0\int_0^z \prod_{j=1}^n (t-z_j)^{-\gamma_j}\,dt, A_0=e^{i\alpha},$$

and only α has to be found. For this we remark that in an interval (θ_{j-1},θ_j) for which $\gamma_j>0$,

$$\arg\varphi'(e^{i\theta})+\theta=\arg f'(e^{i\theta})+\theta-\pi/2=\alpha_j\pi-\pi,$$

from which we find, as in (11), that

$$\alpha=\sum_{j=1}^n(\gamma_j-\mu_j)\arg(e^{i\theta}-z_j)+\arg A-\pi/2, \tag{17}$$

where θ can be chosen arbitrarily in the interval described.

(d) Application of the criteria for close-to-convexity leads to $\mathrm{Re}(f'/\phi')>0$ in Δ and hence to another univalent function F: namely, a primitive F of f'/φ'. For $\mathrm{Re}(F')>0$ in Δ implies that F is univalent [1, p. 47]. This process fails to be invariant, in a certain sense: Let ψ be convex in Δ, say $\psi(\Delta)=D$, and let f be close-to-convex in D: $\mathrm{Re}(f'/\varphi')>0$ for a convex function φ in D. Then again, f is univalent in D, as is F such that $F'=f'/\varphi'$. But $(f\circ\psi)'/(\varphi\circ\psi)'\neq(F\circ\psi)'$ in general. One can take advantage of this lack of invariance to obtain additional univalent functions. For example, if f and φ are Schwarz–Christoffel mappings in Δ, with $\mathrm{Re}(f'/\varphi')>0$ as above, then $F'=f'/\varphi'$ defines a univalent mapping F in Δ which is *not* a Schwarz–Christoffel mapping. But if one maps Δ onto the upper half-plane H, then f becomes a univalent Schwarz–Christoffel mapping f_1 in H and φ becomes a convex Schwarz–Christoffel mapping φ_1 in H. However, $F_1'=f_1'/\varphi_1'$ in H defines a univalent Schwarz–Christoffel mapping in H, which is generally degenerate in that some exponents may be greater than or equal to 1 in absolute value; thus F_1 maps H onto a domain bounded by rays, line segments, and whole straight lines. (The condition $\sum\mu_j=2$ is essential for a Schwarz–Christoffel mapping in Δ. The Schwarz–Christoffel mappings in H have the same form as (8) but $\sum\mu_j=2$ is not essential, since $z=\infty$ plays a special role.)

One can go further, using φ'/f' instead of f'/φ' and using linear combinations of such functions with positive coefficients.

(e) Condition (12) cannot be violated for $n=3,4,5$; that is, *for* $n=3,4,5$ *each Schwarz-Christoffel mapping is univalent.* For $n=6$, violation is possible only if $\mu_j+\mu_{j+1}<-1$ for some j, and one can give specific examples (say with $\mu_1=\mu_2=-0.8$, $\mu_3=\mu_4=\mu_5=\mu_6=0.9$), some having f univalent and some not.

5. Closest Convex Function

One can regard $\sup_\Delta|\arg f'-\arg\varphi'|$ as a measure of how close f and φ are. This leads to the following definitions (see [3]).

DEFINITIONS. Let f be locally univalent in Δ. Then f is close-to-convex of order β, $\beta>0$, if

$$|\arg f'-\arg\varphi'|<\beta\pi/2 \text{ in } \Delta \tag{18}$$

for some convex function φ in Δ (and appropriate choice of the argument functions). The *reentrancy* of f is

$$\operatorname{Ree}(f)=\inf\{\beta\,|\,f \text{ is close-to-convex of order } \beta \text{ in } \Delta\}. \tag{19}$$

If (18) holds for no β then $\operatorname{Ree}(f)=\infty$.

By the first definition, if f is close-to-convex of order β, then it is close-to-convex of order β' for every $\beta'>\beta$. The term "reentrancy" is chosen because, for a mapping onto polygonal domains, it is the "reentrant angles" that cause the mapping to deviate from convexity.

THEOREM 3. *Let f be locally univalent in Δ and let a branch of* $\arg f'$ *be chosen in Δ. Then the following conditions are equivalent:*

(a) *f is close-to-convex of order $\beta>0$ in Δ;*
(b) $\arg f'$ *is bounded in Δ and*

$$S_{f,\rho}(\theta_2)-S_{f,\rho}(\theta_1)>-\beta\pi, \quad 0<\rho<1,\ \theta_1<\theta_2;$$

(c) $\arg f'$ *is bounded in Δ and*

$$S_f(\theta_2)-S_f(\theta_1)\geq-\beta\pi, \quad \theta_1,\theta_2 \text{ in } E_f. \tag{20}$$

COROLLARY. *Under the hypotheses of Theorem 3, let* $\beta_0=\operatorname{Ree}(f)<\infty$. *Then*

$$\beta_0=\frac{1}{\pi}\sup\{S_f(\theta_1)-S_f(\theta_2)\,|\,\theta_1<\theta_2,\theta_1,\theta_2\in E_f\}. \tag{21}$$

Further, f is close-to-convex of order β_0 if $\beta_0>0$; if $\beta_0=0$, f is convex. If $\beta_0\leq 1$ then f is close-to-convex and hence univalent.

REMARK. In general, there is no unique convex φ satisfying (18) for $\beta=\beta_0$, nor is $\arg\varphi'$ unique. This can be seen from simple examples using Schwarz-

Christoffel mappings. Thus we cannot refer to "the convex function closest to f."

The proof of Theorem 3 is like that of Theorem 1, with π replaced by $\beta\pi$ at appropriate places. It is of interest to observe that the construction of the convex function $\varphi = \varphi_\beta$ satisfying (18) leads to the relation

$$\varphi_\beta(z) = \varphi_0(z)\exp(-i\beta\pi/2),$$

where $\varphi_0(z)$ corresponds to the limiting case $\beta = 0$. Thus, by varying β we are simply rotating the convex function.

EXAMPLE. The following Schwarz–Christoffel mapping is considered by Study [4, pp. 76–77]:

$$w = \int_0^z (1+t^5)^{2/5}(1-t^5)^{-4/5}\,dt. \tag{22}$$

Here $n = 10$ and the μ_j are alternately $-2/5$ and $4/5$. The mapping is univalent and the image is as shown in Figure 1. This can be justified (up to a rotation) on symmetry grounds alone. For a Schwarz–Christoffel mapping (8), one sees easily as in Section 3 that (21) becomes

$$\beta_0 = \max(-\mu_p - \cdots - \mu_m), \quad 0 \le p \le m \le 2n, \quad m - p < n. \tag{21'}$$

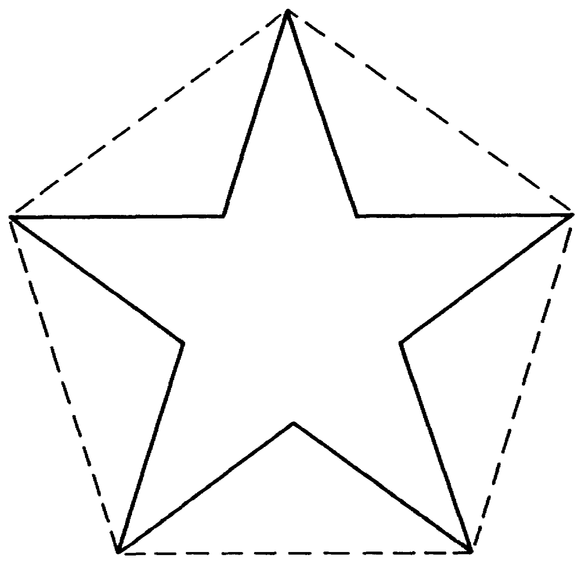

Figure 1 Example (22)

Thus, for the mapping (22), $\beta_0 = 4/5$ (which also proves the univalence). One finds that, for proper numbering, a Schwarz–Christoffel mapping φ constructed as before to satisfy (18) with $\beta = 4/5$ has $\gamma_1 = \gamma_3 = \cdots = \gamma_9 = 2/5$ and $\gamma_2 = \cdots = \gamma_{10} = 0$, and that, for proper choice of the real positive constant A, φ maps $\bar{\Delta}$ onto the pentagonal region shown in Figure 1, the convex hull of $f(\bar{\Delta})$.

REMARKS. For each $\beta_0 \geq 0$ there exist mappings f of reentrancy β_0; in fact, such f can be chosen as univalent Schwarz-Christoffel mappings, with the aid of (21′). The ideas of this section are related to those in Section 5 of [2]. For functions of bounded boundary rotation, Theorem 3 is equivalent to Theorem A of [3].

6. Mapping onto a Domain Convex in One Direction

The plane domains considered here have the property that, for some line L, every line parallel to L meets the domain in a connected set; we say that the domain is convex in the direction of L. We are concerned only with the case of a domain G with polygonal boundary Γ in the w-plane, $w = u + iv$. If L is the v-axis, then the domain G is as suggested in Figure 2. Thus Γ consists of

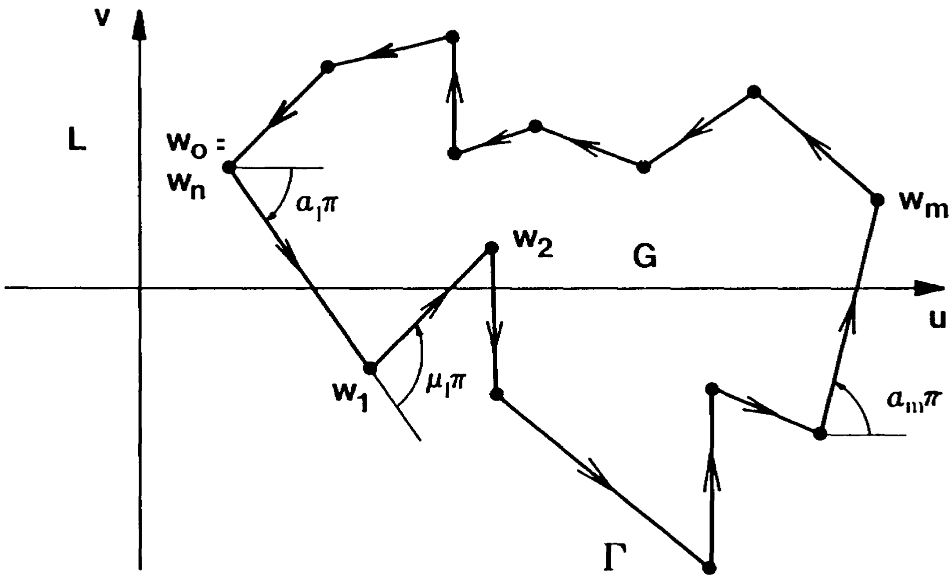

Figure 2

two broken lines $w_0 w_1 \cdots w_m$ and $w_m w_{m+1} \cdots w_n$, as in Figure 2, one consisting of directed segments $w_{j-1} w_j$ with angle of inclination $\alpha_j \pi$ between $-\pi/2$ and $\pi/2$ (inclusive), and the other of similar segments with angle of inclination between $\pi/2$ and $3\pi/2$ (inclusive). The two broken lines meet only at their endpoints and the first lies below the second, in the obvious sense.

THEOREM 4. *Let f be defined by* (8), *where the hypotheses of Section 3 are satisfied: that is,* $n \geq 3$, $A \neq 0$, B *and* $z_j = \exp(i\theta_j)$ *are complex constants, the* μ_j *are real constants and* $0 < |\mu_j| < 1$ $(\mu_1 + \cdots + \mu_n = 2,\ \theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi)$. *Then f is a close-to-convex univalent mapping of Δ onto a domain G convex in one direction if and only if the z_j can be numbered to satisfy the hypotheses of Section 3 in such a way that, for some integer m* $(0 < m < n)$, *the n closed intervals*

$$[a_j - q_j, b_j - q_j], \quad j = 0, 1, \ldots, n-1, \tag{23}$$

have a nonempty intersection, where

$$q_0 = 0, \qquad q_j = \mu_1 + \cdots + \mu_j, \quad j = 1, \ldots, n-1; \tag{24}$$

$$\begin{cases} a_j = -\frac{1}{2} \text{ and } b_j = \frac{1}{2} \text{ for } j = 0, 1, \ldots, m-1, \\ a_j = \frac{1}{2} \text{ and } b_j = \frac{3}{2} \text{ for } j = m, \ldots, n-1. \end{cases} \tag{25}$$

Proof. Sufficiency: Let δ be a common point of the n intervals (23). We can restrict attention to the case in which $B = 0$ and A is chosen so that $\arg(w_1 - w_0) = \delta\pi = \alpha_1 \pi$. Thus, by the hypothesis,

$$a_j \pi - q_j \pi \le \delta\pi \le b_j \pi - q_j \pi, \quad j = 0, 1, \ldots, n-1. \tag{26}$$

From (10) and (24),

$$\alpha_{j+1} = \alpha_1 + q_j = \delta + q_j, \quad j = 0, 1, \ldots, n-1,$$

so that by (23) and (24),

$$\begin{cases} -\pi/2 \le \alpha_j \pi \le \pi/2, & j = 1, \ldots, m, \\ \pi/2 \le \alpha_j \pi \le 3\pi/2, & j = m+1, \ldots, n \end{cases} \tag{27}$$

(see Figure 2). By these inequalities, $u(e^{it}) = \operatorname{Re}(f(e^{it}))$ is nondecreasing as θ increases from θ_0 to θ_m and nonincreasing as θ increases from θ_m to $\theta_n = \theta_0 + 2\pi$. Since f is continuous in $\bar{\Delta}$ and not identically constant, Theorem B applies and f is close-to-convex, hence univalent. By (27), the boundary Γ of $G = f(\Delta)$ is formed of two broken lines as above, so that G is convex in the direction of the v-axis.

Necessity: Let f be given by (8) and be univalent in Δ, and map Δ onto a domain G convex in one direction. After rotation by an angle $\eta\pi$, G becomes a domain convex in the direction of the v-axis, as in Figure 2. This rotation changes the angles $\alpha_j \pi$ to $(\alpha_j + \eta)\pi$. Thus

$$-\pi/2 \le (\alpha_j + \eta)\pi \le \pi/2, \quad j = 1, \ldots, m;$$

$$\pi/2 \le (\alpha_j + \eta)\pi \le 3\pi/2, \quad j = m+1, \ldots, n.$$

Hence η lies in all n intervals

$$[a_j - \alpha_j, b_j - \alpha_j] \quad \text{for } j = 0, \ldots, n-1,$$

so that $\delta = \eta + \alpha_1$ lies in all intervals

$$[a_j - (\alpha_j - \alpha_1), b_j - (\alpha_j - \alpha_1)] \quad \text{for } j = 0, \ldots, n-1$$

or in all intervals (23). □

REMARK. The proof shows that, in general, L is obtained by rotating the v-axis by $-\eta\pi = (\alpha_1 - \delta)\pi$.

References

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