Surfaces Parameterizing Waring Presentations of Smooth Plane Cubics

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1. Introduction

Let V be an n-dimensional vector space over an algebraically closed field k of characteristic 0, and let ϕ be a homogeneous polynomial (form) of degree d on V. A Waring presentation is a presentation of ϕ as a sum of m dth powers of linear forms on V. In this paper we shall study the variety of all such presentations in the case $d=3$, $n=3$, and $m=4$. To introduce the questions we will be discussing, we first consider the case of quadratic forms where the answers are easy and well known.

A nonsingular quadratic form Q on V can be written as a sum of n squares of linear forms $l_i \in V^*$:

$$Q = l_1^2 + \cdots + l_n^2.$$

The variety parameterizing all such n-tuples $(l_1, \ldots, l_n)$ for a given quadratic form Q is isomorphic to the orthogonal group $O_n(k)$. Thus this variety has two isomorphic irreducible components, corresponding to orthogonal matrices of determinant $+1$ and -1 respectively. It follows from the classical Cayley formulas that these components are rational. Indeed, let $so_n(k)$ be the set of all skew-symmetric complex $n\times n$ matrices and let I be the $n\times n$ identity matrix. Then the following mutually inverse rational maps

$$\begin{array}{ccc} so_n(k) & \rightleftharpoons & SO_n(k) \\ A & \to & (I+A)(I-A)^{-1} \\ (B-I)(I+B)^{-1} & \leftarrow & B \end{array}$$

establish a birational isomorphism between $SO_n(k)$ and the affine space $so_n(k)$.

Suppose we start with a cubic form ϕ on a 3-dimensional vector space V which cuts out a smooth curve C in $P(V)$. Assume that the j-invariant of C is nonzero, that is, ϕ cannot be written as a sum of three cubes of linear forms. Then ϕ can be written as a sum of four cubes:

(1) $$\phi = l_1^3 + \cdots + l_4^3;$$

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see Theorem 4.8(c). In this paper we shall study the set X_ϕ of all presentations (1). We show that X_ϕ is an irreducible affine algebraic surface of general type. Moreover, we prove that its minimal compactification $\bar{X}_\phi$ has positive index. For an account of special properties of surfaces of positive index and the place they occupy in the "surface geography" see [8].

The rest of the paper is structured as follows. Section 2 contains some preliminary facts about plane cubics. In Section 3 we prove an irreducibility criterion for abelian coverings of smooth irreducible varieties. In Section 4 we give an explicit description of the surface X_ϕ by relating it to the Hessian curve H of ϕ. The surface X_ϕ is then realized as an abelian cover of an open subset of $H \times H$. In Section 5 we use this construction and the criterion of Section 3 to prove the irreducibility of X_ϕ. We also prove that the generic fiber of the map $X_\phi \to H$ is irreducible.

Next we turn to numerical invariants. In Sections 6 and 7 we present an explicit construction of the minimal compactification $\bar{X}_\phi$ of X_ϕ and show that the genus of the generic fiber of $X_\phi \to H$ is 325; see Theorem 7.4. In Section 8 we show that the Chern numbers of $\bar{X}_\phi$ are given by $c_1^2 = 7{,}452$ (Theorem 8.1) and $c_2 = 2{,}916$ (Theorem 8.2) and thus $\bar{X}_\phi$ has positive index.

While there is a great deal of classical literature on the Waring problem in number theory, we are aware of only a small number of classical results dealing with Waring presentations of forms. For results on binary forms ($n = 2$) we refer the reader to Gundelfinger ([3], [4]; see also [6]). The case of cubic forms in 4 variables was studied by Sylvester [14]. He showed that a form cutting out a smooth cubic surface in $\mathbf{P}^3$ can be written as a sum of five cubes of linear forms. These linear forms are unique up to reordering and multiplication by a cube root of 1.

This work resulted from an attempt to find a geometric interpretation of the algebraic description of X_ϕ presented in [10]. This description is based on an algorithm proposed in [9] which generates Waring presentations of a given cubic form for low values of $m-n$ when $n > 3$.

This paper does not treat the case where ϕ (or, more precisely, the cubic curve that ϕ cuts out in $\mathbf{P}^2$) is singular. If ϕ is nodal then one can describe X_ϕ explicitly and show that it is irreducible in essentially the same way as we do for a smooth ϕ. The calculation of numerical invariants, however, appears to be more complicated. When ϕ is cuspidal, the Hessian curve is no longer irreducible, and the methods of this paper do not apply. In this case X_ϕ is a disjoint union of 12 isomorphic irreducible components; each component is a ruled surface (see [11]).

We also remark that the arithmetic properties of X_ϕ appear to be very interesting. Suppose the cubic form ϕ defined over the rationals cuts out a smooth curve in $\mathbf{P}^2$. It is not known whether or not equation (1) always has a finite number of solutions over a number field. This question is in fact a special case of a conjecture of Voita and Lang about rational points on varieties of general type; see [7, Conjecture 5.8].

The following notations will be used throughout the paper.

k	an algebraically closed base field of characteristic 0.
V	a 3-dimensional vector space over k.
C	a smooth cubic curve in $\mathbf{P}(V)=\mathbf{P}^2$ with a non-zero j-invariant.
ϕ	a cubic form in V which cuts out C.
ω	the symmetric trilinear form obtained from ϕ by polarization.
H	the Hessian curve of C.
O	an inflection point of H.
$\oplus$	the addition operation on H with identity element O.
i	the involution of H given by $\omega(A, i(A), \cdot)\equiv 0$.
P	point of order 2 in H such that $i(A)=A\oplus P$ for any $A\in H$.
X_ϕ	the variety of all presentations of ϕ as a sum of four cubes of linear forms.
f_{ij}	the map $X_\phi \to H$ given by $l_i(f_{ij}(p))=l_j(f_{ij}(p))=0$ where $p=(l_1,\dots,l_4)$.
a_i, b_i, c_i	the three integers between 1 and 4 other than i; here $i=1,\dots,4$.
$\alpha_i, \beta_i, \gamma_i$	the three 2-element subsets of $\{1,2,3,4\}\backslash\{i\}$.

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2. Some Preliminary Considerations

In this section we discuss some basic facts about cubic forms in the projective plane which are used in the sequel. For the convenience of the reader we present complete proofs as well as some references to classical literature.

Let ω be the trilinear symmetric form in V obtained from ϕ by polarization. Recall that $\omega(v,v,v)=\phi(v)$ for every $v\in V$ and that ω is uniquely characterized by this property. In particular, if the linear forms $l_1,\dots,l_4$ satisfy (1) then

$$\omega(x,y,z)=l_1(x)l_1(y)l_1(z)+\cdots+l_4(x)l_4(y)l_4(z). \tag{2}$$

Recall also that the first polar quadric ω_r of ϕ at $r\in V$ is given by

$$\omega_r(x)=\omega(r,x,x). \tag{3}$$

Let H be the set of all $R\in\mathbf{P}(V)$ such that the quadratic form ω_r is degenerate for any affine representative r of R. The following two lemmas are due to Hesse (see [2], [12]).

2.1. Lemma. *H is precisely the zero locus of the Hessian determinant of ϕ in $\mathbf{P}(V)$.*

Proof. Let $\{x_1, x_2, x_3\}$ be a basis of V^*. Let $v_i = x_i(v)$. Then

$$\omega(v, x, x) = \frac{1}{2} \sum_{i=1}^{3} v_i \frac{\partial \phi}{\partial x_i}.$$

The (s, r)th entry of the matrix of this quadratic form is thus given by

$$\sum_{i=1}^{3} v_i \frac{\partial^3 \phi}{\partial x_i \, \partial x_s \, \partial x_r} = \frac{\partial^2 \phi}{\partial x_s \, \partial x_r}(v).$$

Hence, ω_v is singular if and only if the Hessian matrix of ϕ is singular at v. □

Note that the above lemma holds for any cubic form in any number of variables.

2.2. LEMMA. *The Hessian curve H is a smooth cubic in* $\mathbf{P}(V)$.

Proof. For an appropriate choice of coordinates in V, the form ϕ can be written as $x^3 + y^3 + z^3 + 3axyz$. The curve C is smooth if and only if $a^3 \neq -1$; its j-invariant is given by

$$j(C) = a^3 \frac{(8 - a^3)^3}{(a^3 + 1)^3}; \tag{4}$$

see [2, §7.3]. A direct computation shows that H is given by

$$a^2(x^3 + y^3 + z^3) - (a^3 + 4)xyz = 0; \tag{5}$$

see [12, Art. 218]. Since $j(C) \neq 0$, we must have $a \neq 0$. Hence, (2.4) is the same as $x^3 + y^3 + z^3 + 3bxyz = 0$ where $b = -(a^3 + 4)/3a^2$. Therefore, we only need to check that $b^3 \neq -1$. Since

$$b^3 + 1 = -\frac{(a^3 + 1)(a^3 - 8)^2}{27a^6},$$

the lemma now follows from the assumption that $j(C) \neq 0$. □

Lemma 2.2 says that H is an elliptic curve. We fix an inflection point $O \in H$. The group operation on H with the identity element O will be denoted by $\oplus$.

2.3. LEMMA. *Let ω_v be the first polar quadratic form defined in* (3). *Then for any $v \in V$* $\operatorname{rank}(\omega_v) \geq 2$.

Proof. Assume the contrary. Then there exists a 2-dimensional subspace W of V such that $\omega(v, w, \cdot) \equiv 0$ for any $w \in W$. The plane W determines a projective line in $\mathbf{P}(V)$ which lies entirely in the Hessian curve H, contradicting Lemma 2.2. □

By Lemma 2.3, for every point A on the Hessian curve H there exists a unique $B \in \mathbf{P}(V)$ such that $\omega(A, B, \cdot) \equiv 0$. The point B also lies on the Hessian curve H. Following nineteenth-century literature we shall say that B is the point corresponding to A; see [12, Art. 175]. The map which sends a point of H to its corresponding point is an algebraic involution of H. Denote this map by i: for the points A and B above, $i(A)=B$ and $i(B)=A$. We shall now see that i is simply a translation by a point of order 2.

2.4. LEMMA. *i is a fixed point free involution of H.*

Proof. Suppose $i(A)=A$. Since $\phi(A)=\omega(A,A,A)=0$, A lies on the curve C cut out by ϕ in $\mathbf{P}(V)$. Moreover,

$$\omega(A,A,x)=\frac{1}{2}\sum_{i=1}^{3} x_i \frac{\partial\phi}{\partial x_i}(A)$$

vanishes for every $x=(x_1,x_2,x_3)\in V$. Hence, A is a singular point of C, a contradiction. □

2.5. LEMMA (see [12, Art. 178]). *Let $A\in H$. Then the linear form $\omega(A,A,\cdot)$ cuts out the tangent line to H at $i(A)$.*

Proof. For any $A,B\in H$ the line cut out by $\omega(A,B,\cdot)$ passes through $i(A)$ and $i(B)$. □

2.6. LEMMA. *Let $A\in H$. Then the tangent lines to H at A and $i(A)$ intersect on H.*

Proof. Let $B=i(A)$ and let a and $b\in V$ be affine representatives for A and B respectively. Let C be the intersection point of the tangent lines at A and B. Then, by Lemma 2.5,

$$\omega(a,a,C)=\omega(b,b,C)=0.$$

Since we also have $\omega(a,b,\cdot)\equiv 0$,

$$\omega(\alpha a+\beta b,\alpha a+\beta b,C)=0$$

for every $\alpha,\beta\in k$. In other words, the zero locus of the polar quadratic ω_C contains the line joining A and B. Hence, $C\in H$. □

2.7. PROPOSITION. *There exists a point $P\in H$ of order 2 such that for any $A\in H$ we have $i(A)=A\oplus P$.*

Proof. Lemma 2.6 can be restated as follows:

$$2A\ominus 2i(A)=O.$$

Thus $i(A)\to A\ominus i(A)$ is a regular map from H to the set of points of order 2 in H. Since there are only four points of order 2, this map is constant. □

REMARKS. Every fixed point free involution of an elliptic curve is a translation by an element of order 2.

Every smooth cubic in $\mathbf{P}^2$ is realized as the Hessian H of a smooth cubic form ϕ in three different ways; see [12]. The three involutions i induced this way are translations by the three points of order 2. The map

$$\{\text{plane cubics}\} \to \{\text{plane cubics}\}$$

which sends a curve C to its Hessian curve H induces a 3:1 covering map of the j-line to itself given by

$$j(H)=\frac{-j^3(C)+768j^2(C)-196608j(C)+16777216}{27j^2(C)}.$$

We now define the following divisors in $H\times H$ and $H\times H\times H$.

$$(6)\qquad \begin{array}{lll} \text{In } H\times H & D_1: & \{(A,B): A=B\oplus P\}, \\ & D_2: & \{(A,B): 2A\oplus B=0\}; \\ \text{In } H\times H\times H & E_1: & \{(A,B,C): A=B\oplus P\}, \\ & E_2: & \{(A,B,C): A=C\oplus P\}, \\ & E_3: & \{(A,B,C): B=C\oplus P\}, \\ & E_4: & \{(A,B,C): A\oplus B\oplus C=0\}. \end{array}$$

Let $\mathrm{pr}_{ij}: H^j \to \mathbf{P}^2$ be the projection to the ith factor.

2.8. LEMMA.

(a) $\operatorname{div}\omega(A,B,C)=E_1+E_2+E_3+E_4$.
(b) $\operatorname{div}\omega(A,A,B)=2D_1+D_2$.

Here $\omega(A,B,C)$ *and* $\omega(A,A,B)$ *are viewed as global sections of* $\mathrm{pr}_{13}^*\mathcal{O}_{\mathbf{P}^2}(1)\otimes \mathrm{pr}_{23}^*\mathcal{O}_{\mathbf{P}^2}(1)\otimes \mathrm{pr}_{33}^*\mathcal{O}_{\mathbf{P}^2}(1)$ *and* $\mathrm{pr}_{12}^*\mathcal{O}_{\mathbf{P}^2}(2)\otimes \mathrm{pr}_{22}^*\mathcal{O}_{\mathbf{P}^2}(1)$ *respectively.*

Proof. (a) $\omega(A,B,C)$ vanishes if and only if $A=i(B)$ or $A\neq i(B)$ but C lies on the line joining $i(A)$ and $i(B)$. The second possibility translates into $C=i(A)$, $c=i(B)$, or $i(A)+i(B)+C=0$. In view of Proposition 2.7 this implies that $\operatorname{div}\omega(A,B,C)$ is, indeed, a positive integral combination of $E_1,\dots,E_4$; we just have to make sure each E_i occurs with multiplicity 1. Assume $\operatorname{div}\omega(A,B,C)=\sum_{i=1,\dots,4} n_iE_i$. Let L be a line in $\mathbf{P}^2$ which intersects H in three distinct pairwise noncorresponding points Q, R, and S. Let $b,c\in V$ be affine representatives for $B=i(Q)=Q\oplus P$ and $C=i(R)=R\oplus P$. Let $E=\{(A,B,C): B=Q\oplus P,\ C=R\oplus P\}$. Then E intersects each E_i transversely and

$$E_1\cap\{B=Q\oplus P;\ C=R\oplus P\}=\{A=Q\}$$
$$E_2\cap\{B=Q\oplus P;\ C=R\oplus P\}=\{A=R\}$$
$$E_3\cap\{B=Q\oplus P;\ C=R\oplus P\}=\emptyset$$
$$E_4\cap\{B=Q\oplus P;\ C=R\oplus P\}=\{A=S\}$$

Hence, if we view $\omega(A,b,c)$ as a global section of $\mathrm{pr}_{11}^*\mathcal{O}_{\mathbf{P}^2}(1)$ then

$$\operatorname{div}\omega(A,b,c)=n_1Q+n_2R+n_4S.$$

On the other hand, since L intersects H transversely, we have

$$\operatorname{div} \omega(A, b, c) = Q + R + S.$$

Hence, $n_1 = n_2 = n_4 = 1$. The same argument with A and C interchanged yields $n_3 = 1$.

(b) By Lemma 2.5, $\omega(A, A, B)$ vanishes if and only if B lies on the tangent line to $i(A)$, that is, $B = A \oplus P$ or $B + 2A = 0$. Hence,

$$\operatorname{div}(\omega(A, A, B)) = n_1 D_1 + n_2 D_2$$

for some positive integers n_1 and n_2. Fix $Q \in H$ such that $i(Q)$ is not an inflection point of H. Let $r \in V$ be an affine representative of $R = i(Q)$ and let $S = \psi(Q, Q)$. Letting B vary over H and reasoning as in the proof of part (a), we obtain

$$n_1 Q + n_2 S = \operatorname{div}(\omega(a, a, B)) = 2Q + S.$$

Hence, $n_1 = 2$ and $n_2 = 1$. □

3. A Criterion for Irreducibility

In this section we prove an irreducibility criterion for finite abelian covers of smooth irreducible varieties.

3.1. THEOREM. *Suppose Y is a smooth irreducible projective variety, $g_1, \ldots, g_m$ are rational functions on Y, and m, n are positive integers. We define the variety X as the set of all $(p, s_1, \ldots, s_m) \in Y \times k^m$ such that for $i = 1, \ldots, m$ the function g_i does not have a pole at p and $s_i^n = g_i(p)$. Then X is reducible if and only if the divisors of $g_1, \ldots, g_m$ are $(\mathbf{Z}/n\mathbf{Z})$-linearly dependent in* $\operatorname{PDiv}(Y)/n\operatorname{PDiv}(Y)$. *Here* $\operatorname{PDiv}(Y)$ *is the group of principal divisors on Y.*

Proof. Let Y_0 be the complement in Y of the union of the pole sets of $g_1, \ldots, g_m$. Let $\pi: X \to Y_0$ be the natural projection. Denote the irreducible components of X by $X_1, \ldots, X_r$. The group $G = (\mathbf{Z}/n\mathbf{Z})^m$ acts on X by

$$(\xi_1, \ldots, \xi_m)(p, s_1, \ldots, s_m) \to (p, \xi_1 s_1, \ldots, \xi_m s_m). \tag{7}$$ □

3.2. LEMMA. *The action* (7) *induces a transitive permutation action of G on the irreducible components $X_1, \ldots, X_r$.*

Proof. Since π is a finite map, $\pi(X_1), \ldots, \pi(X_r)$ are closed in Y_0. Since they cover Y_0, we may assume without loss of generality that $\pi(X_1) = Y_0$. Then

$$X = \bigcup_{g \in G} gX_1.$$

Hence, each X_i equals gX_1 for some $g \in G$. □

Suppose X is reducible, that is, $r \geq 2$. Let G_0 be the stabilizer of X_1 under this permutation action. By our assumption $r \geq 2$, hence $G_0 \neq G$. This means that there are integers $a_1, \ldots, a_m$, not all divisible by n, such that $\xi_1^{a_1} \cdots \xi_m^{a_n} = 1$ for any $(\xi_1, \ldots, \xi_n) \in G_0$.

3.3. LEMMA. *G_0 acts transitively on the generic fiber of the restricted projection map $\pi : X_1 \to Y$.*

Proof. Observe that $\dim X_i \cap X_j < \dim Y$ for any $i \neq j$. Hence, for a generic point $y \in Y_0$, every point of $\pi^{-1}(y)$ lies in exactly one X_i. Fix one such y and suppose that $\pi(x) = \pi(\tilde{x}) = y$ for some $x, \tilde{x} \in X_1$. Then there exists a $g \in G$ such that $\tilde{x} = gx$. We want to show that $g \in G_0$. Indeed, assume the contrary, say $gX_1 = X_2$. Then $\tilde{x} \in X_1 \cap X_2$, contradicting our choice of y. □

Lemma 3.3 says that the morphism $X_1/G_0 \to Y$ induced by π is birational. Since the regular function

$$h = s_1^{a_1} \cdots s_m^{a_m} \tag{8}$$

on X_1 is G_0-invariant, it is regular on X_1/G_0 and hence rational on Y. Raising (8) to the nth power, we get

$$h^n = g_1^{a_1} \cdots g_m^{a_m} \tag{9}$$

or $n \operatorname{div}(h) = a_1 \operatorname{div}(g_1) + \cdots + a_m \operatorname{div}(g_m)$, as desired.

Conversely, if the divisors of the functions g_i are linearly dependent modulo n then there exists a rational function h on Y such that (9) holds for some integers $a_1, \ldots, a_m$, not all divisible by n. Let

$$s = s_1^{a_1} \cdots s_m^{a_m}.$$

Then $(s-h)(s-\xi h) \cdots (s-\xi^{n-1}h)$ vanishes on X; here ξ is a primitive nth root of unity. This gives a decomposition of X as a union of n closed subsets. Since s is not constant on the fibers of π, each of these subsets is properly contained in X. Hence, X is reducible. □

REMARK. The difficult direction of the theorem (i.e., linear independence of divisors implies irreducibility) holds for any quasi-projective Y. Indeed, by the Hironaka resolution theorem, Y can be embedded as an open subset in a smooth irreducible projective variety Y_1. Now apply Theorem 3.1 to Y_1. For the other direction, however, it is essential that Y should be complete.

4. Configurations

In this section we give an explicit description of X_ϕ as an unramified cover of an open subset of $H \times H$. All double subscripts in this section will be symmetric; for example, f_{ij} will always be equal to f_{ji} and A_{ij} will always be equal to A_{ji}.

Recall that for $i \in \{1,2,3,4\}$ the three elements of $\{1,2,3,4\} \backslash \{i\}$ are denoted by a_i, b_i, and c_i. The three 2-element subsets of $\{1,2,3,4\} \backslash \{i\}$ are denoted by $\alpha_i = \{a_i, b_i\}$, $\beta_i = \{a_i, c_i\}$, $\gamma_i = \{b_i, c_i\}$.

4.1. LEMMA. *Let $(l_1, \ldots, l_4) \in X_\phi$. Then no three of the forms l_i are linearly dependent.*

Proof. Assume the contrary, say l_1, l_2, and l_3 are linearly dependent. Then there exists a $v \in V$ such that $l_1(v) = l_2(v) = l_3(v) = 0$. By (2), the first polar ω_v is a multiple of l_4^2, contradicting Lemma 2.3. □

Let f_{ij} be the morphism from X_ϕ to $\mathbf{P}(V)$ given by $l_i(f_{ij}(p)) = l_j(f_{ij}(p)) = 0$, where $p = (l_1, \ldots, l_4) \in X_\phi$ and $1 \le i < j \le 4$. Since l_i and l_j are linearly independent, f_{ij} is well-defined.

4.2. LEMMA. *$f_{ij}(p)$ lies on H.*

Proof. We may assume without loss of generality that $i = 1$, $j = 2$. Let v be an affine representative of the projective point $f_{12}(p)$. By (2), the first polar quadratic form ω_v is a linear combination of l_3^2 and l_4^2, and hence is singular on V. Hence, $f_{12}(p) \in H$. □

4.3. DEFINITION. Let $\Lambda = (A_{ij})$ be a 6-tuple of points of H; here $1 \le i < j \le 4$. We say that Λ is a *ϕ-configuration if*

(i) there exist four lines $L_1, \ldots, L_4$ in $\mathbf{P}^2$ such that for $i = 1, \ldots, 4$ the three intersection points of L_i with H are A_{ia_i}, A_{ib_i}, and A_{ic_i}.

(ii) $A_{12} = i(A_{34})$, $A_{13} = i(A_{24})$, $A_{23} = i(A_{14})$.

The points A_{ij} and the lines L_i will be called *vertices* and *sides* of Λ respectively. We say that Λ is *nondegenerate* if the six points A_{ij} are distinct or, equivalently the four lines $L_1, \ldots, L_4$ are in general position. A nondegenerate ϕ-configuration is thus simply a quadrangle inscribed in H.

4.4. LEMMA. *Let $p = (l_1, \ldots, l_4) \in X_\phi$, and let $A_{ij} = f_{ij}(p)$ for $1 \le i < j \le 4$. Then $\Lambda = (A_{ij})$ is a nondegenerate ϕ-configuration.*

Proof. For $i = 1, \ldots, 4$ let L_i be the line cut out by l_i. Then the six points $A_{ij} = L_i \cap L_j$ lie on H, by Lemma 4.2. Condition (ii) of the definition is easily verified using (2). Finally, Λ is nondegenerate by Lemma 4.1. □

Thus every Waring presentation $p \in X_\phi$ determines a ϕ-configuration. The next proposition shows that every ϕ-configuration arises in this way.

4.5. PROPOSITION. *Let $\Lambda = (A_{ij})$ be a nondegenerate ϕ-configuration with sides $L_1, \ldots, L_4$. Let l_i be a linear form vanishing on L_i. Then $(l_1, \ldots, l_4)$ lies in X_ϕ if and only if*

$$l_i(A_{\alpha_i})\, l_i(A_{\beta_i})\, l_i(A_{\gamma_i}) = \omega(A_{\alpha_i}, A_{\beta_i}, A_{\gamma_i}) \tag{10}$$

for $i = 1, \ldots, 4$. In particular, any nondegenerate ϕ-configuration is induced by a 4-tuple of linear forms $(l_1, \ldots, l_4) \in X_\phi$.

Proof. If $(l_1, \ldots, l_4) \in X_\phi$ then (10) follows directly from (2). Conversely, suppose (10) holds. Let $\tilde{\omega}$ be the trilinear form defined by

$$\tilde{\omega}(x,y,z)=\sum_{i=1}^{4} l_i(x)l_i(y)l_i(z).$$

We want to show that $\tilde{\omega}=\omega$, or, equivalently,

$$\tilde{\omega}(X,Y,Z)=\omega(X,Y,Z), \tag{11}$$

where X, Y, and Z range over the six points A_{ij}, $1\le i<j\le 4$. By symmetry (i.e., after relabeling the lines if necessary) we may assume $X=A_{12}$. Since A_{23}, A_{24}, and A_{34} span $\mathbf{P}(V)$, we need only check (11) for Y ranging over those three points. If $Y=A_{34}$ then both sides of (11) vanish, since $A_{34}=i(A_{12})$. Hence, we only need to check (11) for $Y=A_{23}$ or A_{24}. Once again, after possibly relabeling the lines L_3 and L_4, we may assume $Y=A_{23}$. Since the points A_{13}, A_{14}, and A_{34} span $\mathbf{P}(V)$, we need only check (11) for Z ranging over those three points. However, if $Z=A_{14}$ or A_{34} then both sides of (11) vanish. Finally, if $X=A_{12}$, $Y=A_{23}$, and $Z=A_{24}$, then (11) reduces to (10) with $i=2$.

The last claim of the proposition is now easily verified: one can start with any l_i vanishing on L_i and normalize it using (10). □

The forms l_i in Proposition 4.5 are determined by the ϕ-configuration up to a multiple of a cube root of 1. Their cubes are thus uniquely determined by the ϕ-configuration.

4.6. COROLLARY. *A 4-tuple of linear forms $l_1,\dots,l_4$ lies in X_ϕ if and only if it induces a nondegenerate ϕ-configuration (A_{ij}) such that*

$$l_i^3(\cdot)=\frac{\omega^3(A_{\alpha_i},A_{\beta_i},\cdot)}{\omega(A_{\alpha_i},A_{\alpha_i},A_{\beta_i})\,\omega(A_{\alpha_i},A_{\beta_i},A_{\beta_i})} \tag{12}$$

for $i=1,\dots,4$.

Proof. Suppose $(l_1,\dots,l_4)\in X_\phi$. Then the induced configuration is nondegenerate by Lemma 4.4. Since both l_i and $\omega(A_{\alpha_i},A_{\beta_i},\cdot)$ vanish at A_{ib_i} and A_{ic_i}, we must have

$$l_i(\cdot)=t_i\,\omega(A_{\alpha_i},A_{\beta_i},\cdot)$$

for some $t_1,\dots,t_4\in k$. Substituting this into (10), we obtain

$$t_i^3\,\omega(A_{\alpha_i},A_{\alpha_i},A_{\beta_i})\,\omega(A_{\alpha_i},A_{\beta_i},A_{\beta_i})=1,$$

as desired.

Conversely, suppose $(l_1,\dots,l_4)$ induces a nondegenerate ϕ-configuration (A_{ij}) and (12) holds. Since $\omega(A_{\alpha_i},A_{\beta_i},\cdot)$ vanishes at A_{ib_i} and A_{ic_i}, it vanishes on L_i. Hence, so does l_i. Conditions (10) of Proposition 4.5 can now be verified directly. □

We now give an explicit description of the variety of all ϕ-configurations. We begin by rewriting Definition 4.3 in terms of the group operation on H. The following lemma is an immediate consequence of Proposition 2.7.

4.7. LEMMA. *Let $\Lambda=(A_{ij})$ be a 6-tuple of points of H where $1\le i<j\le 4$. Then Λ is a ϕ-configuration if and only if*

$$\begin{aligned} A_{14}&=\ominus A_{12}\ominus A_{13},\\ A_{23}&=A_{12}\oplus P,\\ A_{24}&=A_{13}\oplus P,\\ A_{34}&=P\ominus A_{12}\ominus A_{13}. \end{aligned} \tag{13}$$

Moreover, Λ is nondegenerate if and only if the above conditions hold and (A_{12}, A_{13}) does not lie in the union D of the following six curves $D_1,\dots, D_6\subset H^2$:

$$\begin{aligned} D_1&: \{(A,B): A\ominus B=P\},\\ D_2&: \{(A,B): 2A\oplus B=O\},\\ D_3&: \{(A,B): A\oplus 2B=O\},\\ D_4&: \{(A,B): A\ominus B=O\},\\ D_5&: \{(A,B): 2A\oplus B=P\},\\ D_6&: \{(A,B): A\oplus 2B=P\}. \end{aligned} \tag{14}$$

The group $(\mathbf{Z}/3\mathbf{Z})^4$ acts on X_ϕ by

$$(\xi_1,\dots,\xi_4)(l_1,\dots,l_4)\to(\xi_1 l_1,\dots,\xi_4 l_4). \tag{15}$$

The quotient variety for this action is

$$Y_\phi=\{(l_1^3,\dots,l_4^3):(l_1,\dots,l_4)\in X_\phi\}\subset(\mathrm{Sym}^3(V^*))^4.$$

4.8. THEOREM.

(a) *Y_ϕ is isomorphic to $H^2\setminus D$.*
(b) *The quotient map $X_\phi\to Y_\phi$ is an 81:1 unramified covering.*
(c) *X_ϕ is a smooth affine surface. In particular, $X_\phi\neq\emptyset$.*

Proof. (a) For $(A,B)\in H^2\setminus D$, let

$$\begin{aligned} A_{12}&=A,\\ A_{13}&=B,\\ A_{14}&=\ominus A\ominus B,\\ A_{23}&=A\oplus P,\\ A_{24}&=B\oplus P,\\ A_{34}&=\ominus A\ominus B\oplus P \end{aligned} \tag{16}$$

Then (12) gives an isomorphism $H^2\setminus D\to Y_\phi$.

(b) The map is unramified, by Lemma 4.1. Part (c) follows from (b). □

5. Irreducibility of X_ϕ

In this section we prove that X_ϕ is an irreducible surface and that the generic fiber of the map $f_{12}: X_\phi\to H_\phi$ is an irreducible curve.

Let $C \in H_\phi$ and let $c \in V$ be an affine representative of C. For $A, B \in H_\phi$ let $A_{12} = A$, $A_{13} = B$, and A_{14}, A_{23}, A_{24}, and A_{34} be as in (4.4). For $i = 1, \dots, 4$ we define

$$g_i(A,B) = \frac{\omega^3(A_{\alpha_i}, A_{\beta_i}, c)}{\omega(A_{\alpha_i}, A_{\alpha_i}, A_{\beta_i})\,\omega(A_{\alpha_i}, A_{\beta_i}, A_{\beta_i})}. \tag{17}$$

Recall that α_i and β_i are distinct 2-element subsets of $\{1,2,3,4\}\setminus\{i\}$. By Corollary 4.6, the rational function g_i is independent of the way they are chosen. For $C \in H_\phi$, let $X_\phi(C)$ be the set of all $(l_1, \dots, l_4) \in X_\phi$ such that $l_i(C) \neq 0$ for $i = 1, \dots, 4$. The sets $X_\phi(C)$ form an open covering of X_ϕ as C ranges over H_ϕ. Let

$$\begin{aligned} &E_1(C):\ \{(A,B): A = C \oplus P\},\\ &E_2(C):\ \{(A,B): B = C \oplus P\},\\ &E_3(C):\ \{(A,B): A = C\},\\ &E_4(C):\ \{(A,B): B = C\},\\ &E_5(C):\ \{(A,B): A \oplus B \oplus C = O\},\\ &E_6(C):\ \{(A,B): A \oplus B \oplus C = P\}. \end{aligned} \tag{18}$$

5.1. LEMMA. *Let $D_1, \dots, D_6$ be as in* (14), $D = D_1 \cup \dots \cup D_6$, *and let $Y(C) \subset H_\phi \times H_\phi \times k^4$ be given by*

$$Y(C) = \{(A, B; s_1, \dots, s_4) : (A,B) \notin D \cup E_1(C) \cup \dots \cup E_6(C) \text{ and } s_i^3 = g_i(A,B) \text{ for } i = 1, \dots, 4.\}. \tag{19}$$

Then $Y(C)$ and $X_\phi(C)$ are isomorphic as coverings of H_ϕ^2; that is, there exists an isomorphism between them which makes the diagram

$$\begin{array}{ccc} X_\phi(C) & \simeq & Y(C) \\ {\scriptstyle (f_{12}, f_{13})} \searrow & & \swarrow {\scriptstyle \mathrm{pr}} \\ & H^2 & \end{array}$$

commutative. Here pr *is the projection* $(A, B, s_i) \to (A, B)$.

Proof. The isomorphism is given by the following mutually inverse maps:

$$X_\phi \leftrightarrow Y;$$
$$x = (l_1, \dots, l_4) \to (f_{12}(x), f_{13}(x); l_1(c), \dots, l_4(c));$$
$$\left(l_i(\cdot) = \frac{s_i \omega(A_{\alpha_i}, A_{\beta_i}, \cdot)}{\omega(A_{\alpha_i}, A_{\beta_i}, c)} \right)_{i=1,\dots,4} \leftarrow (A, B, s_1, \dots, s_4).$$

The condition $(A,B) \notin E_1(C), \dots, E_6(C)$ ensures that the map $Y \to X_\phi$ is well-defined. □

5.2. THEOREM. *X_ϕ is irreducible.*

Proof. Since the sets $X(C)$ form an open cover of X_ϕ and any two $X(C)$ intersect nontrivially, it is enough to show that $X(C)$ is irreducible for every

$C \in H$. In view of Lemma 5.1, it will suffice to prove that $Y(C)$ is irreducible for every $C \in H$.

5.3. LEMMA. *Let $g_1, \ldots, g_4$ be as in* (17). *Then*

$$(20)\qquad \begin{aligned} \operatorname{div} g_1 &= 3E_3(C)+3E_4(C)+3E_5(C)-D_1-D_5-D_6,\\ \operatorname{div} g_2 &= 3E_2(C)+3E_3(C)+3E_6(C)-D_2-D_4-D_6,\\ \operatorname{div} g_3 &= 3E_1(C)+3E_4(C)+3E_6(C)-D_3-D_4-D_5,\\ \operatorname{div} g_4 &= 3E_1(C)+3E_2(C)+3E_5(C)-D_1-D_2-D_3. \end{aligned}$$

Proof. Recall that

$$g_1(A,B)=\frac{\omega^3(A\oplus P, B\oplus P, c)}{\omega(A\oplus P, A\oplus P, B\oplus P)\,\omega(A\oplus P, B\oplus P, B\oplus P)},$$
$$g_2(A,B)=\frac{\omega^3(A\oplus P, B, c)}{\omega(A\oplus P, A\oplus P, B)\,\omega(A\oplus P, B, B)},$$
$$g_3(A,B)=\frac{\omega^3(A, B\oplus P, c)}{\omega(A, A, B\oplus P)\,\omega(A, B\oplus P, B\oplus P)},$$
$$g_4(A,B)=\frac{\omega^3(A, B, c)}{\omega(A, A, B)\,\omega(A, B, B)}.$$

Formulas (20) now follow from Lemma 2.8. □

We can now easily verify that the divisors of $g_1, \ldots, g_4$ are $(\mathbf{Z}/3\mathbf{Z})$-linearly independent in $\operatorname{div}(H^2)/3\operatorname{div}(H^2)$ and hence in $\operatorname{PDiv}(H^2)/3\operatorname{PDiv}(H^2)$. Therefore, X_ϕ is irreducible by Theorem 3.1. □

A similar argument yields the following stronger statement.

5.4. THEOREM. *Assume that neither A_0 nor $i(A_0)$ is an inflection point of H. Then the fiber $f_{12}^{-1}(A_0)$ of the map $f_{12}: X_\phi \to H$ is an irreducible curve.*

We shall need the following facts about the relative position of the six curves D_i.

5.5. LEMMA. *There are exactly* 36 *points $Q_1, \ldots, Q_{36}$ in H^2 which lie on more than one D_i. Each of them belongs to one of the following disjoint sets of* 9 *points:*

$$\begin{aligned} D_1\cap D_2\cap D_6 &= \{(A,B): B=A\oplus P;\ 3A=P\},\\ D_1\cap D_3\cap D_5 &= \{(A,B): B=A\oplus P;\ 3A=O\},\\ D_2\cap D_3\cap D_4 &= \{(A,B): B=A;\ 3A=O\},\\ D_4\cap D_5\cap D_6 &= \{(A,B): B=A;\ 3A=P\}. \end{aligned}$$

Proof. The lemma is an immediate consequence of the definition (14) of the divisors $D_1, \ldots, D_6$. □

5.6. LEMMA. *The curves* $D_1, \dots, D_6$ *intersect* $\{A_0\} \times H$ *in* 1, 1, 4, 1, 1, *and* 4 *points respectively. All of these intersections are transversal. If* $3A_0 \neq O$ *or* P *then these* 12 *intersection points are distinct. If* $3A_0 = O$ *or* P *then exactly* 8 *of them are distinct.*

Proof. We have

$$\begin{aligned}
D_1 \cap \{A_0\} \times H &= \{(A_0, A_0 \oplus P)\}, \\
D_2 \cap \{A_0\} \times H &= \{(A_0, \ominus 2A_0)\}, \\
D_3 \cap \{A_0\} \times H &= \{(A_0, B) : 2B = \ominus A_0\}, \\
D_4 \cap \{A_0\} \times H &= \{(A_0, A_0)\}, \\
D_5 \cap \{A_0\} \times H &= \{(A_0, P \ominus 2A_0\}), \\
D_6 \cap \{A_0\} \times H &= \{(A_0, B) : 2B = P \ominus A_0\}.
\end{aligned}$$

Transversality is checked directly. □

Proof of Theorem 5.4. It is enough to show that $f_{12}^{-1}(A) \cap X(C)$ is irreducible for every $C \in H \setminus \{A_0, A_0 \oplus P\}$. For $i = 1, \dots, 4$ let $g_i(B)$ be the function g_i defined in (17), viewed as a function of $B \in H$ with $A = A_0$. Then the isomorphism of Lemma 5.1 identifies $f_{12}^{-1}(A_0) \cap X(C)$ with the curve $Y_{A_0}(C) \subset H \times k^4$ given by

$$Y_{A_0}(C) = \{(B; s_1, \dots, s_4) : (A_0, B) \notin D \cup E_1(C) \cup \cdots \cup E_6(C) \text{ and } s_i^3 = g_i(A_0, B) \text{ for } i = 1, \dots, 4.\}. \tag{21}$$

The irreducibility of $Y_{A_0}(C)$ will thus follow from Theorem 3.1 if we can prove that the divisors of the four rational functions $g_1, \dots, g_4$ on $\{A_0\} \times H$ are linearly independent modulo 3. Since $C \neq A_0$, $A_0 \oplus P$, the divisors $E_1(C)$ and $E_3(C)$ do not intersect $\{A_0\} \times H$. On the other hand, each of the divisors $E_2(C)$, $E_4(C)$, $E_5(C)$, and $E_6(C)$ intersects $\{A_0\} \times H$ transversely in a single point. We shall denote these points by Q_2, Q_4, Q_5, and Q_6 respectively. By Lemma 5.6, each of the divisors D_1, D_2, D_4, and D_5 also intersects $\{A_0\} \times H$ transversely in a single point. These points will be denoted by R_1, R_2, R_4, and R_5 respectively. Each of the divisors D_3, D_6 intersects $\{A_0\} \times H$ transversely in four points. They will be denoted by R_3^j and R_6^h, respectively, where j and h range from 1 to 4. By Lemma 5.3 we have

$$\begin{aligned}
\operatorname{div} g_1(B) &= 3(Q_4 + Q_5) - R_1 - R_5 - \textstyle\sum_{h=1}^4 R_6^h, \\
\operatorname{div} g_2(B) &= 3(Q_2 + Q_6) - R_2 - R_4 - \textstyle\sum_{h=1}^4 R_6^h, \\
\operatorname{div} g_3(B) &= 3(Q_4 + Q_6) - \textstyle\sum_{j=1}^4 R_3^j - R_4 - R_5, \\
\operatorname{div} g_4(B) &= 3(Q_2 + Q_5) - R_1 - R_2 - \textstyle\sum_{j=1}^4 R_3^j.
\end{aligned} \tag{22}$$

By Lemma 5.6, the twelve points $R_1, R_2, R_3^j, R_4, R_5, R_6^h$ are distinct. Hence, the four divisors in (22) are linearly independent modulo 3. □

REMARK. If $3A_0 = O$ or P then $f_{12}^{-1}(A_0)$ is reducible. Suppose $3A_0 = P$. Then, by Lemma 5.5, $R_1 = R_2$, $R_4 = R_5$, and the first two divisors in (22) become identical. By Theorem 3.1, $f^{-1}(A_0)\cap X(C)$ is reducible; hence, so is $f^{-1}(A_0)$. Similarly, if $3A_0=O$ then $R_1=R_5$, $R_2=R_4$, the last two divisors in (22) become identical, and we can again apply Theorem 3.1.

6. A Compactification of X_ϕ

Recall that by Lemma 4.8 the map

$$\begin{aligned} & X_\phi \to H^2\backslash D \\ & p=(l_1,\dots,l_4)\to (f_{12}(p), f_{13}(p)) \end{aligned} \tag{23}$$

is an unramified 81 : 1 covering.

In this section we shall construct a normal projective surface $\tilde{X}_\phi^n$ and a map $\pi_1\colon \tilde{X}_\phi^n\to H^2$ such that the diagram

$$\begin{array}{ccc} X_\phi & \hookrightarrow & \tilde{X}_\phi^n \\ \downarrow & & \downarrow \pi_1 \\ H^2\backslash D & \hookrightarrow & H^2 \end{array}$$

is commutative. We shall also give a description of the singularities of $\tilde{X}_\phi^n$ which will be used to construct and study a smooth projective model for X_ϕ in the next two sections.

For $(A,B)\in H^2$, the points A_{ij} will always be given by (16). We now define

$$\tilde{X}_\phi\subset H^2\times \mathbf{P}(V^*\times k)^4$$

as the set of all $(A,B)\times(l_1:d_1)\times\cdots\times(l_4:d_4)$ satisfying

$$\omega(A_{\alpha_i},A_{\alpha_i},A_{\beta_i})\,\omega(A_{\alpha_i},A_{\beta_i},A_{\beta_i})\,l_i^3(\cdot)=d_i\,\omega^3(A_{\alpha_i},A_{\beta_i},\cdot) \tag{24}$$

for every choice of distinct 2-element subsets α_i and $\beta_i\in\{1,2,3,4\}\backslash\{i\}$. We denote the projection $\tilde{X}_\phi\to H^2$ by π_0.

6.1. LEMMA. *For any $(A,B)\times(l_1:d_1)\times\cdots\times(l_4:d_4)\in\tilde{X}_\phi$ and any $i=1,\dots,$ 4, we have $l_i\not\equiv 0$.*

Proof. By Lemma 2.4, the three points $A_{a_ib_i}$, $A_{a_ic_i}$ and $A_{b_ic_i}$ cannot be pairwise corresponding. Hence, $\omega^3(A_{\alpha_i},A_{\beta_i},\cdot)\not\equiv 0$ for some choice of 2-element subsets α_i and β_i of $\{a_i,b_i,c_i\}$. □

Thus we can cover $\tilde{X}_\phi$ by open subsets $\tilde{X}_\phi(C)$ given by $l_i(C)\neq 0$ for $i=1,\dots,4$. Here $C\in H$. If $c\in V$ is an affine representative of C, then $\tilde{X}_\phi(C)$ is isomorphic to the subset of $H^2\times A^4$ given by

$$\begin{aligned} s_1^3&=\frac{\omega(A\oplus P,A\oplus P,B\oplus P)\,\omega(A\oplus P,B\oplus P,B\oplus P)}{\omega^3(A\oplus P,B\oplus P,c)},\\ s_2^3&=\frac{\omega(A\oplus P,A\oplus P,B)\,\omega(A\oplus P,B,B)}{\omega^3(A\oplus P,B,c)},\\ s_3^3&=\frac{\omega(A,A,B\oplus P)\,\omega(A,B\oplus P,B\oplus P)}{\omega^3(A,B\oplus P,c)}, \end{aligned} \tag{25}$$

$$s_4^3 = \frac{\omega(A,A,B)\,\omega(A,B,B)}{\omega^3(A,B,c)}.$$

Here the isomorphism is given by $s_i = d_i/l_i(c)$ for $i = 1, \dots, 4$.

6.2. PROPOSITION.

(a) *$\tilde{X}_\phi$ is the closure of the graph of* (24) *in* $H^2 \times \mathbf{P}(V^* \times k)^4$.
(b) *$\tilde{X}_\phi$ is irreducible.*

Proof. (a) By Corollary 4.6, $\tilde{X}_\phi \setminus \pi_0^{-1}(D)$ is precisely the graph of the map (24). Denote the closure of this set by $\mathrm{cl}(X_\phi)$. Note that the action of $(\mathbf{Z}/3\mathbf{Z})^4$ on $\tilde{X}_\phi$ given by

$$(26) \qquad (\xi_1, \dots, \xi_4) \times (A, B, (l_i : d_i)) \to (A, B, (\xi_i l_i : d_i))$$

preserves $\mathrm{cl}(X_\phi)$. Thus in order to prove part (a), it is enough to show that the $(\mathbf{Z}/3\mathbf{Z})^4$-orbit of every $x \in \tilde{X}_\phi$ intersects $\mathrm{cl}(X_\phi)$. Since $\pi_0(\mathrm{cl}(X_\phi)) = H^2$, this is a consequence of the following lemma.

6.3. LEMMA. *The action* (26) *is transitive on the fibers of* π_0.

Proof. Let $x \in \tilde{X}_\phi(C)$ and $\pi_0(x) = (A, B)$. Then, in the coordinates of (25), the action of $(\mathbf{Z}/3\mathbf{Z})^4$ on $\pi_0^{-1}(\pi_0 x)$ is given by $(A, B, s_i) \to (A, B, \xi_i s_i)$. □

Part (b) of the Proposition follows from part (a) and Theorem 5.2. □

The variety $\tilde{X}_\phi$ we have constructed is thus irreducible, projective, and contains X_ϕ as an open subset. However, it is not smooth and not even normal.

Let π_1 be the composition of the normalization map

$$n\colon \tilde{X}_\phi^n \to \tilde{X}_\phi$$

with π_0.

6.4. LEMMA. *If $Q \neq H^2$ lies on D_i for some $i = 1, \dots, 6$ but not on D_j for $j \neq i$, then the fiber $\pi_1^{-1}(Q)$ of Q in $\tilde{X}_\phi^n$ consists of 27 points. Each of these points is smooth in $\tilde{X}_\phi^n$. If R is one of these points then we can choose $z, w \in \hat{\mathcal{O}}_R(\tilde{X}_\phi^n)$ and $x, y \in \hat{\mathcal{O}}_Q(H^2)$ such that*

(a) $\hat{\mathcal{O}}_Q(H^2) = k[[x, y]]$ *and* $\hat{\mathcal{O}}_R(\tilde{X}_\phi^n) = k[[z, w]]$;
(b) *x is the local equation of D_i*;
(c) *π_1 is given by* $(z, w) \to (x = z^3, y = w)$.

Proof. We shall prove the lemma for $i = 1$; the same argument will work for other i. Choose $C \in H$ so that $Q \in \tilde{X}_\phi(C)$. By (25), $\tilde{X}_\phi(C)$ is given by $s_i^3 = g_i^{-1}(A, B)$, where g_i are as in (17). Let $x \in \mathcal{O}_x(H^2)$ be the local equation of D_1. By Lemma 5.3 this means that over a neighborhood of Q in H^2, $\tilde{X}_\phi$ is given by

$$s_1^3 = e_1 x, \quad s_2^3 = e_2, \quad s_3^3 = e_3, \quad s_4^3 = e_4 x,$$

where e_i is an invertible element of $\mathcal{O}_Q(\tilde{X}_\phi)$ for $i=1,\dots,4$. The normalization of $\tilde{X}_\phi$ over this open neighborhood is given by

$$s_1^3=e_1x,\quad s_2^3=e_2,\quad s_3^3=e_3,\quad t_4^3=e_4e_1^{-1},$$

where $t_4=s_4/s_1$ and $e_4e_1^{-1}$ is again an invertible element of $\mathcal{O}_Q(H^2)$. This shows that there are 27 points in $\pi^{-1}(Q)$. The completion of the local ring of $R\in\pi^{-1}(Q)$ is then given by

$$k[[x,y,s_1]]/(s_1^3=e_1x)=k[[y,z]]/(z^3=x)$$

where z is a unit multiple of s_1. □

Next we investigate the normalization $\tilde{X}_\phi^n$ near $\pi_1^{-1}(Q)$ where Q lies on more than one curve D_i.

6.5. Lemma. *Let Q be one of the 36 points lying on more than one D_i. Then the fiber $\pi_1^{-1}(Q)$ of Q in $\tilde{X}_\phi^n$ consists of 3 points. If R is one of these points then we can choose $u_1,u_2,u_3\in\hat{\mathcal{O}}_R(\tilde{X}_\phi^n)$ and $x,y\in\hat{\mathcal{O}}_Q(H^2)$ such that*

(a) *$\hat{\mathcal{O}}_Q(H^2)=k[[x,y]]$ and x, y, and $x+y$ are the local equations of the three curves D_i passing through Q.*

(b) *$\hat{\mathcal{O}}_R(\tilde{X}_\phi^n)=k[[u_1,u_2,u_3]]/(u_1^3+u_2^3+u_3^3)$.*

(c) *the map π_1 is given by $(u_1,u_2,u_3)\to(x=u_2^3,\ y=u_3^3)$.*

Proof. (a) Q is one of the 36 points in Lemma 5.5. We shall assume that $Q=(O,O)\in D_2\cap D_3\cap D_4$; the same argument will work for any other choice of Q. Choose $C\in H$ so that $Q\in\tilde{X}_\phi(C)$. By (25), $\tilde{X}_\phi(C)$ is given by $s_i^3=g_i^{-1}(A,B)$, where g_i are as in (17). Let f_2, f_3, and $f_4\in\mathcal{O}_Q(H^2)$ be local equations for D_3, D_2, and D_4 respectively. (Transposing D_2 and D_3 at this point simplifies our notation in the rest of the argument.)

By Lemma 5.3 this means that, over a neighborhood of Q in H^2, $\tilde{X}_\phi$ is given by

$$s_1^3=e_1,\quad s_2^3=e_2f_3f_4,\quad s_3^3=e_3f_2f_4,\quad s_4^3=e_4f_2f_3,$$

where each e_i is an invertible element of $\mathcal{O}_Q(\tilde{X}_\phi)$. Let U be an open neighborhood of Q such that each e_1 is invertible in $k[U]$. For $i=2,3,4$ let $t_i=s_if_i/s_js_h$. Here (i,j,h) is a permutation of 2, 3, 4. Then

$$t_2^3=\tilde{e}_2f_2,\quad t_3^3=\tilde{e}_3f_3,\quad t_4^3=\tilde{e}_4f_4,\tag{27}$$

where $\tilde{e}_i=e_i/e_je_h$ is again an invertible element of $k[U]$. Note that

$$k[\pi_1^{-1}(U)]=k[U][s_1,s_2,s_3,s_4]\subset k[U][s_1,t_2,t_3,t_4]$$

because $e_it_jt_h=s_i$.

We claim that $k[U][s_1,t_2,t_3,t_4]$ is the coordinate ring of $\pi_1^{-1}(U)$. There are exactly three closed points of $\operatorname{Spec}k[U][s_1,t_2,t_3,t_4]$ which lie over Q; they correspond to the three cube roots of $e_1(Q)$. The completion of the local ring of any of these points is isomorphic to $\hat{\mathcal{O}}_Q(H^2)[[t_2,t_3,t_4]]$. Hence, it is enough to show that this ring is normal. Since the curves D_2, D_3, and

D_4 meet at Q transversely, we can find units δ_2, δ_3 and δ_4 in $\hat{\mathcal{O}}_Q(H^2)$ such that $f_2\delta_2+f_3\delta_3+f_4=0$. Denote $\delta_2 f_2$ by x and $\delta_3 f_3$ by y. Then, for some unit multiples u_1, u_2, u_3 of t_2, t_3, t_4, we have

$$\begin{aligned}\hat{\mathcal{O}}_Q(H_2)[[t_2,t_3,t_4]] &= k[[x,y,u_1,u_2,u_3]]/(u_1^3=x;\ u_2^3=y;\ u_3^3=x+y)\\ &= k[[u_1,u_2,u_3]]/(u_1^3+u_2^3+u_3^3).\end{aligned}$$

The latter ring is normal. This proves parts (b) and (c) of the lemma. □

7. A Smooth Compactification of X_ϕ

In this section we shall construct and study a smooth compactification $\bar{X}_\phi$ of X_ϕ. We also use this compactification to calculate the genus of the generic fiber of the map $f_{12}: X_\phi \to H$.

We construct X_ϕ by resolving the singularities of $\tilde{X}^n_\phi$. Lemmas 6.4 and 6.5 say that the only singularities of $\tilde{X}^n_\phi$ are the 108 points $P_1, \dots, P_{108}$ which project to the points $Q_1, \dots, Q_{36}$ in H^2. Let $\mathrm{bl}: \bar{X}_\phi \to \tilde{X}^n_\phi$ be the blow-up at $P_1, \dots, P_{108}$. We thus have a tower of surfaces

$$\text{(28)}\qquad \begin{array}{l} \bar{X}_\phi \\ \downarrow \mathrm{bl} \\ \tilde{X}^n_\phi \\ \downarrow n \\ \tilde{X}_\phi \\ \downarrow \pi_0 \\ H^2. \end{array} \quad \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} \pi_1$$

Denote the composite map $\tilde{X}_\phi \to H^2$ by π.

7.1. PROPOSITION.

(a) *$\bar{X}_\phi$ is a smooth surface.*

(b) *The exceptional divisor of* bl *is the union of* 108 *curves C_i such that* $\mathrm{bl}(C_i)=\{P_i\}$. *Each C_i is an elliptic curve with $j(C_i)=0$ and $(C_j, C_j)=-3$.*

(c) *The image of any map $f: \mathbf{P}^1 \to \bar{X}_\phi$ is a point. In particular, $\bar{X}_\phi$ is minimal; that is, it has no exceptional curves of the first kind.*

Proof. Parts (a) and (b) follow from our description of the singularity at P_i in Lemma 6.5. Indeed, let S be the cone $u_1^3+u_3^3+u_3^3=0$ in $\mathbf{A}^3$ at the origin and let $f: T \to S$ be the blow-up of S at the origin. The exceptional curve $E=f^{-1}(0)$ is isomorphic to the cubic curve $u_1^3+u_3^3+u_3^3=0$ in $\mathbf{P}^2$. We want to show that $(E,E)=-3$.

If H in a hyperplane section in S then $(H,H)=3$. Choose distinct hyperplane sections H_1 and H_2 so that they pass through the origin, say $u_1=0$ and $u_2=0$. For $i=1,2$ let H_i' be the strict transform of H_i in T. Then $f^*(H_i)=H_i'+E$. Hence,

$$3=(H_1'+E,H_2'+E)=(H_1',H_2')+(H_1',E)+(H_2',E)+(E,E)$$
$$=0+6+(E,E),$$

as desired.

(c) Composing f with π we obtain a map $\mathbf{P}^1\to H^2$. Since H is an elliptic curve, the image of this map is a point $Q\in H^2$. In other words, $f(\mathbf{P}^1)\subset \pi^{-1}(Q)$. By part (b), $\pi^{-1}(Q)$ is finite or a disjoint union of three elliptic curves. In either case, $f(\mathbf{P}^1)$ is a point. □

Let D_i' be the strict transform of D_i, that is, the closure of

$$\pi^{-1}(D_i\setminus\{Q_1,\dots,Q_{36}\})$$

in $\bar{X}_\phi$.

7.2. LEMMA.

(a) *D_i' is a smooth curve.*
(b) $\pi^*(D_i)=3D_i'+3\sum_{\tau(C_j)\in D_i}C_j$ *in* $\operatorname{div}(\bar{X}_\phi)$.
(c) *If $P_j\in C_i$ then $(D_i',C_j)=3$; otherwise $(D_i',C_j)=0$.*

Proof. Let R be a point of D_i' away from the exceptional curves. Then, by Lemma 6.4, there are formal coordinates z,w near R such that π^*(local equation for D_i) $=z^3$. Here we are identifying $\bar{X}_\phi$ with $\tilde{X}_\phi^n$ since bl is an isomorphism away from the exceptional locus. This shows that D_i is smooth at R and that D_i' enters in $\pi^*(D_i)$ with coefficient 3.

Now let R be one of the points $P_1,\dots,P_{108}$. Let x be the local equation of D_i near C_j and let u_1,u_2,u_3 be as in Lemma 6.5. Then locally $\bar{X}_\phi$ is given by

$$(u_1')^3+(u_2')^3+1=0,$$

where $u_1'=u_1/u_3$, $u_2'=u_2/u_3$, and $u_3'=u_3$. The local equation of C_j in this coordinate system is given by $u_3'=0$. The local equation of $\pi^*(D_i)$ is $x=0$ or, equivalently, $u_2^3=0$ or $u_3^3(u_2')^3=0$. This, in turn, can be rewritten as $(u_3')^3(1+(u_1')^3)^3=0$. The local equation of D_i' is thus $u_2'=0$; D_i' intersects C_j at the three points $(\mu_i,0,0)$, where μ_1,μ_2,μ_3 are the three cube roots of -1. Moreover, in this coordinate system, D_i' is given as the union of the three lines $\{u_2'=0;\ u_3'=\mu_i\}$ for $i=1,2,3$. Hence D_i' is smooth at each of its three intersection points with C_j, and the intersection is transversal at these points. □

7.3. LEMMA. *The canonical divisor K of $\bar{X}_\phi$ is given by*

$$2\sum_{i=1}^{6}(D_i)'+5\sum_{j=1}^{108}C_j.$$

Proof. Let Ω be a nonvanishing regular differential form on H^2. Then $K\equiv \operatorname{div}\pi^*(\Omega)$. Since ϕ is an unramified covering over $H^2\setminus D$, $\operatorname{div}\pi^*(\Omega)$ is supported on

$$D_1'\cup\cdots\cup D_6'\cup C_1\cup\cdots\cup C_{108}.$$

To compute the coefficient of D_i' in div($\pi^*(\Omega)$), we use the coordinate system of Lemma 6.4 near a point $R \in D_i'$ away from the exceptional divisors. We have

$$\pi^*(dx \wedge dy) = d(z^3) \wedge dw = 3z^2 dz \wedge dw.$$

Recall that z is a local equation of D_i. Hence, D_i' enters in $\pi^*(\Omega)$ with coefficient 2.

On the other hand, let R be a point of C_j for some $j = 1, \dots, 108$ away from all D_i'. Let u_1, u_2, and u_3 be as in Lemma 6.5. We may assume that a formal neighborhood of R is given by

$$\{(u_1', u_2', u_3') : (u_1')^3 + (u_2')^3 + 1 = 0\}.$$

Here $u_1' = u_1/u_3$, $u_2' = u_2/u_1$, and $u_3' = u_3$ as in the proof of Lemma 7.2. In particular, u_2' and u_3' are formal coordinates near R. Then, by Lemma 6.5,

$$\pi^*(dx \wedge dy) = d(u_2^3) \wedge d(u_3^3) = d(u_3' u_2')^3 \wedge d(u_3')^3 = 9(u_3')^5 (u_2')^2 du_2' \wedge du_3'.$$

Recall that in this coordinate system u_3' is a local equation for C_j. Since $R \notin D_i'$, we have $u_2'(R) \neq 0$. Hence, C_j enters in $\pi^*(\Omega)$ with coefficient 5. □

REMARK. One can use the adjunction formula to obtain the coefficient 5 in Lemma 7.3. Suppose

$$K \equiv 2 \sum_{i=1}^{6} D_i' + \sum_{j=1}^{108} N_j C_j.$$

The adjunction formula says that $1 = g(C_h) = 1 + (C_h, C_h + K)/2$. By Lemma 7.1(b), $(C_h, C_h) = -3$. Thus we must have $(C_h, K) = 3$; that is,

$$2 \sum_{i=1}^{6} (C_1, D_i') + \sum_{j=1}^{108} N_j (C_h, C_j) = 3.$$

By Lemma 7.2(c), three terms in the first sum are equal to 3, and the other three are equal to zero. In the second sum, the hth term is $-3N_h$, and the other 107 terms are equal to zero. We thus obtain

$$2 \cdot 9 + (-3) N_h = 3;$$

that is, $N_h = 5$.

The adjunction formula can also be used to calculate the genus of the generic fiber of the map $f_{12} : X_\phi \to H$ defined in the beginning of Section 4. Recall that by Theorem 5.4 $f_{12}^{-1}(A_0)$ is irreducible, provided that neither A_0 nor $i(A_0)$ is an inflection point of H. By our construction $f_{12}^{-1}(A_0)$ is an open subset in $F_{A_0} = \pi^{-1}(\{A_0\} \times H)$.

7.4. THEOREM. *Assume that neither $A_0 \in H$ nor $i(A_0)$ is an inflection point of H. Then F_{A_0} is a smooth irreducible projective curve of genus* 325.

Proof. By our assumption on A_0, the curve F_{A_0} does not intersect the exceptional curves $C_1, \dots, C_{108}$. Hence, bl maps F_{A_0} isomorphically to

$$\pi_1^{-1}(\{A_0 \times H\}) \subset \tilde{X}_\phi^n.$$

Thus we only need to prove smoothness and irreducibility for this curve. Denote it by F.

Irreducibility: We argue as in the proof of Proposition 6.2. Since

$$F^* = F \setminus \pi_1^{-1}(D)$$

is isomorphic to $f_{12}^{-1}(A_0)$, it is irreducible by Theorem 5.4. Denote the closure of F^* in F by $\mathrm{cl}(F^*)$. It is sufficient to show that $\mathrm{cl}(F^*) = F$. The action (26) of the group $(\mathbf{Z}/3\mathbf{Z})^4$ on $\tilde{X}_\phi$ induces an action on $\tilde{X}_\phi^n$ which preserves F and F^*. Since $\pi_1(\mathrm{cl}\,F^*) = \{A_0\} \times H = \pi_1(F)$, we only need to show that $(\mathbf{Z}/3\mathbf{Z})^4$ acts transitively on every fiber of π_1. By Lemma 6.3, $(\mathbf{Z}/3\mathbf{Z})^4$ acts transitively on the fibers of π_0. This means that it acts transitively on almost every fiber of π_1. In other words, π_1 induces a finite map $\tilde{X}_\phi^n/(\mathbf{Z}/3\mathbf{Z})^4 \to H^2$ which is one-to-one almost everywhere. Such a map must be one-to-one everywhere. Thus $(\mathbf{Z}/3\mathbf{Z})^4$ acts transitively on every fiber of π_1, as desired.

Smoothness: Since F^* is an unramified 81:1 cover of $\{A_0\} \times H \setminus D$, it is smooth. Let R be a point of F lying over a point $Q \in D_i$. By our choice of A_0, no other D_j passes through Q. By Lemma 6.4, we can find formal coordinates x, y near Q and z, w near R such that π_1 is given by $(z, w) \to (x = z^3, y = w)$. Since D_i intersects $\{A_0\} \times H$ transversely at R, we may also assume that y is a defining equation for $\{A_0\} \times H$ in this coordinate system. Then C is cut out by w near R, and hence is smooth at R.

We now proceed to calculate the genus of F_{A_0}. By the adjunction formula the genus is given by

$$g(F_{A_0}) = 1 + (F_{A_0}, F_{A_0} + K)/2.$$

Let A_1 be another point of H; then

$$(F_{A_0}, F_{A_0}) = (F_{A_0}, F_{A_1}) = 0.$$

On the other hand,

$$(F_{A_0}, K) = 2\sum_{i=1}^{6} (F_{A_0}, D_i') + 5 \sum_{j=1} (F_{A_0}, C_j).$$

Each term in the second sum is zero, since F_{A_0} does not intersect any exceptional curve C_j. Hence,

$$g(F_{A_0}) = 1 + \sum_{i=1}^{6} (F_{A_0}, D_i').$$

Recall that $\{A_0\} \times H$ intersects each curve D_i transversely. The number of points of intersection of $\{A_0\} \times H$ with $D_1, \ldots, D_6$ is 1, 1, 4, 1, 1, and 4 respectively. By Lemma 6.4, each of these points gives rise to 27 points of intersection of F_{A_0} with the appropriate D_i'; each intersection will be transversal. This yields $g(F_{A_0}) = 1 + (27 \cdot 12) = 325$. □

8. Numerical Invariants

In this section we calculate the Chern numbers c_1^2 and c_2 of the surface $\bar{X}_\phi$.

8.1. Theorem. $c_1^2(\bar{X}_\phi) = (K, K) = 7{,}452.$

Proof. By Lemma 7.2(b), we have

$$\sum_{i=1}^{6} \pi^* D_i = 3 \sum_{i=1}^{6}\Big(D_i' + \sum_{\pi(C_j)\in D_i} C_j\Big) = 3\Big(\sum_{i=1}^{6} D_i' + \sum_{i=1}^{6}\sum_{\pi(C_j)\in D_i} C_j\Big).$$

Since there are three D_i passing through each Q_h (see Lemma 5.5), each C_j appears three times in the double sum above. Thus

$$\sum_{i=1}^{6} \pi^* D_i = 3\sum_{i=1}^{6} D_i' + 9\sum_{j=1}^{108} C_j.$$

By Lemma 7.3 we have

$$3K \equiv 2\sum_{i=1}^{6} \pi^* D_i - 3\sum_{j=1}^{108} C_j.$$

Let $Y = \pi^* \sum_{i=1}^{6} D_i$ and $Z = \sum_{j=1}^{108} C_j$. Note that

$$D_1 \equiv D_4, \quad D_2 \equiv D_5, \quad D_3 \equiv D_6.$$

Thus $(Y,Y) = 81\cdot 4(D_1+D_2+D_3)^2$. Since $(D_i, D_i) = 0$ and

$$(D_1, D_2) = (D_1, D_3) = (D_2, D_3) = 9,$$

we have $(Y,Y) = 81\cdot 4\cdot 54 = 17{,}496$.

On the other hand, by Lemma 7.1(b),

$$(Z,Z) = \Big(\sum_{j=1}^{108} C_j, \sum_{h=1}^{108} C_h\Big) = 108\cdot(-3) = -324.$$

For any $i = 1, \dots, 6$ and $j = 1, \dots, 108$ we have

$$(\pi^* D_i, C_j) = (\pi^* \tilde{D}_i, C_j) = 0.$$

Here $\tilde{D}_i$ is D_i translated by an element of H^2 so that $Q_1, \dots, Q_{36} \notin \tilde{D}_i$. Hence, $(Y,Z) = 0$ and $(K,K) = \frac{1}{9}(4(Y,Y) - 6(Y,Z) + 9(Z,Z)) = 7{,}452$. □

Recall that by Proposition 7.1, $\bar{X}_\phi$ is a minimal surface. Since (K,K) is positive, it is a surface of general type; see [1, Part 3, Thm. 5.4].

In the sequel, χ will denote the Euler characteristic.

8.2. THEOREM. $c_2(\bar{X}_\phi) = \chi(\bar{X}_\phi) = 2{,}916$.

Proof. (a) Our calculation is similar to that in [5, 2.2]. We use the principle that the Euler characteristic behaves as cardinality of sets. Recall that the map $\pi : \bar{X}_\phi \to H^2$ is an 81:1 cover over $H^2 \backslash (D_1 \cup \cdots \cup D_6)$ and a 27:1 cover over $D \backslash \{Q_1, \dots, Q_{36}\}$. The fiber over each of the Q_i is the union of three elliptic curves and hence has Euler characteristic 0. Thus

$$\chi(\bar{X}_\phi) = 81(\chi(H^2) - \chi(D)) + 27(\chi(D) - \chi(\{Q_1, \dots, Q_{36}\})) + 0\cdot\chi(\{Q_1, \dots, Q_{36}\}).$$

Since $\chi(H^2) = 0$, we obtain

$$\chi(\bar{X}_\phi) = -54\chi(D) - 27\cdot 36. \tag{29}$$

It remains to calculate $\chi(D)$. Consider the projection $\mathrm{pr}_1: H^2 \to H$ restricted to D. By Lemma 5.6, the fibers of $A_0 \in H$ consist of 8 points if $3A_0 = O$ or P and 12 points otherwise. Denote by T the set of all $A_0 \in H$ such that $3A_0 = O$ or P. Note that T is a finite subset consisting of 18 points: 9 inflection points of H and their corresponding 9 points. We have

$$\chi(D) = 12(\chi(H) - \chi(T)) + 8\chi(T) = 12(0-18) + 8\cdot 18 = -72.$$

Substituting this into (29), we obtain

$$\chi(\bar{X}_\phi) = -54\cdot(-72) - 27\cdot 36 = 2{,}916. \qquad \square$$

We summarise our results in the following theorem.

8.3. THEOREM.

(a) *$\bar{X}_\phi$ is a minimal surface of general type which contains X_ϕ as a Zariski-open subset.*

(b) *The Chern numbers of $\bar{X}_\phi$ are given by $c_1^2 = 7{,}452$, $c_2 = 2{,}916$.*

Note that the invariant c_1^2/c_2 of the surface $\bar{X}_\phi$ is equal to $2\frac{5}{9}$. The index of the intersection form is given by $i = \frac{1}{3}(c_1^2 - 2c_2) = 540$.

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