Polynomial Approximation in the Mean with Respect to Harmonic Measure on Crescents II

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For which planar domains G are the polynomials dense in the Hardy space $H^s(G)$, $1 \leq s < \infty$? Roan [9] has shown that if $G = \varphi(\mathbf{D})$, where $\mathbf{D}$ is the unit disk and φ is a weak-star generator of H^∞, then the polynomials are dense in $H^s(G)$. What if G is not the image of the unit disk under such a φ? Among the "simplest" of such regions are crescents (see [10]). In [1] the author examines the question of density of the polynomials in $H^s(G)$ for crescents G of a rather limited variety. This paper is much more conclusive in that it completely answers this question for a large class of crescents; indeed, it gives three necessary and sufficient conditions for density of the polynomials in $H^s(G)$. We begin with some definitions.

1. DEFINITION. A *crescent* is a region (in the complex plane) bounded by two Jordan curves which intersect in a single point such that one of the Jordan curves is internal to the other.

If G is any crescent, $z_0 \in G$ and $1 \leq s < \infty$, then let $\omega(\cdot, G, z_0)$ (or $\nu(\cdot, G, z_0)$ etc.) denote harmonic measure on ∂G evaluated at z_0 and $P^s(\omega)$ be the closure of the polynomials in $L^s(\omega)$; $\omega := \omega(\cdot, G, z_0)$. The Hardy space $H^s(G)$ is the collection of all functions f which are analytic in G such that $|f|^s$ has a harmonic majorant on G. With z_0 as before, define $\|\cdot\|_{z_0}: H^s(G) \to \mathbf{R}$ by $\|f\|_{z_0} = (u_f(z_0))^{1/s}$, where u_f is the least harmonic majorant of $|f|^s$ on G. Then $\|\cdot\|_{z_0}$ is a norm on $H^s(G)$ and under this norm $H^s(G)$ forms a Banach space. If the polynomials are dense in $H^s(G)$, then $P^s(\omega)$ is isometrically isomorphic to $H^s(G)$; we shall indicate this by $P^s(\omega) = H^s(G)$. Denote the "outer boundary" of G by $\partial_\infty G$ ($\partial_\infty G := \partial(\bar{G}\hat{\ })$, where $\bar{G}\hat{\ }$ is the polynomially convex hull of the closure of G), and let $\mathbf{C}$, H^+ and $\mathbf{D}$ denote the complex plane, the upper half-plane ($\{z \in \mathbf{C}: \operatorname{Im}(z) > 0\}$) and the unit disk ($\{z \in \mathbf{C}: |z| < 1\}$) respectively.

2. DEFINITION. Let $\mathfrak{A}$ be the collection of all functions f such that:

(i) $f: \mathbf{R} \to [0, \infty)$, $f(x) = 0$ if and only if $x = 0$ and $f(-x) = f(x)$ for all x in $\mathbf{R}$;

(ii) $\int_0^1 \log(f(x))\,dx > -\infty$, and there exists $\alpha > 1$ such that $f(x) \leq x^\alpha/(\alpha - 1)$ whenever $0 \leq x \leq 1$;

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(iii) there exists $M>0$ such that, for all x and y, $|f(x)-f(y)|\leq M|x-y|$ and, whenever $0<x\leq y$, $y/f(y)-x/f(x)\leq M$.

In Definition 2, an implication of condition (i) and the second part of (ii) is that $t/f(t)\to\infty$ (not necessarily monotonically) as $t\to 0$. Therefore, "most often", $y/f(y)-x/f(x)\leq 0$ when $0<x\leq y$. Thus, the second part of condition (iii) is nearly redundant except when viewed as a moderate restriction on the oscillation of $t/f(t)$ near zero.

The restrictions on the class of functions $\mathfrak{A}$, and hence on the ensuing collection of crescents, are quite natural and not without precedent in the literature (cf. [4, Thm. 5.5]) even though our approach to the problem addressed by this paper is somewhat nonstandard. One should note that $\mathfrak{A}$ contains and is much larger than $\mathfrak{F}$ (as defined in [1]).

For $0<a<b<\infty$ and $f\in\mathfrak{A}$, let $\tau(a,f)=\inf\{t:t>0 \text{ and } |(t,f(t))|=a\}$ and $G(f,a,b)=\{z\in H^+:|z|<b\}\setminus\{z=x+iy:|z|\leq a, |x|\leq\tau(a,f) \text{ and } y\geq f(x)\}$ (see Figure 1). The collection of crescents that interests us here is defined in terms of crescents of the sort $G(f,a,b)$.

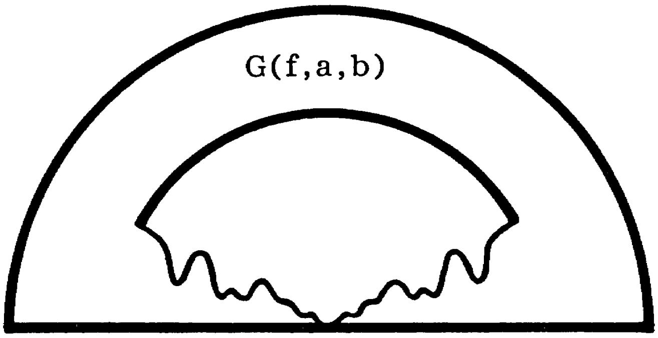

Figure 1

3. DEFINITION. Let $\mathfrak{B}$ be the collection of all crescents E for which there exist f in $\mathfrak{A}$, $0<a<b<\infty$, and $c>1$ such that $G(f/c,a,b)\subseteq E\subseteq G(cf,a,b)$.

Observe that $\{G(f,a,b): f\in\mathfrak{A} \text{ and } 0<a<b<\infty\}\subseteq\mathfrak{B}$, and that $\mathfrak{B}$ is much larger than $\mathfrak{C}$ (as defined in [1]). Moreover, the size of $\mathfrak{B}$ is actually not limited by the fact that for each E in $\mathfrak{B}$ there exists $b>0$ such that $\partial_\infty E=[-b,b]\cup\{z\in H^+:|z|=b\}$. Indeed, by a conformal mapping argument, one can show that for any crescent G there exists a crescent G^*, where $\partial_\infty(G^*)=\partial(\mathbf{D}\cap H^+)$, such that the polynomials are dense in $H^s(G)$ if and only if they are dense in $H^s(G^*)$.

Now for the main result.

4. THEOREM. *Suppose* $E\in\mathfrak{B}$, $z_0\in E$, $\omega:=\omega(\cdot,E,z_0)$, $\omega_\infty:=\omega|_{\partial_\infty E}$, *$m$ is arclength measure on* $\partial_\infty E$, $\ell(r)=\text{length}(E\cap\{z:|z|=r\})$, *and* $1\leq s<\infty$. *Then the following are equivalent:*

(i) $P^s(\omega)=H^s(E)$.

(ii) $\displaystyle\int_0^\epsilon \frac{r}{\ell(r)}\,dr=\infty$.

(iii) $\int \log\left[\frac{d\omega_\infty}{dm}\right] dm = -\infty.$

(iv) $P^s(\omega_\infty) = L^s(\omega_\infty)$.

The proof of Theorem 4 is postponed until certain estimates on harmonic measure have been made. However, before launching into any technicalities, we should consider the statement of the theorem.

It is apparent from (ii) or (iii) of Theorem 4 that density of the polynomials in $H^s(E)$, for E in $\mathcal{B}$ and $1 \le s < \infty$, is independent of s. Szegö's theorem gives the equivalence between (iii) and (iv); that (i) and (iii) are equivalent says similarly that density of the polynomials in $H^s(E)$ is completely determined by the nature of $\omega(\cdot, E, z_0)$ on $\partial_\infty E$; of course, the distribution of $\omega(\cdot, E, z_0)$ on $\partial_\infty E$ depends on the geometry of E. Condition (ii) has a counterpart in the context of area measure (cf. [4, Thm. 5.5]) which relates the density of the polynomials in $L_a^s(E)$ to the divergence of $\int_0^\epsilon \log(\ell(r))\,dr$ (in the area measure case, $\ell(r)$ is defined in such a way that it respects "one-sided" behavior of the geometry of E near the multiple boundary point). Evidently, therefore, density of the polynomials in $H^s(E)$ is much more common than in $L_a^s(E)$.

Although $\mathcal{B}$ is large, there are some easily defined crescents to which Theorem 4 does not apply. For example, let $\Omega = \{z = x+iy : y \ge |x| \text{ if } x \le 0 \text{ and } y \ge x^2 \text{ if } x \ge 0\}$ and $G = \{z \in H^+ : |z| < 1\} \setminus \{z \in \Omega : |z| \le \frac{1}{2}\}$. For $\omega = \omega(\cdot, G, z_0)$ and $1 \le s < \infty$ one can use standard estimates on harmonic measure to show that $P^s(\omega_\infty) = L^s(\omega_\infty)$ and yet $P^s(\omega - \omega_\infty) \ne L^s(\omega - \omega_\infty)$. So G satisfies (iii) and (iv) of Theorem 4; however, $P^s(\omega) \ne H^s(G)$. Notice that $G \notin \mathcal{B}$ primarily because the inner boundary of G is the graph of a function that is far from being even. Just as easily one can show that, in general, (i) and (ii) of Theorem 4 are not equivalent.

Now let us get to the technical details of the paper. We shall lead off with a definition and two lemmas. The definition and the first of the two lemmas each have a close relative in [1].

5. DEFINITION. If $f \in \mathcal{A}$, $0 < \theta < \pi/2$, and n is any positive integer, then let

$$\rho_n := \rho_n(\theta, f) = \begin{cases} 0 & \text{if } \{r > 0 : r\sin(\theta/n) \le f(r\cos(\theta/n))\} = \emptyset, \\ \inf\{r > 0 : r\sin(\theta/n) \le f(r\cos(\theta/n))\} & \text{otherwise.} \end{cases}$$

Let $x_n := x_n(\theta, f) = \rho_n \cos(\theta/n)$.

Notice that this definition is consistent with its counterpart in [1] and for any f in $\mathcal{A}$ and any θ, $0 < \theta < \pi/2$, there exists N such that whenever $n \ge N$, $\rho_n \ne 0$. Furthermore, if $\rho_n \ne 0$, then $\rho_k \ne 0$ for all $k \ge n$ and $\rho_n > \rho_{n+1} > \cdots > \rho_{n+j} \to 0$ as $j \to \infty$. Since f is a function, the same holds for x_n.

6. LEMMA. *If $f \in \mathcal{A}$ and $\int_0^\epsilon (t/f(t))\,dt = \infty$ for some $\epsilon > 0$, then*

$$\sum_{n=1}^{\infty} \rho_n(\theta, f) = \infty \quad \textit{whenever } 0 < \theta < \pi/2.$$

Proof. Suppose that $\sum_{n=1}^{\infty} \rho_n(\theta, f) < \infty$ for some θ, $0 < \theta < \pi/2$. Then $\sum_{n=1}^{\infty} x_n < \infty$, where $x_n := x_n(\theta, f)$. Now since $f \in \mathfrak{A}$, there exists $M > 0$ such that $y/f(y) - x/f(x) \leq M$ whenever $0 < x \leq y$. Moreover, as noted previously, there exists a positive integer N such that $x_n \neq 0$ for all $n \geq N$ and $x_N > x_{N+1} > \cdots > x_{N+j} \to 0$ as $j \to \infty$. Consequently,

$$\begin{aligned}\sum_{n=N}^{\infty} x_n &\left[\max_{x_{n+1} \leq s \leq x_n} \left(\frac{s}{f(s)} - \frac{x_n}{f(x_n)}\right)\right] \\ &= \sum_{n=N}^{\infty} x_n \left[\left(\frac{x_{n+1}}{f(x_{n+1})} - \frac{x_n}{f(x_n)}\right) + \max_{x_{n+1} \leq s \leq x_n} \left(\frac{s}{f(s)} - \frac{x_{n+1}}{f(x_{n+1})}\right)\right] \\ &\leq \sum_{n=N}^{\infty} x_n \left[\frac{1}{\theta} + M\right] < \infty.\end{aligned}$$

It follows that $\int_0^{\epsilon} (t/f(t))\, dt < \infty$ for any $\epsilon > 0$. □

7. LEMMA. *Suppose that $f \in \mathfrak{A}$, $0 < a < b < \infty$, $z_0 \in G := G(f, a, b)$, $\omega := \omega(\cdot, G, z_0)$, and m is arclenth measure on $\partial_\infty G$. Then there exists a constant $M > 1$ such that*

$$\frac{d\omega}{dm}(x) \geq \frac{1}{Mf(x)} \cdot \exp\left[-\int_x^a \frac{M}{f(t)}\, dt\right]$$

whenever $0 < x \leq \tau := \tau(a, f)$.

Proof. Since f is Lipschitz, there exists a constant c, $0 < c < 1$, such that $D_x := \{z \in H^+ : |x - z| \leq cf(x)\} \subseteq G$ whenever $0 < x \leq \tau$. Moreover, there exist continuously differentiable functions h_1 and h_2 defined on $[0, \tau]$ such that $(2c/3)f(x) \leq h_1(x) \leq (3c/4)f(x)$ and $(c/4)f(x) \leq h_2(x) \leq (c/3)f(x)$ whenever $0 \leq x \leq \tau$. Extend h_1 and h_2 to the nonnegative real line by letting $h_1(x) = h_1(\tau)$ and $h_2(x) = h_2(\tau)$ if $x > \tau$.

For any x, $0 < x \leq \tau$, let $G_x = \{z = s + it : x < s \text{ and } h_2(s) < t < h_1(s)\} \cap G$ and $B_x = \{z \in \partial G_x : \operatorname{Re}(z) = x\}$ (see Figure 2). By Harnack's inequality we may assume that $z_0 = \tau + (i/2)(h_1(\tau) + h_2(\tau))$, which is in G_x. Let $\mathbf{D}^+ = \mathbf{D} \cap H^+$. Applying a standard conformal mapping argument, we can find a positive constant d such that if I is any interval contained in $[-\frac{1}{2}, \frac{1}{2}]$ and $\frac{1}{4} \leq r \leq \frac{3}{4}$, then $\omega(I, \mathbf{D}^+, ir) \geq d|I|$; $|I| := \text{length}(I)$. So, by the obvious conformal map of

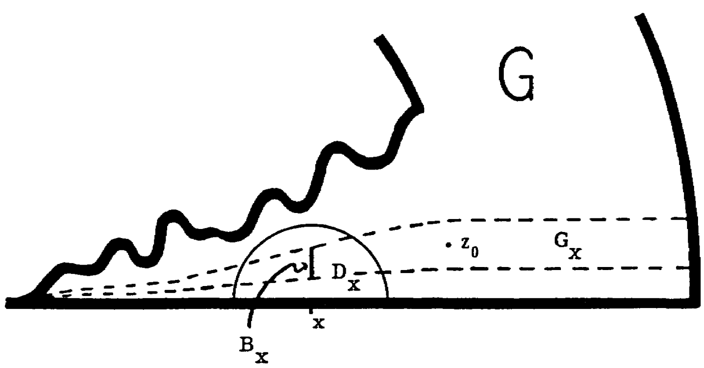

Figure 2

D_x onto $\mathbf{D}^+$, for any interval I contained in $[x-(c/2)f(x), x+(c/2)f(x)]$, $\omega(I, D_x, z) \geq (d/cf(x))|I|$ whenever $z \in B_x$.

Now there exists a constant k, $0<k<1$, such that $h_1(x)-h_2(x) \geq kf(x)$ whenever $0 \leq x \leq \tau$. So, by [3, Thm. C, p. 380] (one may also refer to [7, §6(17)]), there is a positive constant α (independent of x) such that

$$\omega(B_x, G_x, z_0) \geq \text{const} \cdot \exp\left[-\int_x^\tau \frac{\alpha}{f(t)}\, dt\right].$$

Therefore, if I is any interval contained in $[x-(c/2)f(x), x+(c/2)f(x)]$ and u is the harmonic function in G with boundary values χ_I (i.e., $u(z) = \omega(I, G, z)$), then by the maximum principle

$$\begin{aligned}\omega(I, G, z_0) &= \int_{\partial G_x} u(z)\, d\omega(z, G_x, z_0) \\ &\geq [\inf_{z \in B_x} u(z)] \cdot \omega(B_x, G_x, z_0) \\ &\geq [\inf_{z \in B_x} \omega(I, D_x, z)] \cdot \omega(B_x, G_x, z_0) \\ &\geq \text{const} \cdot \frac{d|I|}{cf(x)} \exp\left[-\int_x^\tau \frac{\alpha}{f(t)}\, dt\right]. \qquad \square\end{aligned}$$

Proof of Theorem 4. First observe that since $E \in \mathcal{B}$, there exists f in $\mathcal{A}$, $c>1$, and $0<a<b<\infty$ such that

$$V := G(f/c, a, b) \subseteq E \subseteq G(cf, a, b) := W. \tag{4.1}$$

A consequence of this is that

$$\int_0^\epsilon \frac{r}{\ell(r)}\, dr = \infty \quad \text{if and only if} \quad \int_0^\epsilon \frac{t}{f(t)}\, dt = \infty. \tag{4.2}$$

Moreover, if $\sigma := \sigma(\cdot, \text{inside}(\partial_\infty E), z_0)$ then, by a conformal mapping argument (left to the reader), there exists a constant $k>1$ such that

$$\frac{1}{k}|z^2-b^2| \leq \frac{d\sigma}{dm}(z) \leq k|z^2-b^2|. \tag{4.3}$$

To prove the theorem, let us show that (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Suppose $P^s(\omega) = H^s(E)$. Then $P^s(\omega_\infty) = L^s(\omega_\infty)$ and so, by Szegö's theorem [6, p. 136],

$$\int \log\left(\frac{d\omega_\infty}{d\sigma}\right) d\sigma = -\infty.$$

Therefore, by (4.3), $\int \log(d\omega_\infty/dm)\, dm = -\infty$. Consequently, (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

Now assume that $\int \log(d\omega_\infty/dm)\, dm = -\infty$. Applying (4.3) and [1, Prop. 1.3], we have

$$\int_{-\epsilon}^{\epsilon} \log\left(\frac{d\omega}{dm}\right) dm = -\infty$$

whenever $0<\epsilon<a$. So, if $\nu := \nu(\cdot, V, z_0)$ and $0<\epsilon<a$ then, by (4.1),

$$\int_{-\epsilon}^{\epsilon} \log\left(\frac{d\nu}{dm}\right) dm = -\infty.$$

By the symmetry of V it follows that

$$\int_0^\epsilon \log\left(\frac{d\nu}{dm}\right) dm = -\infty$$

whenever $0 < \epsilon < a$. Choose $\epsilon = \tau(a, f/c)$ and apply Lemma 7 to obtain

$$\int_0^\epsilon \log\left(c \exp\left[-c\int_x^\epsilon \frac{M\,dt}{f(t)}\right] \Big/ Mf(x)\right) dx = -\infty. \tag{4.4}$$

Since $f \in \mathfrak{A}$, $\int_0^\epsilon \log(f(x))\,dx > -\infty$, and so by (4.4) and Tonelli's theorem,

$$\int_0^\epsilon \frac{t}{f(t)}\,dt = \infty.$$

From (4.2) we now have that

$$\int_0^\epsilon \frac{r}{\ell(r)}\,dr = \infty.$$

Finally, suppose that $\int_0^\epsilon (r/\ell(r))\,dr = \infty$. Then, by (4.2), $\int_0^\epsilon (t/cf(t))\,dt = \infty$. Combining Lemma 6 with [1, Lemma 2.4, proof of Theorem 2.5] we have $P^s(\mu) = H^s(W)$, where $\mu := \mu(\cdot, W, z_0)$. From (4.1) and [1, Prop. 2.2] it follows that $P^s(\omega) = H^s(E)$. □

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