Divergence-Free Vector Wavelets

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1. Introduction

Wavelet decompositions have found a number of applications in recent years, particularly in harmonic analysis, quantum field theory, and signal analysis. In close parallel with this new development has been the construction of bases of wavelets satisfying regularity and decay conditions of one kind or another [1; 5; 10; 12]. Now there seems to be a growing interest in wavelet decompositions tailored to specific problems – a basis of wavelets for a space of functions defined by some differential constraint, for example. Our own interest is in constructing such orthonormal bases in the space of vector fields with vanishing divergence. Such bases should be useful in several contexts, including the study of incompressible fluids [7].

Since the divergence-free condition is invariant with respect to scaling and translation, one might suppose the construction of such wavelets to be straightforward. Surprisingly enough, this does not seem to be the case, except in two dimensions [3]. The 3-dimensional case is already a complicated case – far from trivial – and we solve the problem here. We construct a divergence-free wavelet orthonormal basis, where the wavelets are class C^N with exponential decay – ideal for many applications.

We are currently purusing the 4-dimensional case because we expect space-time wavelets satisfying the continuity equation to be useful in the analysis of conserved currents. As far as other directions are concerned, one can consider other regularity and decay properties. For example, our method of construction may extend to divergence-free Meyer wavelets (i.e., wavelets with Fourier transform in C_0^∞), but we have not pursued that possibility. Nor have we considered the construction of C_0^N wavelets with vanishing divergence. Lemarié has constructed non-orthonormal divergence-free wavelet bases of compact support [11].

Let I be a finite index set, and for each t in I let ψ_t be a vector-valued function defined on 3-dimensional real space. For $\alpha=(r,n,t)$ in $Z\times Z^3\times I$, the wavelet ψ_α is defined by

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$$\psi_\alpha(x) = 2^{3r/2}\psi_t(2^r x - n).$$

The wavelet $\psi_{(0,0,t)}(x) = \psi_t(x)$ is said to be a "mother wavelet" of $\{\psi_\alpha(x)\}$. Thus, sets $\{\psi_\alpha\}$ of wavelets are generated by scalings and translates of a finite set of real valued functions ψ_t. If $\alpha = (r, n, t)$ then we will write $r = r(\alpha)$, $n = n(\alpha)$, and $t = t(\alpha)$. The *length scale* of a wavelet is $2^{-r(\alpha)}$; $r = 0$ is the *unit scale;* wavelets of the same r value have the *same scale.* If the wavelets are in an inner product space of functions, $\mathfrak{s}$, the set of wavelets is said to form a basis if they are linearly independent and their span is dense in $\mathfrak{s}$. They form an orthonormal basis if they are in addition orthonormal.

Our construction in three dimensions is closely related to a 4-dimensional gauge field construction, due to Federbush and Williamson in [6] and [8]. There one implicitly arrived at the set of vector wavelets $\{A_\alpha\}$ and 2-form wavelets $\{B_\alpha\}$. We are using the notation

$$B_\alpha = dA_\alpha = \nabla \times A_\alpha.$$

(Properly we should define $B_\alpha = 2^{-r(\alpha)}\, dA_\alpha$, but we suppress this normalization factor.) The $\{B_\alpha\}$ wavelets implicit in [8] satisfied:

(1a) The B_α are interscale orthogonal with respect to the inner product

$$\langle B_1, B_2 \rangle = \int B_1 \cdot B_2.$$

That is, if B_1 and B_2 are not of the same scale, their inner product is zero; the dot product is that of 2-forms.

(1b) B_α is $C^{1-\epsilon}$.

(1c) B_α has exponential fall-off.

(1d) The B_α form a non-orthonormal basis for the set of closed 2-forms.

It is the observation that in three dimensions closed 2-forms correspond to divergence-free vector fields that motivated the construction of this paper. In Sections 2–4 of this paper we convert the constructions of [6] and [8] to three dimensions with an important generalization giving additional smoothness. Specifically, for each positive integer N we find 3-vector wavelets $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\alpha\}$ with $\mathbf{B}_\alpha = \nabla \times \mathbf{A}_\alpha$ such that the $\{\mathbf{B}_\alpha\}$ satisfy:

(2a) The $\mathbf{B}_\alpha$ are interscale orthogonal with respect to the inner product

$$\langle \mathbf{B}_1, \mathbf{B}_2 \rangle = \int \mathbf{B}_1 \cdot \mathbf{B}_2.$$

(2b) $\mathbf{B}_\alpha$ is $C^{N-\epsilon}$.

(2c) $\mathbf{B}_\alpha$ has exponential fall-off.

(2d) The $\mathbf{B}_\alpha$ form a non-orthonormal basis for the set of divergence-free vector fields.

This construction for $N = 1$ is that of [6] and [8] converted to three dimensions. In [6] and [8] "gauge transformations" on A_α were performed that are not necessary for our purposes. In Section 5 we carry out the rather standard translation-invariant orthogonalization inside each level, and we verify that this results in a wavelet orthonormal basis $\{\mathbf{u}_\alpha\}$ satisfying:

(A) The $\mathbf{u}_\alpha$ are an orthonormal set in the inner product

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \int \mathbf{u}_1 \cdot \mathbf{u}_2.$$

(B) $\mathbf{u}_\alpha$ is $C^{N-\epsilon}$.
(C) $\mathbf{u}_\alpha$ has exponential fall-off.
(D) The $\mathbf{u}_\alpha$ are an orthonormal basis for the set of divergence-free vector fields.
(E) All moments of the $\mathbf{u}_\alpha$ of degree at most N are zero. That is,

$$\int \mathbf{u}_\alpha x^\beta = 0 \quad \text{if } |\beta| \leq N.$$

The wavelets are constructed in Sections 2 and 3. The analyticity and fall-off properties in Fourier transform space are investigated in Section 4; these lead to properties (2c) and (2b) respectively. Property (E) follows directly from properties (A) and (B) by a theorem of one of the authors in [2]. Proving property (C) is the essential burden of Section 5.

We now describe the generalization, mentioned above, of the Federbush–Williamson construction so that $\mathbf{B}_\alpha$ can have any degree of smoothness we wish. Their averaging transformation, which takes a continuum vector field to an antisymmetric tensor-valued lattice configuration, has the form

$$\mathbf{A} \mapsto \int_{\mathcal{B}(\mathbf{m})} \int_{\Gamma + x} \mathbf{A} \cdot d\mathbf{r}\, dx, \quad \mathbf{m} \in \mathbb{Z}^3,$$

where $\mathcal{B}(\mathbf{m})$ is the unit block with minimum coordinate vertex at $\mathbf{m}$, and Γ is some canonical oriented loop at the origin associated with a given pair of coordinate directions. These are (1.13)–(1.14) of [6]. This block average of loop integrals can be written as

$$\text{(1.1)} \qquad \mathbf{A} \mapsto \int \eta(x - \mathbf{m}) \int_{\Gamma + x} \mathbf{A} \cdot d\mathbf{r}\, dx, \quad \mathbf{m} \in \mathbb{Z}^3, \qquad \text{where} \quad \eta(x) = \prod_\mu \chi(x_\mu),$$

and χ is the characteristic function of $[0,1]$. The desired generalization is obtained by the replacement of (1.1) by an N-fold convolution (N a parameter we fix)

$$\hat{\eta}(p) = \prod_\mu \hat{\chi}(p_\mu)^N.$$

This smoothed-out type of averaging will be useful because $\eta(x)$ satisfies a functional equation of the form

$$\text{(1.2)} \qquad \eta(x) = \sum_{\mathbf{m}} c_{\mathbf{m}} \eta(2x - \mathbf{m}).$$

Any such function is also an input function to the Meyer–Mallat construction machine for scalar wavelets described so well in [5]. We show how (1.2) is relevant to our own purpose in Section 2.

The technique used in [6] and [8] to construct gauge field wavelets or used in [1] to construct scalar field wavelets is called *constrained minimization.* It naturally leads to wavelets that are interscale orthogonal. This is the technique that arose naturally out of the interaction between the renormalization group and constructive quantum field theory. It might pay the reader to skim references [1], [6], and [8] in order to become familiar with the technique of constrained minimization. The first construction of wavelets by this technique was by Gawedzki and Kupiainen in a lattice situation [9].

Our construction is rather technical, but one should remember that in many applications, only the existence of a basis with the above properties is actually used. The most important estimates on $\mathbf{u}_\alpha$ easily follow from these properties. Only in numerical work would one be interested in an explicit formula for $\mathbf{u}_\alpha$.

2. The Calculation

We apply the constrained minimization technique for constructing wavelets. We minimize the quadratic form $\int(\nabla\times\mathbf{A})^2$ with respect to the conditions

$$\nabla\cdot\mathbf{A}=0 \quad \text{and}$$

$$\int \eta(x-\mathbf{m})\int_{\partial P_{\mu\nu}(x)}\mathbf{A}\cdot d\mathbf{r}\,dx=b_{\mu\nu}(\mathbf{m}),$$

where $\partial P_{\mu\nu}(x)$ denotes the boundary of the oriented unit plaquette in the (μ,ν)-direction with minimum coordinate vertex at x. Obviously, the solution depends on the antisymmetric tensor configuration $\mathbf{b}$ on the lattice (which is a pseudo-vector configuration in three dimensions) chosen for the constraints, but $\mathbf{b}$ itself must satisfy the lattice exterior derivative condition

$$b_{23}(\mathbf{m}+\mathbf{e}_1)-b_{23}(\mathbf{m})+b_{31}(\mathbf{m}+\mathbf{e}_2)-b_{31}(\mathbf{m})+b_{12}(\mathbf{m}+\mathbf{e}_3)-b_{12}(\mathbf{m})=0 \tag{2.1}$$

together with the averaging condition

$$\begin{aligned}\sum_{\mathbf{m}} c_{\mathbf{m}}[b_{\mu\nu}(\mathbf{m}+2\mathbf{n})+b_{\mu\nu}(\mathbf{m}+\mathbf{e}_\mu+2\mathbf{n})\\ +b_{\mu\nu}(\mathbf{m}+\mathbf{e}_\nu+2\mathbf{n})+b_{\mu\nu}(\mathbf{m}+\mathbf{e}_\mu+\mathbf{e}_\nu+2\mathbf{n})]=0.\end{aligned} \tag{2.2}$$

Here

$$c_{\mathbf{m}}=\prod_{\mu=1}^{3}\binom{N}{m_\mu} \quad \text{for } 0\le m_\mu\le N,$$

and $c_{\mathbf{m}}$ is zero otherwise. The lattice derivative condition is necessary for the consistency of the constraints, while the averaging condition insures orthogonality of solutions on different scales. For example, assuming the above constraints, consider the same plaquette averaging for the next scale up. Let $\partial Q_{\mu\nu}(x)$ denote the boundary of the oriented 2×2 plaquette in the (μ,ν)-direction with minimum coordinate vertex at x. Since

$$\int_{\partial Q_{\mu\nu}(x)} \mathbf{A}\cdot d\mathbf{r} = \int_{\partial P_{\mu\nu}(x)} \mathbf{A}\cdot d\mathbf{r} + \int_{\partial P_{\mu\nu}(x+\mathbf{e}_\mu)} \mathbf{A}\cdot d\mathbf{r} + \int_{\partial P_{\mu\nu}(x+\mathbf{e}_\nu)} \mathbf{A}\cdot d\mathbf{r} + \int_{\partial P_{\mu\nu}(x+\mathbf{e}_\mu+\mathbf{e}_\nu)} \mathbf{A}\cdot d\mathbf{r},$$

we have

$$\int \eta(\tfrac{1}{2}x-\mathbf{n}) \int_{\partial Q_{\mu\nu}(x)} \mathbf{A}\cdot d\mathbf{r}\,dx = \sum_{\mathbf{m}} c_{\mathbf{m}} \int \eta(x-2\mathbf{n}-\mathbf{m}) \int_{\partial Q_{\mu\nu}(x)} \mathbf{A}\cdot d\mathbf{r}\,dx$$
$$= \sum_{\mathbf{m}} c_{\mathbf{m}}[b_{\mu\nu}(\mathbf{m}+2\mathbf{n}) + b_{\mu\nu}(\mathbf{m}+\mathbf{e}_\mu+2\mathbf{n}) + b_{\mu\nu}(\mathbf{m}+\mathbf{e}_\nu+2\mathbf{n}) + b_{\mu\nu}(\mathbf{m}+\mathbf{e}_\mu+\mathbf{e}_\nu+2\mathbf{n})] = 0.$$

Thus the given $\mathbf{A}$ lies in the kernels of the linear functionals defining the constraints used on the larger length scale, so the solution to minimizing $\int(\nabla\times\mathbf{A})^2$ with respect to a set of larger-scale constraints is automatically orthogonal to the given $\mathbf{A}$ with respect to the quadratic form. In particular, it is orthogonal to the solution of a constrained minimization on the unit scale. The geometric principle applied here depends on the continuity of the linear functionals with respect to $\int(\nabla\times\mathbf{A})^2$, which follows immediately from their definition in terms of plaquettes.

We now turn to the actual solution of the problem. A standard method is to use Lagrange multipliers, but instead we choose to minimize the α-dependent quantity

(2.3)
$$\int(\nabla\times\mathbf{A})^2 + \alpha^2\int(\nabla\cdot\mathbf{A})^2 + \alpha^2\sum_{\mathbf{m}}\sum_{\mu<\nu}\left[\int\eta(x-\mathbf{m})\int_{\partial P_{\mu\nu}(x)}\mathbf{A}\cdot d\mathbf{r}\,dx - b_{\mu\nu}(\mathbf{m})\right]^2$$

and take its limit as $\alpha\to\infty$. We first interchange order of integration to write

$$\int\eta(x-\mathbf{m})\int_{\partial P_{\mu\nu}(x)}\mathbf{A}\cdot d\mathbf{r}\,dx = \int\mathbf{A}(x)\cdot\zeta^{\mu\nu}(x-\mathbf{m})\,dx;$$
$$\zeta^{\mu\nu}(x) = -\int_{\partial P_{\mu\nu}(x-\mathbf{e}_\mu-\mathbf{e}_\nu)}\eta\,d\mathbf{r},$$

and then we observe that for finite α the integral-differential equation determining minimization of (2.3) is

$$-\Delta A_\lambda + (1-\alpha^2)\partial_\lambda(\nabla\cdot\mathbf{A}) + \alpha^2\sum_{\mathbf{m}}\sum_{\mu<\nu}\zeta_\lambda^{\mu\nu}(\cdot-\mathbf{m})\int\mathbf{A}(x)\cdot\zeta^{\mu\nu}(x-\mathbf{m})\,dx$$
$$= \alpha^2\sum_{\mathbf{m}}\sum_{\mu<\nu}\zeta_\lambda^{\mu\nu}(\cdot-\mathbf{m})\,b_{\mu\nu}(\mathbf{m}).$$

Next we consider the differential operator given by the first two terms and let C_α be its inverse. The equation becomes the integral equation

$$\mathbf{A} + \alpha^2\sum_{\mathbf{m}}\sum_{\mu<\nu}C_\alpha R_{\zeta_{\mathbf{m}}^{\mu\nu}}\mathbf{A} = \alpha^2\sum_{\mathbf{m}}\sum_{\mu<\nu}b_{\mu\nu}(\mathbf{m})C_\alpha\zeta_{\mathbf{m}}^{\mu\nu},$$

where $R_{\mathbf{f}}$ is just the unnormalized L^2-projection onto $\mathbf{f}$. Solving by multiplication of Fourier series and matrices is routine. In momentum space we have

$$\hat{\mathbf{A}}(p)=\alpha^2 G(p)^*\cdot[1+\alpha^2 M(p)]^{-1} C_\alpha(p)\hat{\zeta}(p), \tag{2.4}$$

$$C_\alpha(p)=\Delta_\alpha(p)^{-1},$$

$$\Delta_\alpha(p)_{\lambda\sigma}=p^2\delta_{\lambda\sigma}+(\alpha^2-1)p_\lambda p_\sigma,$$

$$\hat{\zeta}_\lambda^{\mu\nu}(p)=\hat{\eta}(p)[\delta_{\nu\lambda}(1-e^{ip_\mu})+\delta_{\mu\lambda}(e^{ip_\nu}-1)]\hat{\chi}(p_\lambda), \tag{2.5}$$

$$G_{\mu\nu}(k)=\sum_{\mathbf{m}} b_{\mu\nu}(\mathbf{m})e^{i\mathbf{m}\cdot k},$$

$$M_{\mu\nu,\mu'\nu'}(k)=\sum_{\mathbf{m}} a_{\mu\nu,\mu'\nu'}(\mathbf{m})e^{i\mathbf{m}\cdot k},$$

$$(C_\alpha \zeta_{\mathbf{m}}^{\mu\nu}, \zeta_{\mathbf{m}'}^{\mu'\nu'})=a_{\mu\nu,\mu'\nu'}(\mathbf{m}-\mathbf{m}'),$$

where $C_\alpha(p)$ acts on the ordinary vector-valued plaquette vector $\hat{\zeta}(p)$ as an ordinary vector and $[1+\alpha^2 M(p)]^{-1}$ acts on $\hat{\zeta}(p)$ as a plaquette vector. Let $G(p)^*$ denote the complex conjugation of the plaquette vector $G(p)$. By Poisson summation we obtain

$$M_{\mu\nu,\mu'\nu'}(p)=\sum_{\mathbf{l}} \hat{\zeta}^{\mu'\nu'}(p+2\pi\mathbf{l})^* C_\alpha(p+2\pi\mathbf{l})\hat{\zeta}^{\mu\nu}(p+2\pi\mathbf{l}). \tag{2.6}$$

On the other hand, it is easy to infer from the well-known inversion

$$C_\alpha(p)_{\lambda\sigma}=\frac{1}{p^2}\delta_{\lambda\sigma}+\left(\frac{1}{\alpha^2}-1\right)\frac{1}{p^4}p_\lambda p_\sigma$$

that

$$C_\alpha(p)_{\lambda\sigma}\zeta^{\mu\nu}(p)=\frac{1}{p^2}\zeta^{\mu\nu}(p).$$

Hence (2.4) reduces to

$$\hat{\mathbf{A}}(p)=\frac{\alpha^2}{p^2}G(p)^*\cdot[1+\alpha^2 M(p)]^{-1}\hat{\zeta}(p).$$

Next we observe that (2.5) may be written as

$$\hat{\zeta}_\lambda^{\mu\nu}(p)=\frac{i\hat{\eta}(p)}{p_\lambda}(e^{ip_\mu}-1)(e^{ip_\nu}-1)(\delta_{\mu\lambda}-\delta_{\nu\lambda})$$

and introduce the shorthand [8]

$$f_\mu(p)\equiv e^{+ip_\mu}-1,$$

$$\hat{\chi}(p_\mu+2\pi l_\mu)=\frac{if_\mu(p)}{p_\mu+2\pi l_\mu}, \quad \text{and}$$

$$\langle h(p)\rangle\equiv\sum_{\mathbf{l}}\frac{1}{(p+2\pi\mathbf{l})^2}|\hat{\eta}(p+2\pi\mathbf{l})|^2 h(p+2\pi\mathbf{l})$$

to observe that (2.6) reduces to

$$M_{\mu\nu,\mu'\nu'}(p)=f_\mu(p)f_\nu(p)f_{\mu'}(p)^*f_{\nu'}(p)^*\left\langle\frac{1}{p_\mu^2}(\delta_{\mu\mu'}-\delta_{\mu\nu'})-\frac{1}{p_\nu^2}(\delta_{\mu'\nu}-\delta_{\nu\nu'})\right\rangle.$$

Following [8], we let $D(p)$ denote the diagonal matrix in plaquette directions such that

$$D_{\mu\nu,\mu\nu}(p)=f_\mu(p)f_\nu(p). \tag{2.7}$$

We then have $M(p)=D(p)\langle M_0(p)\rangle D(p)^*$ and

$$M_0(p)=\begin{bmatrix} \frac{1}{p_2^2}+\frac{1}{p_3^2} & \frac{1}{p_3^2} & -\frac{1}{p_2^2} \\ \frac{1}{p_3^2} & \frac{1}{p_3^2}+\frac{1}{p_1^2} & \frac{1}{p_1^2} \\ -\frac{1}{p_2^2} & \frac{1}{p_1^2} & \frac{1}{p_1^2}+\frac{1}{p_2^2} \end{bmatrix}, \tag{2.8}$$

where the ordering of our plaquette directions is $(3,2)$, $(3,1)$, $(2,1)$.

3. The Inversion

It is easy to check that $M_0(p)$ is a singular matrix of rank 2, and so is the nonnegative sum $\langle M_0(p)\rangle$, for the zero-eigenvalue eigenspace is generated by

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

which is independent of p. This means that the limit as $\alpha\to\infty$ of

$$\alpha^2(1+\alpha^2D(p)\langle M_0(p)\rangle D(p)^*)^{-1}$$

cannot exist. The same sort of problem is encountered in [6; 8], and we circumvent it in the same way as therein. First note that the zero-eigenvalue eigenspace of $D(p)\langle M_0(p)\rangle D(p)^*$ is generated by

$$\begin{bmatrix} f_1(p)^* \\ -f_2(p)^* \\ f_3(p)^* \end{bmatrix},$$

while the dot product of this plaquette vector with $G(p)^*$ is *zero*. This latter observation is just the lattice exterior derivative condition (2.1) on $b_{\mu\nu}(\mathbf{m})$ written in terms of Fourier series. Thus the limit as $\alpha\to\infty$ of the whole momentum expression (2.4) exists, and our next task is to derive the momentum expression for this limit.

Let U be any unitary matrix which maps

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{ to } \begin{bmatrix} 0 \\ 0 \\ \sqrt{3} \end{bmatrix}.$$

For example, pick

$$U=\begin{bmatrix} \frac{1}{3}\sqrt{6} & \frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \end{bmatrix}.$$

We have

$$UM_0(p)U^{-1}=\frac{1}{2}\begin{bmatrix} \dfrac{3}{p_2^2}+\dfrac{3}{p_3^2} & \dfrac{\sqrt{3}}{p_3^2}-\dfrac{\sqrt{3}}{p_2^2} & 0 \\ \dfrac{\sqrt{3}}{p_3^2}-\dfrac{\sqrt{3}}{p_2^2} & \dfrac{4}{p_1^2}+\dfrac{1}{p_2^2}+\dfrac{1}{p_3^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathcal{S}$ denote the orthogonal complement of

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

This two-dimensional subspace is invariant under $\langle M_0(p)\rangle$ and to invert $\langle M_0(p)\rangle$ on $\mathcal{S}$ is equivalent to inverting $U\langle M_0(p)\rangle U^{-1}$ on $U\mathcal{S}$. The latter is just the inversion of a 2×2 matrix. We obtain

$$I(p)=2\mathfrak{D}(p)^{-1}\begin{bmatrix} 4q_1+q_2+q_3 & \sqrt{3}(q_2-q_3) & 0 \\ \sqrt{3}(q_2-q_3) & 3(q_2+q_3) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$q_\mu=\left\langle\frac{1}{p_\mu^2}\right\rangle, \quad \text{and}$$

(3.1) $$\mathfrak{D}(p)=12(q_1q_2+q_3q_1+q_2q_3),$$

so the inversion of $\langle M_0(p)\rangle$ on $\mathcal{S}$ yields

(3.2) $$L(p)=U^{-1}I(p)U$$

$$=\frac{4}{3}\mathfrak{D}(p)^{-1}\begin{bmatrix} 4q_1+q_2+q_3 & 2q_1+2q_2-q_3 & -2q_1+q_2-2q_3 \\ 2q_1+2q_2-q_3 & q_1+4q_2+q_3 & -q_1+2q_2+2q_3 \\ -2q_1+q_2-2q_3 & -q_1+2q_2+2q_3 & q_1+q_2+4q_3 \end{bmatrix}.$$

Let $\mathcal{S}(p)$ denote the orthogonal complement of

$$\begin{bmatrix} f_1(p)^* \\ -f_2(p)^* \\ f_3(p)^* \end{bmatrix}.$$

Then the inversion of $D(p)\langle M_0(p)\rangle D(p)^*$ on $\mathcal{S}(p)$ yields

$$D(p)^{-1*}L(p)D(p)^{-1}.$$

In summary, the solution $\mathbf{A}$ in the $\alpha=\infty$ limit is given by

$$\hat{\mathbf{A}}(p)=\frac{1}{p^2}G(p)^*\cdot D(p)^{-1*}L(p)D(p)^{-1}\hat{\mathfrak{f}}(p);$$

$$\hat{\xi}_\lambda(p) = \frac{\hat{\eta}(p)}{ip_\lambda}\begin{bmatrix} f_3(p)f_2(p)(\delta_{3\lambda}-\delta_{2\lambda}) \\ f_3(p)f_1(p)(\delta_{3\lambda}-\delta_{1\lambda}) \\ f_2(p)f_1(p)(\delta_{2\lambda}-\delta_{1\lambda}) \end{bmatrix}.$$

But as we have already indicated in the introduction, $\nabla\times\mathbf{A}$ is the function we are after. A straightforward calculation yields

$$ip\times\mathbf{A}(p) = \frac{\hat{\eta}(p)}{p^2}G(p)^{+*}D(p)^{-1*}L(p)M_0(p)E(p); \tag{3.3}$$

$$E(p) = \begin{bmatrix} p_2p_3 & 0 & 0 \\ 0 & -p_3p_1 & 0 \\ 0 & 0 & p_1p_2 \end{bmatrix}, \tag{3.4}$$

where we have written the (plaquette) dot product with the column vector $G(p)^*$ as the matrix product with the row vector $G(p)^{+*}$.

4. Real Analyticity in Momentum Space

To establish exponential decay of $\mathbf{B}=\nabla\times\mathbf{A}$, we show that $\hat{\mathbf{B}}(p)$ is real-analytic and satisfies the appropriate decay bounds. We discuss the momentum decay leading to the class $C^{N-\epsilon}$ smoothness of $\mathbf{B}(x)$ at the end of the section. Consider $\hat{B}_1(p)$; the analysis of $\hat{B}_2(p)$ and $\hat{B}_3(p)$ is essentially the same. Now, by equations (2.7), (2.8), (3.1), (3.2), (3.3) and (3.4) together with the exterior derivative condition,

$$G_{32}(p)f_1(p)-G_{31}(p)f_2(p)+G_{21}(p)f_3(p)=0,$$

and we have the decomposition

$$B_1(x)=B_1'(x)+B_1''(x),$$

$$\hat{B}_1'(p) = \frac{\hat{\eta}(p)p_2p_3}{p^2f_2(p)^*f_3(p)^*}\mathfrak{D}(p)^{-1}G_{32}(p)^*\left(q_1\frac{1}{p_3^2}+q_2\frac{1}{p_3^2}+q_1\frac{1}{p_2^2}\right), \tag{4.1}$$

$$\hat{B}_1''(p) = \frac{\hat{\eta}(p)p_2p_3}{p^2f_1(p)^*f_2(p)^*}\mathfrak{D}(p)^{-1}G_{21}(p)^*\left(q_2\frac{1}{p_3^2}-q_3\frac{1}{p_2^2}\right). \tag{4.2}$$

First we control (4.1) – the easier quantity to control.

In addition to establishing analyticity on a strip $\Omega_\epsilon\times\Omega_\epsilon\times\Omega_\epsilon$ with

$$\Omega_\epsilon=\mathbb{R}+i[-\epsilon,\epsilon],$$

we require decay with respect to the variables $\operatorname{Re}z_\mu$. Our first observation is that the factor

$$z_2z_3\frac{\hat{\eta}(z)}{f_2(-z)f_3(-z)}$$

satisfies all of the desired properties on such a strip because

$$\hat{\eta}(z)=\sum_{\mu=1}^{3}\left(\frac{f_\mu(z)}{iz_\mu}\right)^N. \tag{4.3}$$

$\hat{\eta}(z)=0$ for $z_\mu=2\pi m$, $m\in\mathbb{Z}\backslash\{0\}$, while the zero of $f_\mu(z)=e^{iz_\mu}-1$ at $z_\mu=0$ is handled by the elementary cancellation

$$\lim_{z_\mu\to 0}\frac{z_\mu}{f_\mu(-z)}=i. \tag{4.4}$$

Therefore we may focus on

$$\Big(\sum_\mu z_\mu^2\Big)^{-1}\mathfrak{D}(z)^{-1}\Big(\frac{1}{z_3^2}\Big\langle\frac{1}{z_1^2}+\frac{1}{z_2^2}\Big\rangle+\frac{1}{z_2^2}\Big\langle\frac{1}{z_1^2}\Big\rangle\Big) \tag{4.5}$$

with the aim of obtaining only a uniform bound over a strip $\Omega_\epsilon\times\Omega_\epsilon\times\Omega_\epsilon$. Our next observation is that

$$z_\nu^2\Big\langle\frac{1}{z_\nu^2}\Big\rangle\sum_\mu z_\mu^2=\hat{\eta}(z)\hat{\eta}(-z)(1+\delta_\nu(z));$$

$$\delta_\nu(z)=z_\nu^2\Big(\sum_\mu z_\mu^2\Big)\sum_{\mathbf{l}\neq 0}\prod_\mu\Big(\frac{z_\mu}{z_\mu+2\pi l_\mu}\Big)^{2N}\frac{1}{\sum_\mu(z_\mu+2\pi l_\mu)^2}\frac{1}{(z_\nu+2\pi l_\nu)^2}.$$

Thus

$$z_1^2z_2^2z_3^2\mathfrak{D}(z)\Big(\sum_\mu z_\mu^2\Big)^2=\hat{\eta}(z)^2\hat{\eta}(-z)^2\sum_{\nu>\lambda}\sum_{\mu\neq\nu,\lambda}z_\mu^2(1+\delta_\nu(z))(1+\delta_\lambda(z)),$$

and so

$$z_1^2z_2^2z_3^2\mathfrak{D}(z)\sum_\mu z_\mu^2$$
$$=\hat{\eta}(z)^2\hat{\eta}(-z)^2\Big[\Big(\sum_\mu z_\mu^2\Big)^{-1}\sum_{\nu>\lambda}\sum_{\mu\neq\nu,\lambda}z_\mu^2(1+\delta_\nu(z)+\delta_\lambda(z)+\delta_\nu(z)\delta_\lambda(z))\Big].$$

Similarly,

$$z_1^2z_2^2z_3^2\Big(\frac{1}{z_3^2}\Big\langle\frac{1}{z_1^2}+\frac{1}{z_2^2}\Big\rangle+\frac{1}{z_2^2}\Big\langle\frac{1}{z_1^2}\Big\rangle\Big)$$
$$=\hat{\eta}(z)\hat{\eta}(-z)\Big[1+\Big(\sum_\mu z_\mu^2\Big)^{-1}(z_2^2\delta_1(z)+z_1^2\delta_2(z)+z_3^2\delta_1(z))\Big].$$

These equations motivate the idea of multiplying both numerator and denominator of (4.5) by $z_1^2z_2^2z_3^2$. The point is that, on a narrow strip $\Omega_\epsilon\times\Omega_\epsilon\times\Omega_\epsilon$, the quantity (4.5) is clearly analytic except possibly in a small neighborhood of each $z=2\pi\mathbf{l}$, and our algebraic trick is designed to establish analyticity in the $\mathbf{l}=0$ neighborhood, where we have to worry about the manifolds $z_\mu=0$, $\sum_\mu z_\mu^2=0$. But analyticity there follows now from the observation that $(\sum_\mu z_\mu^2)^{-1}\delta_\nu(z)$ is small in neighborhoods of these manifolds for small $\operatorname{Im} z_\mu$. The analyticity in any other $2\pi\mathbf{l}$-neighborhood becomes manifest by multiplying numerator and denominator by $(z_1+2\pi l_1)^2(z_2+2\pi l_2)^2(z_3+2\pi l_3)^2$ instead. Indeed, this is less interesting because the z_μ and $\sum_\mu z_\mu^2$ are bounded away from zero in this case. Finally, since (4.5) consists of factors that are either periodic or have decay, it follows that (4.5) is also uniformly bounded on the strip.

We now turn to the problem of controlling (4.2), which is a little different. First consider only *one* of the two terms. We have

$$\frac{\hat{\eta}(z)z_2z_3}{(\sum z_\mu^2)f_1(-z)f_2(-z)}\mathfrak{D}(z)^{-1}\frac{1}{z_2^2}\left\langle\frac{1}{z_3^2}\right\rangle, \tag{4.6}$$

but this time we can only separate out $\hat{\eta}(z)z_2z_3/f_2(-z)$ and must analyze

$$f_1(z)^{-1}\left(\sum_\mu z_\mu^2\right)^{-1}\mathfrak{D}(z)^{-1}\frac{1}{z_2^2}\left\langle\frac{1}{z_3^2}\right\rangle.$$

As before, the only possible regions for analyticity problems are small neighborhoods of the points $z=2\pi\mathbf{l}$, provided $\operatorname{Im} z_\mu$ is sufficiently small, and the most interesting case is $\mathbf{l}=0$. We multiply numerator and denominator by $z_1^2z_2^2z_3^2$ to obtain

$$\frac{(\sum_\mu z_\mu^2)^{-1}z_1^2(1+\delta_3(z))}{\hat{\eta}(z)\hat{\eta}(-z)f_1(z)[1+(\sum_\mu z_\mu^2)^{-1}\sum_{\nu>\lambda}\sum_{\mu\neq\lambda,\nu}z_\mu^2(\delta_\nu+\delta_\lambda+\delta_\mu\delta_\lambda)]},$$

which is *not* analytic on the manifold $\sum_\mu z_\mu^2=0$. However, the other quantity subtracted from (4.6) involves *only* the replacement $z_2^{-2}\langle z_3^{-2}\rangle\to z_3^{-2}\langle z_3^{-2}\rangle$, so that the combined expression arising from our reduction is

$$\frac{(\sum_\mu z_\mu^2)^{-1}z_1^2(\delta_3(z)-\delta_2(z))}{\hat{\eta}(z)\hat{\eta}(-z)f_1(z)[1+(\sum_\mu z_\mu^2)^{-1}\sum_{\nu>\lambda}\sum_{\mu\neq\lambda,\nu}z_\mu^2(\delta_\nu+\delta_\lambda+\delta_\mu\delta_\lambda)]},$$

This expression *is* analytic in the neighborhood. For any other $2\pi\mathbf{l}$-neighborhood we multiply numerator and denominator of the ratios by

$$(z_1+2\pi l_1)^2(z_2+2\pi l_2)^2(z_3+2\pi l_3)^2$$

instead, but, as before, the cancellations that fail are harmless because they involve the factors z_μ and $\sum_\mu z_\mu^2$, which are now bounded away from zero.

Having established the exponential fall-off of $B_1(x)$, we turn to the smoothness property. Considering $B_1'(x)$ first, we separate out the factor

$$\frac{\hat{\eta}(p)p_2p_3}{p^2f_2(p)^*f_3(p)^*} \tag{4.7}$$

from $\hat{B}_1'(p)$ because we have already shown in our analyticity argument that the rest of the expression is bounded. By (4.3) and (4.4) we know that (4.7) has no singularities away from $p=0$ and that the cancellation of poles against the zeros of $\hat{\eta}(p)$ does not affect the $\prod_\mu|p_\mu|^{-N}$ decay contributed by $\hat{\eta}(p)$ because $f_\mu(p)^*$ is periodic. Now clearly we have the decay bound

$$cp^{-2}\prod_\mu(1+|p_\mu|)^{-N+1},$$

and for $N\geq 2$ this yields class $C^{N-\epsilon}$ smoothness in position space because

$$|p|^{N-\epsilon}\left[p^{-2}\prod_\mu(1+|p_\mu|)^{-N+1}\right]$$
$$=\left[|p|^{N-2-\epsilon}\prod_\mu(1+|p_\mu|)^{-N+2+\epsilon N^{-1}}\right]\prod_\mu(1+|p_\mu|)^{-1-\epsilon N^{-1}}$$

is integrable. The point is that

$$\prod_{\mu}(1+|p_\mu|)^{\epsilon N^{-1}} \le c(1+|p|)^{\epsilon};$$

$$|p_\mu|^{N-2} \le c\prod_{\mu}(1+|p_\mu|)^{N-2}, \quad N\ge 2.$$

For the special case $N=1$ we must estimate more carefully. In the region where $|p_2|\le 1$ and p_3 is large, we have the bound

$$cp^{-2}(1+|p_1|)^{-1},$$

while in the region where $|p_3|\le 1$ and p_2 is large, the same estimate holds. In each instance

$$|p|^{1-\epsilon}[p^{-2}(1+|p_1|)^{-1}]=|p|^{-1-\epsilon}(1+|p_1|)^{-1}]$$

is integrable over the given region. Now to handle the region where both p_2 and p_3 are large, we consider the three terms of $\hat{B}_1'(p)$ separately, extracting the factor

$$\frac{\hat{\eta}(p)p_2p_3}{p_\nu^2p^2f_2(p)^*f_3(p)^*}, \tag{4.8}$$

where $\nu=2,3$, as the case may be, say, $\nu=2$. The residual expression is periodic and regular by our analyticity arguments, and (4.8) has the decay bound

$$cp^{-2}(1+|p_1|)^{-1}(1+|p_2|)^{-2}.$$

Clearly,

$$|p|^{1-\epsilon}[p^{-2}(1+|p_1|)^{-1}(1+|p_2|)^{-2}]=|p|^{-1-\epsilon}(1+|p_1|)^{-1}(1+|p_2|)^{-2}$$

is integrable, so we have established class $C^{N-\epsilon}$ smoothness of $B_1'(x)$ in the special case $N=1$ as well.

The arguments for the contribution $B_1''(x)$ are similar: We separate out the factor $\hat{\eta}(p)p_2p_3/p^2f_2(p)^*$ in all cases except $N=1$ with p_2 and p_3 both large, and in that special case we look at the two terms of $\hat{B}_1''(x)$ and extract $\hat{\eta}(p)p_2p_3/p_\nu^2p^2f_2(p)^*$, where $\nu=2,3$.

5. Orthogonalization on a Single Scale

Our (non-orthonormal) basis of wavelets for divergence-free vector fields is only interscale orthogonal as it stands. To orthogonalize on each scale in a translation-invariant way is routine, provided that our translation-invariant overlap matrix is bounded below by some $\epsilon>0$. We address this issue here.

As usual, we consider the unit-scale wavelets without loss of generality, but we first notice that the "unit-scale" wavelets in this paper are actually *two*-unit translates of one another. We regard them as "unit-scale" wavelets because they are defined by linear constraints associated with *one*-unit translates of $\eta(x)$, so our convention differs from the usual one. Now let $\{\mathbf{b}^{(i)}\}$ be a linearly independent set of antisymmetric tensor-valued unit lattice configurations satisfying both (2.1) and (2.2); in three dimensions such a set has fourteen elements, and we shall give an example below. If $\mathbf{B}^{(i)}(\mathbf{x})$ is the "mother wavelet" constructed from $\mathbf{b}^{(i)}(\mathbf{m})$, then our "unit-scale" wavelets are the vector-valued functions

$$\mathbf{B}^{(i)}(x+2\mathbf{n}), \quad \mathbf{n}\in\mathbb{Z}^3, \ 1\le i\le 14,$$

so the overlap matrix for that scale is

$$S^{ii'}_{\mathbf{nn}'}=\int \mathbf{B}^{(i)}(x+2\mathbf{n})\cdot\mathbf{B}^{(i')}(x+2\mathbf{n}')\,dx.$$

Obviously,

$$S^{ii'}_{\mathbf{nn}'}=\int e^{i(2\mathbf{n}-2\mathbf{n}')\cdot p}S^{ii'}(p)\,dp,$$

$$S^{ii'}(p)=\hat{\mathbf{B}}^{(i)}(p)\cdot\hat{\mathbf{B}}^{(i')}(p)^*,$$

and so, by the continuity of $S^{ii'}(p)$, the positive lower bound on the overlap matrix exists if the periodic matrix-valued function

$$\sum_{\mathbf{l}\in\mathbb{Z}^3} S(p+\pi\mathbf{l})$$

is nonsingular everywhere. This condition certainly holds if we can establish that

$$\sum_{\mathbf{l}}\left|\sum_i t_i\hat{\mathbf{B}}^{(i)}(p+\pi\mathbf{l})\right|^2>0$$

everywhere for arbitrary t_i not all zero. Let

$$G^{(i)}_{\mu\nu}(p)=\sum_m b^{(i)}_{\mu\nu}(m)e^{im\cdot p},$$

and following the notation of Section 3, let

$$G^{(i)}(p)=\begin{bmatrix} G^{(i)}_{32}(p)\\ G^{(i)}_{31}(p)\\ G^{(i)}_{21}(p)\end{bmatrix}.$$

By (3.3) the desired condition becomes

$$\sum_{\lambda\in\{0,1\}^3} H(p+\pi\lambda)>0,$$

where

$$H(p)=\left(\sum_i t_i G^{(i)}(p)^{\dagger *}\right)D(p)^{-1*}L(p)$$
$$\times\left\langle\frac{1}{p^2}M_0(p)E(p)^2M_0(p)\right\rangle L(p)D(p)^{-1}\sum_i t_i^* G^{(i)}(p),$$

with matrix multiplication understood here and the dagger representing the transpose matrix. Now, by inspection of our momentum expressions, it is easy to see that the 3×3 matrix

$$D(p)^{-1*}L(p)\left\langle\frac{1}{p^2}M_0(p)E(p)^2M_0(p)\right\rangle L(p)D(p)^{-1}$$

is certainly nonsingular on the orthogonal complement of

$$\begin{bmatrix} f_1(p) \\ -f_2(p) \\ f_3(p) \end{bmatrix}$$

when no coordinate of p is an integer multiple of 2π. In particular,

$$M_0(p)E(p)^2M_0(p)$$
$$= \begin{bmatrix} 2+\frac{p_3^2}{p_2^2}+\frac{p_2^2}{p_3^2}+\frac{p_1^2}{p_3^2}+\frac{p_1^2}{p_2^2} & 1+\frac{p_2^2}{p_3^2}+\frac{p_1^2}{p_3^2} & -1-\frac{p_3^2}{p_2^2}-\frac{p_1^2}{p_2^2} \\ 1+\frac{p_2^2}{p_3^2}+\frac{p_1^2}{p_3^2} & 2+\frac{p_2^2}{p_3^2}+\frac{p_1^2}{p_3^2}+\frac{p_3^2}{p_1^2}+\frac{p_2^2}{p_1^2} & 1+\frac{p_3^2}{p_1^2}+\frac{p_2^2}{p_1^2} \\ -1-\frac{p_3^2}{p_2^2}-\frac{p_1^2}{p_2^2} & 1+\frac{p_3^2}{p_1^2}+\frac{p_2^2}{p_1^2} & 2+\frac{p_3^2}{p_2^2}+\frac{p_3^2}{p_1^2}+\frac{p_1^2}{p_2^2}+\frac{p_2^2}{p_1^2} \end{bmatrix}$$

annihilates

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

and is nonsingular on the orthogonal complement. Hence

$$H(p) \geq c\left|\sum_i t_i G^{(i)}(p)\right|^2$$

on cubes bounded away from these planes of bad points. This follows from the fact that every $G^{(i)}(p)$ lies in the orthogonal complement of

$$\begin{bmatrix} f_1(p) \\ -f_2(p) \\ f_3(p) \end{bmatrix}.$$

Thus we may finally conclude that

$$\sum_{\lambda\in\{0,1\}^3} H(p+\pi\lambda) \geq c \max_{\lambda\in\{0,1\}^3}\left|\sum_i t_i G^{(i)}(p+\pi\lambda)\right|^2$$

everywhere. It is straightforward to show that this lower bound is strictly positive once $\{G^{(i)}(p)\}$ has been chosen. A natural choice is

$$G^{(\sigma 1)}(p) = \begin{bmatrix} f_2(p)f_3(p) \\ -f_1(p)f_3(p) \\ 0 \end{bmatrix} S_\sigma$$

and

$$G^{(\sigma 2)}(p) = \begin{bmatrix} f_2(p)f_3(p) \\ 0 \\ -f_1(p)f_2(p) \end{bmatrix} S_\sigma,$$

where

$$S_\sigma = \sum_{\mathbf{m}} (-1)^{\sum_\mu m_\mu} c_{\sigma - \mathbf{m}} e^{i\mathbf{m}\cdot p} = \prod_\mu [e^{ip_\mu}(1-e^{ip_\mu})^N]^{\sigma_\mu}. \tag{5.1}$$

The c_σ in (5.1) have been defined after equation (2.2), and σ ranges over the seven elements of $\{0,1\}^3\backslash\{0,0,0\}$ (and indexes the "mother wavelet"). Note that both (2.1) and (2.2) are satisfied.

Finally, we observe that the wavelets we obtain when we apply the inverse square root of the overlap matrix have exponential decay. This follows from an argument of Thomas and Combes [4].

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