Lusin's Condition (N) and Mappings with Nonnegative Jacobians

O. MARTIO & WILLIAM P. ZIEMER*

1. Introduction

Let G be a domain in $\mathbf{R}^n$. A continuous mapping $f: G \to \mathbf{R}^n$ is said to satisfy Lusin's condition (N) if $|f(A)| = 0$ whenever $A \subset G$ and $|A| = 0$. Here, $|A|$ denotes the Lebesgue measure of A. Smooth mappings, say C^1, locally Lipschitzian or even continuous mappings in the Sobolev space $W^{1,p}(G, \mathbf{R}^n)$, $p > n$, satisfy condition (N). However, it is well known that for $p < n$, such mappings fail to satisfy condition (N) (cf. [Po]). For $n = 1$ the condition (N) is well understood (see [Sa]), but for $n \geq 2$ necessary and sufficient analytic conditions for (N) are not so clear. The most important recent studies in this area have been made by Reshetnyak [R3] who proved that a quasiregular mapping $f: G \to \mathbf{R}^n$ satisfies the condition (N). For continuous mappings in $W^{1,n}(G, \mathbf{R}^n)$ he also gave, in [R2], a topological condition that implies condition (N).

Because of the importance of condition (N) in applications, in this paper we investigate how this property is related to mappings in the Sobolev space $W^{1,n}(G, \mathbf{R}^n)$. We focus our attention on this class of mappings because of its application to quasiregular mappings and nonlinear elasticity (cf. [Ba] and [Mu]). We use multiplicity functions, defined in terms of topological degree and related to the topological condition given by Reshetnyak; to characterize those mappings that satisfy condition (N) (see Theorem 3.10). We also introduce Sard's condition (SA): A continuous mapping $f: G \to \mathbf{R}^n$ with partial derivatives a.e. satisfies the condition (SA) if $Jf(x) = 0$ a.e. in an open set $A \subset G$ yields $|f(A)| = 0$. This condition can be regarded as a weak counterpart of the Sard-type result for mappings f with rank $f'(x) < n$: If $f: G \to \mathbf{R}^n$ is C^1 then $|f(A)| = 0$ for $A = \{x \in G: Jf(x) = 0\}$. In general, continuous mappings in $W^{1,n}(G, \mathbf{R}^n)$ that satisfy condition (SA) do not satisfy (N); however, if a continuous map $f \in W^{1,n}(G, \mathbf{R}^n)$ has the property that Jf is of one sign almost everywhere, then we show that conditions (N) and (SA) are equivalent (Theorem 3.12). It is easy to see that a quasiregular mapping satisfies (SA), and hence the condition (N) for quasiregular mappings fol lows from the above result.

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It follows from our results that continuous mappings $f \in W^{1,n}(G, \mathbf{R}^n)$ that satisfy condition (N) are, in some sense, nearly open. It is not known whether continuous, open mappings $f \in W^{1,n}(G, \mathbf{R}^n)$ satisfy condition (N), but we show in Section 4 that this is true if $n=2$. In fact, we show that a mapping f satisfies condition (N) if and only if f is *almost open* (Theorem 4.4).

Our methods exploit the interaction between certain multiplicity functions that are determined by the topological character of the mapping f and its Jacobian Jf, which is analytic in character. Much of this interaction has been developed in the theory of Lebesgue area of surfaces (cf. [F3; F4]). The multiplicity functions are intimately related to the notion of topological degree which is carefully discussed in the classical reference [RR] by Radó and Reichelderfer.

2. Notation and Preliminaries

The open ball in $\mathbf{R}^n$ centered at x with radius r is denoted by $B(x,r)$. The following notion will be central to our development.

2.1. DEFINITION. Suppose X is a metric space and $f: X \to \mathbf{R}^n$ is a continuous mapping. If $C \subset X$ then y is an *unstable* value of $f \mid C$ if, for every $\delta > 0$, there is a continuous mapping $g: C \to \mathbf{R}^n$ satisfying $|f(x)-g(x)| < \delta$ for each $x \in C$ and $g(C) \subset \mathbf{R}^n - y$. A point y is called *stable* if it is not unstable. We denote by

$$S(f, X, y) \tag{2.1}$$

the supremum of the set of all nonnegative integers m with the property that X contains m disjoint compact sets C such that y is a stable value of $f \mid C$.

2.2. REMARK. One can easily verify that in case $n=1$ and C is a connected subset of X, y is then a stable value of $f \mid C$ if and only if y is an interior point of the interval $f(C)$. Moreover, in this case, $S(f, X, y)$ is the supremum of the set of all nonnegative integers m with the property that X contains m disjoint continua C such that y is an interior point of the interval $f(C)$.

We also introduce two more multiplicity functions. The first is

$$N(f, X, y) \tag{2.2}$$

which is the number (possibly infinite) of points of $f^{-1}(y) \cap X$. In the next definition, we assume that X is an open, connected subset G of $\mathbf{R}^n$. Assuming that $f: G \to \mathbf{R}^n$ is continuous, the components V of $f^{-1}[B(y,r)]$ are open, connected subsets of $\mathbf{R}^n$. If $\bar{V}$ is compact in G, then we consider the induced homomorphism

$$f^*: H^n(\mathbf{R}^n, \mathbf{R}^n - B(y,r)) \to H^n(\bar{V}, \partial V)$$

of the n-dimensional Čech cohomology groups with integer coefficients. The groups are infinite cyclic [ES, p. 312 and Thm. 6.8(iv), p. 315]. Thus, f^* maps a generator of the first group onto the integral multiple, call it $d(f,r,V)$, of the second group; in the terminology of Radó and Reichelderfer [RR, p. 123], $d(f,r,V)=\mu(y,f,V)$. Let $\mathfrak{F}(r)$ be the family of all components $V \subset\subset G$ of $f^{-1}[B(y,r)]$. As in [F3, p. 327], we define

$$M(f,G,y)=\lim_{r\to 0}\sum_{V\in\mathfrak{F}(r)}|d(f,r,V)|.$$

Recall from properties of topological degree that the sum of the right of the above expression is a nonincreasing function of r. Also recall that if for some $r>0$ there are k distinct components of $f^{-1}(B(y,r))$, say $V_1, V_2, \ldots, V_k$, with the property that $d(f,r,V_i)\neq 0$, $i=1,2,\ldots,k$, then

$$S(f,G,y)\geq k. \tag{2.3}$$

Another important fact concerning these multiplicity functions is

$$S(f,G,y)\leq M(f,G,y) \tag{2.4}$$

for all $y\in\mathbf{R}^n$ [DF, Thm. 3.11]. We recall a fact that follows from the properties of topological degree:

$$\text{if } M(f,G,y)\neq 0 \text{ then } y\in \text{interior } f(G). \tag{2.5}$$

The importance of the multiplicity function $M(f,G,y)$ is the role it plays in Lebesgue area. If $X\subset\mathbf{R}^n$ is a finitely triangulable set (such as the closure of a smoothly bounded domain), the Lebesgue area of a continuous mapping $f: X\to\mathbf{R}^n$ is defined by

$$\mathfrak{L}(f,X)=\inf\left\{\liminf_{i\to\infty}\int_X |Jf_i(x)|\,dx\right\},$$

where the infimum is taken over all Lipschitz maps $f_i: X\to\mathbf{R}^n$ that converge uniformly to f on X. A fundamental result due to Federer (see [F1, §4] or [F4, Cor. 7.9]) states that

$$\mathfrak{L}(f,X)=\int_{\mathbf{R}^n} M(f,X,y)\,dy. \tag{2.6}$$

The assumption that X is finitely triangulable can be removed by defining the Lebesgue area of $f: G\to\mathbf{R}^n$, where $G\subset\mathbf{R}^n$ is an open set, by

$$\mathfrak{L}(f,G)=\sup\mathfrak{L}(f,X)$$

where the supremum is taken over all finitely triangulable subsets X of G. The extension of (2.6) still remains valid in this context [F3, Lemma 6.4]:

$$\mathfrak{L}(f,G)=\int_{\mathbf{R}^n} M(f,G,y)\,dy. \tag{2.7}$$

Another formula we will need concerns Lipschitz mappings (cf. [F5, Thm. 3.2.3]). If $f: G\to\mathbf{R}^n$ is a Lipschitz mapping then

$$\int_A |Jf(x)|\,dx = \int_{\mathbf{R}^n} N(f, A, y)\,dy \tag{2.8}$$

whenever $A \subset G$ is a measurable set.

We now consider Sobolev mappings $f \in W^{1,n}(G, \mathbf{R}^n)$ whose coordinate functions belong to $W^{1,n}(G)$. Note that Jf exists almost everywhere and is integrable on G. In case such a mapping is also continuous, it was proved in [GZ, Thm. 4.1] that

$$\mathcal{L}(f, G) = \int_G |Jf(x)|\,dx. \tag{2.9}$$

Therefore, in view of (2.7), we have

$$\int_G |Jf(x)|\,dx = \int_{\mathbf{R}^n} M(f, G, y)\,dy \tag{2.10}$$

whenever $f \in W^{1,n}(G, \mathbf{R}^n)$ is continuous.

3. Mappings on $\mathbf{R}^n$ and Condition (N)

3.1. THEOREM. *Suppose $f \in W^{1,n}(G, \mathbf{R}^n)$. Then*

$$\int_G |Jf(x)|\,dx \le \int_{\mathbf{R}^n} N(f, G, y)\,dy.$$

Moreover, equality holds if and only if f satisfies condition (N) *on G.*

Proof. We appeal to the result of [F2] which states that any real-valued function g defined on G that has partial derivatives almost everywhere on G possesses a surprising amount of regularity. That is, G has a partition

$$G = \bigcup_{i=0}^{\infty} E_i,$$

where each E_i is measurable, $i = 0, 1, \ldots$, where $|E_0| = 0$, and where $g \mid E_i$ is Lipschitz, $i = 1, 2, \ldots$. Applying this result to each of the coordinate functions of f, we obtain a similar conclusion. Thus, we may use the same notation with g replaced by f. Since $f \mid E_i$ is Lipschitz, $i > 0$, we apply Kirzbraun's theorem to obtain a Lipschitz extension of $f \mid E_i$ which is defined on $\mathbf{R}^n$. Denote this extension by f_i. By (2.8) we have that

$$\int_{E_i} |Jf_i(x)|\,dx = \int_{\mathbf{R}^n} N(f_i, E_i, y)\,dy$$

for $i > 0$. By basic methods, it can be shown that $Jf_i = Jf$ almost everywhere on E_i. Since $f_i \mid E_i = f \mid E_i$ and the E_i are disjoint, it follows that

$$\int_G |Jf(x)|\,dx = \int_A |Jf(x)|\,dx = \int_{\mathbf{R}^n} N(f, A, y)\,dy \le \int_{\mathbf{R}^n} N(f, G, y)\,dy, \tag{3.1}$$

where $A = \bigcup_{i=1}^{\infty} E_i$. If f is assumed to satisfy condition (N), then clearly

$$\int_G |Jf(x)|\,dx = \int_{\mathbf{R}^n} N(f, G, y)\,dy. \tag{3.2}$$

On the other hand, if we assume (3.2) then (3.1) implies that f satisfies condition (N). □

3.2. THEOREM. *Let $f \in W^{1,n}(G, \mathbf{R}^n)$ be a continuous mapping. Then $f^{-1}(y)$ is totally disconnected for almost all $y \in \mathbf{R}^n$; that is, each component of $f^{-1}(y)$ is a point.*

Proof. We use here the fact that, on almost all hyperplanes H orthogonal to the coordinate axes, $|f(H)| = 0$ (cf. [BI] or [R1, Lemma 6.3, p. 177]). For $i = 1, 2, \ldots, n$, let P_i be a countable, dense set of hyperplanes orthogonal to the ith coordinate direction on which f has the stated property. Then $|f(P)| = 0$ where $P = \bigcup_{i=1}^n P_i$. Now, for each $y \in \mathbf{R}^n - f(P)$ we have that $f^{-1}(y)$ is totally disconnected. Indeed, if $C \subset f^{-1}(y)$ were a nondegenerate continuum for some $y \in \mathbf{R}^n - f(P)$, then its orthogonal projection onto some coordinate hyperplane would also be a nondegenerate continuum. This implies that $C \cap P \neq \emptyset$, a contradiction. □

3.3. DEFINITION. Suppose that $f \in W^{1,n}(G, \mathbf{R}^n)$ is continuous. We let $\mathcal{B} \subset \mathbf{R}^n$ be the set of those points y for which $f^{-1}(y)$ is not totally disconnected. Also, the *branch set* B_f of f consists of all points $x \in G$ such that f is not a local homeomorphism at x. Clearly, B_f is a relatively closed set in G.

For the next lemma, observe that if $y \notin \mathcal{B}$ and if $x \in f^{-1}(y)$, then there exists $r_0 > 0$ such that for all $0 < r \le r_0$, the x-component V of $f^{-1}(B(y, r))$ satisfies $V \subset\subset G$. Thus, $d(f, r, V)$ is defined for all $0 < r \le r_0$.

3.4. LEMMA. *Suppose, for almost all $y \notin \mathcal{B}$ and each $x \in f^{-1}(y)$, that $d(f, r, V) \neq 0$ for all sufficiently small $r > 0$. Then f satisfies condition* (N) *on G.*

Proof. Theorem 3.2 states that $\mathcal{B}$ has measure zero. Let $\{x_1, x_2, \ldots, x_k\} \in f^{-1}(y)$ be distinct point-components of y, where y is chosen as in the statement of the lemma. Choose $r > 0$ so small that there exist disjoint components $V_1, V_2, \ldots, V_k$ of $f^{-1}(B(y, r))$ and $d(f, r, V_i) \neq 0$ for $i = 1, 2, \ldots, k$. Therefore, $M(f, G, y) \ge N(f, G, y)$ for almost all $y \in \mathbf{R}^n$, and (2.10) implies that

$$\int_G |Jf(x)|\,dx = \int_{\mathbf{R}^n} M(f, G, y)\,dy \ge \int_{\mathbf{R}^n} N(f, G, y)\,dy.$$

Reference to Theorem 3.1 yields the desired conclusion. □

3.5. THEOREM. *If $f \in W^{1,n}(G, \mathbf{R}^n)$ is a local homeomorphism, then f satisfies condition* (N) *on G and $|f(A)| = 0$, where $A = G \cap \{x : Jf(x) = 0\}$.*

Proof. Since f is a local homeomorphism, the local topological degree is ± 1 and thus f satisfies condition (N) on G by the previous result. Now let $U \supset A$ be an open set with the property that

$$\int_U |Jf(x)|\, dx < \epsilon$$

for some arbitrarily chosen $\epsilon > 0$. Then, by Theorem 3.1, it follows that

$$\epsilon > \int_U |Jf(x)|\, dx = \int_{\mathbf{R}^n} N(f, U, y)\, dy \geq |f(U)|.$$

Thus, $|f(A)| = 0$ as required. □

3.6. THEOREM. *Let $f \in W^{1,n}(G, \mathbf{R}^n)$ be a continuous mapping with $Jf = 0$ almost everywhere on G. Then $B_f = G$.*

Proof. If $G - B_f$ were not empty, there would exist a nonempty open set $U \subset G - B_f$ on which f would be a homeomorphism. Use (2.10) to conclude that

$$0 = \int_U |Jf(x)\, dx| = \int_{\mathbf{R}^n} M(f, U, y)\, dy.$$

But $M(f, U, y) = 1$ for all $y \in f(U)$ since f is a homeomorphism on U. This would imply that $|f(U)| = 0$, a contradiction since $f(U)$ is a nonempty open set. □

3.7. REMARKS. (a) The first part of Theorem 3.5 was proved by Reshetnyak [R3, Cor. 1, p. 182] using a different method.

(b) Note that a homeomorphism $f \in W^{1,n}(G, \mathbf{R}^n)$ need not satisfy $Jf \neq 0$ a.e. in G. A simple example can be constructed as follows: Let E be a Cantor set of positive 1-dimensional measure in the unit interval, and write

$$g(x) = \int_0^x \chi_{\tilde{E}}(t)\, dt$$

where $\chi_{\tilde{E}}$ is the characteristic function of the component of E. Then $f(x) = (g(x), x_2, \ldots, x_n)$, $x = (x_1, x_2, \ldots, x_n)$, has $Jf(x) = 0$ a.e. in $E \times \mathbf{R}^{n-1}$, and f is even an 1-Lipschitz homeomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^n$.

3.8. DEFINITION. Consider a mapping $f: G \to \mathbf{R}^n$, fix a point $x_0 \in G$, and let $L: \mathbf{R}^n \to \mathbf{R}^n$ denote a linear mapping. Let Q denote the n-cube centered at the origin with side length 2. For small $t > 0$ and $z \in Q$, define

$$f_{x_0}(t, z) = \frac{f(x_0 + tz) - f(x_0)}{t} - L(z)$$

$$\gamma_{x_0}(t) = \sup\{|f_{x_0}(t, z)| : z \in \partial Q\}.$$

The mapping f is said to have a *regular approximate differential* at x_0 if there is a linear map L and a set $A \subset \mathbf{R}$ such that 0 is a point of metric density of A and that $\gamma_{x_0}(t) \to 0$ as $t \to 0$, $t \in A$. If

$$\liminf_{t \to 0} \gamma_{x_0}(t) \to 0$$

then L is called the *weak differential* of f at x_0. The set A is said to have metric density at 0 provided

$$\lim_{r \to 0} \frac{|A \cap [0, r]|}{r} = 1.$$

We will let $Df(x_0)$ denote the linear mapping L.

While it is true that a function $f \in W^{1,n}(G)$ has partial derivatives almost everywhere, it is not true in general that it possesses a total differential almost everywhere. However, the following result, proved in [GZ], states that it does have a regular approximate differential almost everywhere.

3.9. THEOREM. *If $f \in W^{1,p}(G, \mathbf{R}^n)$, $p > n-1$, then f has a regular approximate differential almost everywhere in G.*

This result was essential in proving (2.9). However, it was Reshetnyak [R4] who first proved that functions in $W^{1,n}(G)$ have a weak differential almost everywhere. Part of the significance of this result resides in the relationship between the approximate differential and the local topological degree. Indeed, we recall the following fundamental fact concerning mappings with weak differentials [RR, p. 329]: If $f \in W^{1,n}(G, \mathbf{R}^n)$ has a weak differential at a point $x_0 \in G$, then

$$\mu(y, f, B(x_0, r)) = \operatorname{sgn} Jf(x_0) = \pm 1 \tag{3.3}$$

for all $r > 0$ sufficiently small and all y close to $f(x_0)$. Here, $\mu(y, f, B(x_0, r))$ denotes the topological degree of f relative to the ball $B(x_0, r)$.

This information is used to establish the following result, which is a slight improvement of the result by Reshetnyak [R2] concerning a sufficient condition for a mapping to satisfy condition (N).

3.10. THEOREM. *Suppose $f \in W^{1,n}(G, \mathbf{R}^n)$ is continuous. Then f satisfies condition* (N) *if and only if*

$$S(f, G, y) = M(f, G, y) = N(f, G, y) \quad \textit{almost everywhere in } \mathbf{R}^n. \tag{3.4}$$

Proof. If f satisfies condition (N) then—since f possesses a regular approximate differential almost everywhere in G—we may appeal to [RR, Thm. 2, p. 359], which uses (3.3), to conclude $M(f, G, y) = N(f, G, y)$ almost everywhere in $\mathbf{R}^n$. In order to show $S(f, G, y) = N(f, G, y)$ almost everywhere, let $A = G \cap \{x : Jf(x) = 0\} \cap \{x : Df(x) \text{ exists}\}$ and note that since f satisfies condition (N), we have $|f(A)| = 0$ from the proof of Theorem 3.1 and (2.8). Appealing also to Theorem 3.2, it therefore follows that almost all points $y \in \mathbf{R}^n$ satisfy the following: $N(f, G, y) < \infty$, $f^{-1}(y)$ is totally disconnected, $Df(x)$ exists, and $Jf(x) \neq 0$ for each $x \in f^{-1}(y)$. Note also that if $B = G \cap \{x : Df(x) \text{ does not exist}\}$, then $|f(B)| = 0$ by Theorem 3.9 and the fact that f satisfies condition (N). Now use (3.3) and (2.3) to conclude that $S(f, G, y) = N(f, G, y)$.

On the other hand, if (3.4) is satisfied, we use (2.10) and Theorem 3.1 to find that f satisfies condition (N) on G. $\square$

3.11. THEOREM. *Suppose $f \in W^{1,n}(G, \mathbf{R}^n)$ is continuous with $Jf \geq 0$ almost everywhere on G. For each $y \in \mathbf{R}^n$ and each $B(y, r)$, consider a component $V \subset\subset G$ of $f^{-1}(B(y, r))$. Then either $Jf = 0$ almost everywhere on V or $d(f, r, V) > 0$.*

Proof. Using the sets E_i that appear in the proof of Theorem 3.1, let

$$D = G \cap \{x : Df(x) \text{ exists}\} \cap \left(\bigcup_{i=1}^{\infty} E_i \right).$$

Also, let $P = V \cap D \cap \{x : Jf(x) > 0\}$ and assume that $|P| > 0$. We will show that $d(f, r, V) > 0$. If A is any measurable subset of P, it follows from (2.8) that

$$\int_A Jf(x)\, dx = \int_{\mathbf{R}^n} N(f, A, y)\, dy.$$

Now, if we take $B = f(P) \cap \{y : N(f, P, y) = \infty\}$, then it follows that $|B| = 0$. Therefore, taking $A = P \cap f^{-1}(B)$, we have that $|A| = 0$ since $Jf > 0$ on A. Consequently, after subtracting a set of measure zero from P if necessary, we may assume that $N(f, P, y) < \infty$ for all $y \in \mathbf{R}^n$. Now select $y' \in f(P) \cap B(y, r)$ and note that $f^{-1}(y') \cap P = \{x_1, x_2, \ldots, x_k\}$ for some positive integer k. Moreover, from (3.3) we see that the local topological degree of f at each x_i is positive. Hence, from properties of the topological degree, it follows that $d(f, r, V) > 0$. $\square$

3.12. THEOREM. *Suppose $f \in W^{1,n}(G, \mathbf{R}^n)$ is continuous with $Jf \geq 0$ almost everywhere on G. Then the conditions* (SA) *and* (N) *are equivalent.*

Proof. Assume first that f satisfies the condition (SA) on G. Define $T \subset f(G)$ to be the set of points y with the property that $Jf = 0$ almost everywhere on some component $V \subset\subset G$ of $f^{-1}(B(y, r))$ for some $r > 0$. Let $\mathfrak{F}$ denote the family of all such components V and define

$$D = \bigcup_{V \in \mathfrak{F}} V.$$

We will show that $|T| = 0$. In fact, we will show that $|f(D)| = 0$. For this purpose consider the family of all closed balls B with the property that B is centered at some point of D and that B is contained in some $V \in \mathfrak{F}$. The family of all such balls B produces a Vitali covering of D, and therefore there exists a countable, disjoint subfamily $\{B_1, B_2, \ldots,\}$ such that

$$\left| D - \bigcup_{i=1}^{\infty} B_i \right| = 0.$$

Since each B_i is a subset of some $V \in \mathfrak{F}$, it follows that $Jf = 0$ almost everywhere on B_i. Therefore $Jf = 0$ almost everywhere on D, and consequently $|f(D)| = 0$ since f satisfies condition (SA).

With $\mathcal{B}$ as in Definition 3.3, choose $y \notin \mathcal{B} \cup T$ so that $f^{-1}(y)$ is totally disconnected. Let $x_1, x_2, \dots, x_k$ be a set of point-components of $f^{-1}(y)$. Choose $r > 0$ so small that there exist disjoint components $V_1, V_2, \dots, V_k$ of $f^{-1}(B(y,r))$ such that $x_i \in V_i \subset\subset G$, $i = 1, 2, \dots, k$. Then Jf cannot be zero almost everywhere in V_i and thus, by Theorem 3.11, we have $d(f, r, V_i) > 0$. Now refer to Lemma 3.4 to conclude that f satisfies condition (N) on G.

Now assume that f satisfies condition (N) on G. Let $B \subset G$ be a set on which $Jf = 0$. Then, with the notation as in the proof of Theorem 3.1, it follows from (2.8) that

$$0 = \int_{B \cap A} |Jf(x)|\, dx = \int_{\mathbf{R}^n} N(f, B \cap A, y)\, dy,$$

which implies that $|f(B \cap A)| = 0$. But $B = (B \cap E_0) \cup (B \cap A)$ where $|E_0| = 0$, and therefore $|f(B)| = 0$. □

Note that if we assume $Jf > 0$ almost everywhere on G, then by Theorem 3.11 $d(f, r, V) > 0$ whenever V is a component of $f^{-1}(B(y,r))$. Referring again to Lemma 3.4, we have that f satisfies condition (N) on G. We state this as the next corollary.

3.13. COROLLARY. *If $f \in W^{1,n}(G, \mathbf{R}^n)$ is continuous and has the property that $Jf > 0$ almost everywhere on G, then f satisfies condition* (N) *on G.*

3.14. REMARK. The result above remains true even if f is not assumed to be continuous. See [GV] or [Sv].

3.15. REMARK. Theorem 3.12 is not true without the assumption $Jf \geq 0$ almost everywhere. In [R2] Reshetnyak provides quite an elementary example of a continuous mapping $f \in W^{1,2}(\mathbf{R}^2, \mathbf{R}^2)$ with the following properties:

(i) f has a total differential almost everywhere;
(ii) $Jf \neq 0$ almost everywhere; and
(iii) f does not satisfy condition (N).

For a similar example in $\mathbf{R}^n$, $n \geq 3$, see [V4]. The authors do not know of any example of a continuous mapping $f \in W^{1,n}(\mathbf{R}^n, \mathbf{R}^n)$ with $Jf \geq 0$ almost everywhere that does not satisfy condition (N). Recently, Malý [Ma] has constructed a continuous mapping $f \in W^{1,n}(\mathbf{R}^n, \mathbf{R}^n)$ such that $Jf = 0$ almost everywhere and f does not satisfy condition (N); the mapping f is produced by a Peano-type construction. Hence, there exist mappings $f \in W^{1,n}(\mathbf{R}^n, \mathbf{R}^n)$ that do not satisfy (SA).

3.16. THEOREM. *Suppose that $f \in W^{1,n}(G, \mathbf{R}^n)$ is a continuous mapping with $Jf \geq 0$ almost everywhere. If* $\operatorname{int} B_f = \emptyset$, *then f satisfies condition* (N) *on G.*

Proof. By Theorem 3.12, we need only show that f satisfies condition (SA). Let A be an open, nonempty set with $Jf=0$ almost everywhere on A. Since $\operatorname{int} B_f=\emptyset$, the set $A-B_f$ is nonempty. But this is impossible by Theorem 3.6 with G replaced by A. □

3.17. REMARK. The previous result is not true without the assumption that $Jf \geq 0$ almost everywhere. The example in Example 3.15 has the additional property $\operatorname{int} B_f=\emptyset$.

A mapping $f\colon G \to \mathbf{R}^n$ is called *discrete* if $f^{-1}(y)$ consists of isolated points in G for each $y \in \mathbf{R}^n$.

3.18. THEOREM. *Suppose that $f \in W^{1,n}(G, \mathbf{R}^n)$ is continuous, discrete, and open. Then f satisfies condition* (N) *on G.*

Proof. By a result of Chernavskii ([C1; C2], see also [V2]), the topological dimension of B_f is no more than $n-2$. Hence, $\operatorname{int} B_f=\emptyset$ and $G-B_f$ is a domain. This result also establishes that f is either sense-preserving or sense-reversing which, because of Theorem 3.9 and (3.3), implies that Jf is either nonnegative or nonpositive almost everywhere in $G-B_f$. But now our result follows from Theorem 3.16. □

4. Mappings on $\mathbf{R}^2$

It is not known whether a continuous, open mapping $f \in W^{1,n}(G, \mathbf{R}^n)$ satisfies condition (N). Note that Jf may change sign for these mappings. However, in case $n=2$, we are able to supply an affirmative answer by appealing to a result of Demers and Federer [DF] that is useful in determining when the stable multiplicity function is nonzero.

4.1. THEOREM. *Let $G \subset \mathbf{R}^n$ be an open set with smooth boundary, and suppose*

$$u\colon G \to \mathbf{R}, \quad v\colon G \to \mathbf{R}^{n-1}, \quad f\colon G \to \mathbf{R} \times \mathbf{R}^{n-1}$$

are continuous maps such that $f(x)=(u(x), v(x))$ for $x \in G$. Then there is a countable set $D \subset \mathbf{R}$ such that

$$S[f, G, (s,t)] \geq S[v, u^{-1}(s), t] \tag{4.1}$$

for $(s,t) \in (\mathbf{R}-D) \times \mathbf{R}^{n-1}$.

The following definition is somewhat artificial, but is introduced in order to characterize certain mappings that appear in Theorem 4.4.

4.2. DEFINITION. A mapping $f\colon G \to R^n$ is called *almost open* if, for almost each $y \in R^n$, there exists $r>0$ such that each component W of $f^{-1}(U)$ that contains a point-component of $f^{-1}(y)$ is mapped onto U whenever $U \subset B(y,r)$ is an open, connected set containing y.

4.3. THEOREM. *Suppose $G \subset \mathbf{R}^2$ is a smoothly bounded open set and that $f \in W^{1,2}(G, \mathbf{R}^2)$ is a continuous, almost open mapping. Then*

$$S(f, G, y) \geq N(f, G, y)$$

for almost all $y \in \mathbf{R}^2$.

Proof. By Theorems 3.2 and 4.1, we need to consider only those $y = (s, t) \in \mathbf{R}^2$ for which $f^{-1}(y)$ is totally disconnected and (4.1) holds. Thus, for all such y, it will be sufficient to show

$$S[v, u^{-1}(s), t] \geq N[f, G, y]. \tag{4.2}$$

Now select $y = (s, t)$ satisfying the conditions stated above, and let $x_1, x_2, \ldots, x_k$ denote k distinct point-components of $f^{-1}(y)$. Since $f^{-1}(y)$ is totally disconnected, the x_i-components V_i of $f^{-1}(B(y, r))$ for small $r > 0$ are separate and $V_i \subset\subset G$. Choosing r smaller if necessary, we conclude that since f is almost open, each component W_i of $f^{-1}(U)$ that contains x_i is mapped onto U whenever $U \subset B(y, r)$ is an open, connected set containing y. Hence, it follows that if I_y is a closed nondegenerate vertical interval centered at y, then the x_i-components C_i of $f^{-1}(I_y)$ are disjoint and $f(C_i) = I_y$ for $i = 1, 2, \ldots, k$. Thus, by Remark 2.2, it follows that

$$S(v, u^{-1}(s), t) \geq k.$$

This establishes (4.2) since k is arbitrary. □

4.4. THEOREM. *Let $G \subset \mathbf{R}^2$ be an open set and suppose $f \in W^{1,2}(G, \mathbf{R}^2)$ is continuous. Then f satisfies condition* (N) *on G if and only if f is almost open on G.*

Proof. We may assume that G is a smoothly bounded open set. If f is almost open on G, then from (2.10), (2.4), and the previous result we obtain

$$\int_G |Jf(x)|\, dx = \int_{\mathbf{R}^n} M(f, G, y)\, dy \geq \int_{\mathbf{R}^n} S(f, G, y)\, dy \geq \int_{\mathbf{R}^n} N(f, G, y)\, dy.$$

Now refer to Theorem 3.1 to conclude that f satisfies condition (N) on G.

Now assume that f satisfies condition (N) on G. Then, by Theorem 3.1, we have that $N(f, G, y) < \infty$ for almost all $y \in \mathbf{R}^n$. Hence, by (3.4), it follows that

$$S(f, G, y) = M(f, g, y) = N(f, G, y) < \infty \tag{4.3}$$

for almost all y. Select such a y and let $\{x_1, x_2, \ldots, x_k\} = f^{-1}(y)$. In view of (4.3), there exists $r > 0$ such that the components V_i of $f^{-1}(B(y, r))$ that contain x_i have the property that

$$|d(f, r, V_i)| = 1, \quad i = 1, 2, \ldots, k. \tag{4.4}$$

Now let $U \subset B(y, r)$ be an open, connected set and let W_i denote that component of $f^{-1}(U)$ containing x_i, $i = 1, 2, \ldots, k$. Then, in view of (4.4), $\mu(y, f, W_i) \neq 0$ (cf. [RR, Thm. 3, p. 126]) and therefore $f(W_i) = U$. □

We mentioned earlier that it is not known whether a mapping in $W^{1,n}(G, \mathbf{R}^n)$ with $Jf \geq 0$ almost everywhere satisfies condition (N). In case $n=2$ and f is a light mapping, we will show that f does satisfy condition (N). A mapping $f: G \to \mathbf{R}^n$ is said to be *light* if $f^{-1}(y)$ is totally disconnected for each $y \in \mathbf{R}^n$.

4.5. THEOREM. *If $f \in W^{1,2}(G, \mathbf{R}^2)$ is continuous, light, and satisfies $Jf \geq 0$ almost everywhere in G, then f satisfies condition* (N).

Proof. We may as well assume that $|f(G)| > 0$. For each $y_0 \in f(G)$, let $V \subset\subset G$ be a component (which is open) of $f^{-1}(B(y_0, r))$. In this connection, see the remark preceding Lemma 3.4. By Lemma 3.4, it suffices to show that $d(f, r, V) > 0$. This will be established by appealing to Theorem 3.11, which states that it is sufficient to show $Jf > 0$ on some set of positive measure on V. This, in turn, will be established with the help of (2.10) by showing that

$$\int_{\mathbf{R}^n} M(f, V, y)\, dy > 0. \tag{4.5}$$

In order to prove (4.5), consider the connected set $f(V) \subset B(y_0, r)$. Note that $f(V)$ is not a point since f is assumed to be light. We may assume that the projection of $f(V)$ onto the x-axis is an interval, call it J, for if not then this could be arranged by employing a suitable rotation R of $\mathbf{R}^2$ about the point y_0. This would have the effect of replacing f with $R \circ f$ in (4.5), which is immaterial. Let I denote an open (vertical) interval whose endpoints lie on $\partial B(y_0, r)$ and whose projection onto the x-axis is a point in $J - D$, where D is the countable set that appears in Theorem 4.1. Let $K = f^{-1}(I) \cap V$ and observe that K cannot be totally disconnected, for otherwise $V - K$ and therefore $f(V-K)$ would be connected (cf. [HW, §II 4, Thm. IV 4]). But, on the other hand, $f(V-K) = f(V) - I$ is not connected. Thus, since K is not totally disconnected, K must contain a continuum C and consequently its image under f is a nondegenerate interval, since f is light. Now refer to Theorem 4.1 to conclude that $S(f, V, y') > 0$ for some $y' \in I \cap B(y_0, r)$. Technically, to apply Theorem 4.1 we need to know that V is smoothly bounded. This can be avoided by approximating V from within by smoothly bounded open sets. Thus, this establishes (4.5) since $0 < S(f, V, y') \leq M(f, V, y')$ and $M(f, V, y)$ is lower semicontinuous in y. □

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O. Martio
Department of Mathematics
University of Jyväskylä
SF-40100 Jyväskylä
Finland

William P. Ziemer
Department of Mathematics
Indiana University
Bloomington, IN 47405