On the Quasidiagonality of Direct Sums of Normal Operators and Shifts

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1. Introduction

1.0. Let $\mathcal{H}$ be a complex, infinite-dimensional, separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators acting on $\mathcal{H}$. By $\mathcal{K}(\mathcal{H})$ we denote the compact operators, and $\mathcal{A}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra. The canonical map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{A}(\mathcal{H})$ is denoted π, and for $T \in \mathcal{B}(\mathcal{H})$ the spectrum of T is written $\sigma(T)$; $\sigma_e(T) = \sigma(\pi(T))$ will denote the essential spectrum of T.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to belong to the class (QD) of *quasidiagonal* operators if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of finite-dimensional projections converging strongly to the identity operator I for which $\{\|TP_n - P_nT\|\}_{n=1}^{\infty}$ converges to 0. If $TP_n = P_nT$ for all $n \geq 1$, then T is said to be *block-diagonal* and we write $T \in (BD)$. It is well known that (QD) is closed, invariant under compact perturbations, and given $\epsilon > 0$ and $T \in (QD)$, there exist $B \in (BD)$ and $K \in \mathcal{K}(\mathcal{H})$, $\|K\| < \epsilon$, such that $T = B + K$.

An operator T is called *quasitriangular* ($T \in (QT)$) if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ as above for which $\lim_{n\to\infty} \|(I-P_n)TP_n\| = 0$, and T is *bi-quasitriangular* ($T \in (BQT)$) if both T and T^* are quasitriangular. Both (QT) and (BQT) have been studied extensively (cf. [12] and its references) and much is known about these classes. It is not hard to see that $(QD) \subseteq (BQT)$ and $T \in (QD)$ if and only if there is a sequence $\{P_n\}_{n=1}^{\infty}$ as above which simultaneously implements the quasitriangularity of T and T^*.

1.1. Recall that $W \in \mathcal{B}(\mathcal{H})$ is called a *bilateral* (resp. *unilateral*) weighted shift if there exists an orthonormal basis $\{e_n\}_{n\in\mathbf{Z}}$ (resp. $\{e_n\}_{n\geq 1}$) of $\mathcal{H}$ and a bounded sequence of scalars $\{\omega_n\}_{n\in\mathbf{Z}}$ (resp. $\{\omega_n\}_{n\geq 1}$) such that $We_n = \omega_n e_{n+1}$ for all n. Up to unitary equivalence (under which quasidiagonality is invariant), we may and do assume that all weights are positive. The bilateral (resp. unilateral) shift with all weights equal to 1 is denoted by B (resp. S).

If W is a bilateral weighted shift, then we say that W is *block-balanced* if, given $\epsilon > 0$ and $n > 0$ an integer, there exist integers p and q such that $p+n < 0 < q$ and

$$\|(\omega_p, \omega_{p+1}, \omega_{p+2}, \dots, \omega_{p+n}) - (\omega_q, \omega_{q+1}, \omega_{q+2}, \dots, \omega_{q+n})\|_\infty < \epsilon.$$

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We have the following theorem due to Smucker.

1.2. THEOREM [21]. *Let W be a bilateral weighted shift with weights $\{\omega_n\}_{n\in\mathbf{Z}}$. Then $W\in(QD)$ if and only if either*

(1) $0=\liminf_{n\geq 0}|\omega_n|=\liminf_{n\leq 0}|\omega_n|$;

or

(2) *W is block-balanced.*

1.3. In the next section we determine necessary and sufficient conditions for an operator of the form $N\oplus W$, with W a weighted shift and N a normal operator with $\sigma(N)\supseteq\sigma(W)$, to be quasidiagonal. We then use these conditions to produce an example for each $n\geq 1$ of a biquasitriangular operator T_n for which T_n^k is quasidiagonal if and only if n divides k.

The reader may compare this result with that of Brown [8] and Salinas [18], who exhibit an example for each $n\geq 1$ of an operator A_n such that $A_n, A_n\oplus A_n, \ldots, A_n^{(n-1)}$ are not quasidiagonal, although $A_n^{(n)}$ *is.*

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2

2.0. It is an important theorem of Apostol, Foiaş, and Voiculescu [2] that $T\in(BQT)$ if and only if the index $\operatorname{ind}(T-\lambda)=\operatorname{nul}(T-\lambda)-\operatorname{nul}(T-\lambda)^*=0$ for all $\lambda\in\rho_{sF}(T)$, the semi-Fredholm domain of T. In other words, the only obstruction to membership in (BQT) is index. It follows immediately that if N is normal and $\sigma(T)\subseteq\sigma_e(N)$, then $N\oplus T\in(BQT)$.

The operator

$$T=\begin{bmatrix} 0 & I & S \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}$$

acting on $\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}$, where S is the unilateral shift, is known not to be quasidiagonal ([20], see also [13]). Since $T^3=0$, we have $\sigma(T)=\{0\}$, and yet if $0\oplus T\in(QD)$ it would follow that $T\in(QD)$ (cf. [16]). This contradiction shows that index is not the only obstruction to membership in (QD).

If W is an essentially normal (unilateral or bilateral) weighted shift, then Berg [4] has shown that, given N normal such that $\sigma(N)\supseteq\sigma(W)$, $N\oplus W$ can be written in the form $U^*(N+K)U$, where U is unitary and $K\in\mathcal{K}(\mathcal{H})$. (This also follows from [9].) Noting that an operator $T\in(QD)$ if and only if $V^*TV+L\in(QD)$ for all V unitary and L compact, it is clear then that $N\oplus W\in(QD)$. However, as we shall demonstrate, it is possible for $N\oplus W$ to be quasidiagonal even if W is neither essentially normal nor quasidiagonal. An example of this will be the unilateral weighted shift W with weights $\{1,\frac{1}{2},1,1,\frac{1}{2},\frac{1}{2},1,1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2},1,\ldots\}$ and the normal operator N with $\sigma(N)=\{z\,|\,|z|\leq 1\}$. One of the main ingredients in Berg's proof is the existence of arbitrarily long blocks of almost constant weights for W. It is this observation, combined with an argument of Smucker's, that leads to Theorem 2.5. Before that, however, we shall reduce our operators N and W to a suitable form.

2.1. We shall first deal with the case where W is a unilateral shift with weights $\{\omega_n\}_{n=1}^{\infty}$, $0 \le \omega_n \le 1$. If $\liminf_{n\ge 1}\{\omega_n\} = 0$, then it is easy to see that $W \in (QD)$ and so $N \oplus W \in (QD)$ for any normal operator N. The nontrivial case therefore is where $\liminf_{n\ge 1}\{\omega_n\} = \delta > 0$. Up to a compact perturbation of W (not affecting quasidiagonality), we may then assume that $\delta \le \omega_n \le 1$ for all $n \ge 1$. Then

$$\sigma(W) = D_{\mathrm{spr}(W)} = \{z \mid |z| \le \mathrm{spr}(W)\},$$

where $\mathrm{spr}(W) = \lim_{n\to\infty}\|W^n\|^{1/n}$ is the spectral radius of W. We also define

$$r_e(W) = \inf_{m\ge 1} \lim_{n\to\infty} \inf_{k\ge m} |\omega_{k+1}\omega_{k+2}\cdots\omega_{k+n}|^{1/n}.$$

Then $r_e(W) \ge \delta$ and $r_e(W)$ can be thought of as the inner radius of the annulus $\sigma_e(W)$.

Recall that B denotes the (unweighted) bilateral shift, and let $\{r_n\}_{n=1}^{\infty}$ denote a countable dense subset of $[0, \mathrm{spr}(W)]$, with each term repeated infinitely often. If N is any normal operator with spectrum $D_{\mathrm{spr}(W)}$, then it follows from the Weyl–von Neumann–Berg theorem that N is approximately unitarily equivalent to $M = \bigoplus_{n=1}^{\infty} r_n B$ (we write $N \cong_a M$). Clearly $N \oplus W \in (QD)$ if and only if $M \oplus W \in (QD)$. Let $0 \le \lambda \le \mathrm{spr}(W)$. Since $M \oplus \lambda S^*$ is an essentially normal operator with no nonzero index on its semi-Fredholm domain, $M \oplus \lambda S^* \cong M + K$ for some $K \in \mathcal{K}(\mathcal{H})$, by the Brown–Douglas–Fillmore theorem [9]. Then $M \oplus W \in (QD)$ if and only if $M \oplus \lambda S^* \oplus W \in (QD)$.

Before stating our theorem, we recall the following notation and theorems from [21].

2.2. Let $m \ge 1$ and define X to be the Cartesian product of $[0,1]$ with itself $(m+1)$ times, so that $\vec{x} \in X$ is written $\vec{x} = (x_0, x_1, \dots, x_m)$. Denote by $\mathcal{C}^r(X)$ the normed space of real-valued functions on X with the supremum norm.

2.3. THEOREM [21]. *If u is a function in $\mathcal{C}^r(X)$ that vanishes identically on the coordinate faces $x_j = 0$ $(0 \le j \le m)$ of X, then the weighted shift V_u with weights $\{u(v_i, v_{i+1}, \dots, v_{i+m})\}_{i=-\infty}^{\infty}$ is in the C^*-algebra generated by the weighted shift V with weights $\{v_i\}_{i=-\infty}^{\infty}$ $(0 \le v_i \le 1)$.*

2.4. THEOREM [21]. *If $T \in (QD)$, then $C^*(T) \subseteq (QD)$.*

We are now in a position to state and prove our theorem.

2.5. THEOREM. *Let W be a norm-one unilateral weighted shift with positive weights, and let N be a normal operator such that $\sigma(N) \supseteq \sigma(W)$. Then $N \oplus W \in (QD)$ if and only if either $W \in (QD)$ or $\lambda S^* \oplus W \in (QD)$ is block-balanced for some $r_e(W) \le \lambda \le \mathrm{spr}(W)$.*

Proof. We first consider the case where $\sigma(N)=\{z \,|\, |z|\leq \mathrm{spr}(W)\}$.

If $W\in(QD)$, clearly $N\oplus W\in(QD)$. Moreover, assume there exists λ, $r_e(W)\leq\lambda\leq\mathrm{spr}(W)$, such that $\lambda S^*\oplus W$ is block-balanced. From above, $N\oplus W\in(QD)$ if and only if $M\oplus\lambda S^*\oplus W\in(QD)$. Since $M\in(QD)$ is clear and $\lambda S^*\oplus W\in(QD)$ by Smucker's theorem (1.2), we have $N\oplus W\in(QD)$.

To prove the converse, we can assume that $\delta\leq\omega_n\leq 1$ for $n\geq 1$ and some $\delta>0$. Also, there exist $\epsilon>0$ and $m>1$ such that

$$\|(\omega_n,\omega_{n+1},\ldots,\omega_{n+m})-(\lambda,\lambda,\ldots,\lambda)\|_\infty\geq\epsilon \quad \text{for all } n\geq 1 \text{ and } 0\leq\lambda\leq\mathrm{spr}(W).$$

Otherwise, for $k\geq 1$, let $\epsilon=1/k$, $m=k$, and choose $0\leq\lambda_k\leq\mathrm{spr}(W)$ and $n_k\geq 1$ such that $\|(\omega_{n_k},\omega_{n_k+1},\ldots,\omega_{n_k+k})-(\lambda_k,\lambda_k,\ldots,\lambda_k)\|_\infty<1/k$. If λ is any accumulation point of $\{\lambda_k\}_{k=1}^\infty$, it is easy to check that $\lambda S^*\oplus W$ is block-balanced. (In this case, from the definition of $r_e(W)$, it is not hard to verify that in fact we would have $\lambda\geq r_e(W)$.) Now the proof follows the lines of Smucker's.

Let Y be the closure in X of $\{(\omega_n,\omega_{n+1},\ldots,\omega_{n+m})\}_{n=1}^\infty$, and let Z be the closure in X of the set consisting first of the points $(\lambda,\lambda,\ldots,\lambda)$, $0\leq\lambda\leq\mathrm{spr}(W)$, and secondly of those points $\vec{x}=(x_0,x_1,\ldots,x_m)$ in X for which $x_j=0$ for at least one j, $0\leq j\leq m$. Then $\mathrm{dist}(Y,Z)\geq\min(\epsilon,\delta)>0$ and so $Y\cap Z=\emptyset$. By Urysohn's lemma, there exists $u\in\mathcal{C}^r(X)$ with $u(Y)=1$ and $u(Z)=0$. It follows that

$$\begin{aligned}(M\oplus(\mathrm{spr}(W)\cdot S^*\oplus W))_u &= (M_u)\oplus(\mathrm{spr}(W)\cdot S^*\oplus W)_u\\ &= \left[\bigoplus_{n=1}^{\infty} r_n(B_u)\right]\oplus(\mathrm{spr}(W)\cdot S^*\oplus W)_u.\end{aligned}$$

Now $(B_u)=0$ and $(\mathrm{spr}(W)\cdot S^*\oplus W)_u$ is a weighted shift V, with weights $v_n=0$ if $n<-m$ and $v_n=1$ if $n>0$. By changing (if necessary) $v_{-m},v_{-m+1},\ldots,v_0$ to 0 (a compact perturbation of V), we see that

$$(M\oplus(\mathrm{spr}(W)\cdot S^*\oplus W))_u=0\oplus(0\oplus S)+K.$$

But $0\oplus(0\oplus S)+K\in(QD)$ if and only if $S\in(QD)$, which is obviously false by index considerations. We conclude from Theorem 2.4 that

$$(M\oplus(\mathrm{spr}(W)\cdot S^*\oplus W))\notin(QD),$$

hence $N\oplus W\notin(QD)$.

If $\sigma(N)\supsetneqq\sigma(W)$, we simply remark the following. Since

$$\mathrm{spr}(W)=\lim_{n\to\infty}\sup_{i\geq 1}|\omega_i\omega_{i+1}\cdots\omega_{i+(n-1)}|^{1/n},$$

$\lambda S^*\oplus W$ is not block-balanced for any $\lambda>\mathrm{spr}(W)$. Thus the proof goes through as above for any normal satisfying $\sigma(N)=D_r=\{z\,|\,|z|\leq r\}$ with $r>\mathrm{spr}(W)$. If $\sigma(N_{\mathrm{spr}(W)})=D_{\mathrm{spr}(W)}\subsetneqq\sigma(N)\subsetneqq D_r=\sigma(N_r)$ for normals N_r and $N_{\mathrm{spr}(W)}$, then if $N\oplus W\notin(QD)$ it follows that $N_{\mathrm{spr}(W)}\oplus W\notin(QD)$ (because $N\oplus W\cong_a(N_{\mathrm{spr}(W)}\oplus W)\oplus N$), and so the conclusion of the theorem still holds. If $N\oplus W\in(QD)$, then clearly $N_r\oplus W\in(QD)$ and again the conclusion of the theorem holds. $\square$

2.6. Conjecture 2 of [13] is the following.

If $T \in \mathcal{B}(\mathcal{H})$, N is normal, and $\sigma(N)$ is a spectral set for T, then $T \oplus N \in (QD)$. Moreover, if $\sigma(N)$ is connected and contains the origin, then $T \oplus N$ is the limit of block-diagonal nilpotents.

Using the above theorem, it is now not difficult to produce a counterexample to this conjecture, where T is a contractive weighted shift and $\sigma(N)$ is the unit disc. In particular, the operator T_2 of Section 3.1 will be shown to do the job (cf. §3.2).

For the following corollary, we need an extension of Smucker's theorem to n-tuples of weighted shifts.

2.7. COROLLARY. *Let $W_1, W_2, \ldots, W_n$ be unilateral forward weighted shifts with (positive) weights $\{\omega_{k,j}\}_{j\geq 1}$ for W_k, $1 \leq k \leq n$, and let N be a normal operator such that $\sigma(N) \supseteq \sigma(W)$ where $W = \bigoplus_{k=1}^{n} W_k$. Assume furthermore that for each k $(1 \leq k \leq n)$ we have $N \oplus W_k \notin (QD)$. Then $N \oplus W \notin (QD)$.*

Proof. The proof is identical to that above if we replace Y by the closure of $\{(\omega_{k,r}, \omega_{k,r+1}, \ldots, \omega_{k,r+m})\}_{r\geq 1,\, 1\leq k\leq n}$ for the appropriate m, also chosen as above. First consider $\sigma(N) = \sigma(W)$. Then we can show as before that $N \oplus W \in (QD)$ if and only if $\bigoplus_{k=1}^{n} (M \oplus \mathrm{spr}(W) \cdot S^* \oplus W_k) \in (QD)$. But we find that $(\bigoplus_{k=1}^{n} (M \oplus \mathrm{spr}(W) \cdot S^* \oplus W_k))_u$ is a finite-rank perturbation of $0 \oplus S^{(n)}$, which is not quasidiagonal as $S^{(n)}$ is not quasidiagonal by index considerations. The case $\sigma(N) \supsetneq \sigma(W)$ is also handled as before. □

2.8. REMARK. Using the results of Sivaramakrishnan [19], we can extend the characterization of Theorem 2.5 to arbitrary direct sums of shifts $W = \bigoplus_{i=1}^{n} W_i$. However, the formulation is long, and since the original motivation for obtaining the above results was to show that the operator T_n of Example 3.1 actually does what it claims to do, we omit this general case. Nevertheless we shall include the case of a single bilateral shift, since it helps lead to a conjecture of what is perhaps "really" going on.

2.9. In the case of a bilateral shift W with weights $\{\omega_n\}_{n\in\mathbb{Z}}$, we change the weight ω_0 to 0 allowing us to write W as $W_1^* \oplus W_2$, where W_1 (resp. W_2) is a forward unilateral shift acting on

$$\mathcal{H}_1 = \overline{\mathrm{span}}\{e_{-n}\}_{n=0}^{\infty} \quad (\text{resp. } \mathcal{H}_2 = \overline{\mathrm{span}}\{e_n\}_{n=1}^{\infty}).$$

The reader is left to verify that the above arguments produce the following result.

2.10. PROPOSITION. *Let $W = W_1^* \oplus W_2$ be a bilateral shift with weight $\omega_0 = 0$ and let N be a normal operator with $\sigma(N) \supseteq \sigma_e(W)$. Then $N \oplus W \in (QD)$ if and only if either*

(1) $W \in (QD)$;

(2) $W_1 \in (QD)$, $W, W_2 \notin (QD)$, *and* $N \oplus W_2 \in (QD)$;

(3) $W_2 \in (QD)$, $W, W_1 \notin (QD)$, *and* $N \oplus W_1 \in (QD)$; *or*
(4) $W_1, W_2, W \notin (QD)$ *and both* $N \oplus W_1$ *and* $N \oplus W_2 \in (QD)$.

The bilateral and unilateral cases will share a common formulation based on the following observation. The proof uses a technique due to Berg [4]. We use a more general version as found in [10]. The reader will find several closely related results in [14].

2.11. PROPOSITION. *Let W be a unilateral weighted shift and assume there exists $0 \le \lambda \le \operatorname{spr}(W)$ such that $\lambda S^* \oplus W$ is block-balanced. Then for each $\epsilon > 0$ there exist U unitary, V an essentially normal weighted shift, and F block-diagonal (all depending upon ϵ) such that $W - U(V \oplus F)U^* \in \mathcal{K}(\mathcal{H})$ and $\|W - U(V \oplus F)U^*\| < \epsilon$.*

Proof. Let $\epsilon > 0$ and choose $M > 0$ an integer such that $\pi/M < \epsilon/2$. For each integer $n \ge M$ define $k_n = 2^{M+n}$, and choose sequences

$$\vec{\omega}_{i_n} = (\omega_{i_n+1}, \omega_{i_n+2}, \ldots, \omega_{i_n+k_n})$$

acting on orthogonal spaces such that $\|\vec{\omega}_{i_n} - \vec{\lambda}_n\| < \epsilon/2^n$, where $\vec{\lambda}_n = (\lambda, \lambda, \ldots, \lambda) \in \mathbf{R}^{k_n}$. By perturbing each $\vec{\omega}_{i_n}$ to $\vec{\lambda}_n$, we obtain a new shift Y with weights $\{y_i\}_{i=1}^{\infty}$ such that $W - Y$ is compact and $\|W - Y\| < \epsilon/2$.

As in the proof of [17, Thm. 2.2.11], we can apply the unitaries U_n from Berg's technique to the subspaces acted upon by

$$(y_{i_n+k_n/2+1}, y_{i_n+k_n/2+2}, \ldots, y_{i_n+k_n}) \text{ and } (y_{i_{n+1}+1}, y_{i_{n+1}+2}, \ldots, y_{i_{n+1}+k_n/2}).$$

Since the perturbation at the nth step is on the order of $\pi/2(k_n/2) < \epsilon/(2 \cdot 2^n)$ and since all the perturbations are taking place on orthogonal spaces, the result here of applying Berg's technique simultaneously to all the subspaces is an operator T which is a *compact* perturbation of norm less than $\epsilon/2$. Let U be the unitary operator which acts as U_n when restricted to the domain (= the range) of U_n and acts as the identity operator on the orthogonal complement of the closed span of these domains. Then we have $\|Y - UTU^*\| < \epsilon/2$ and $Y - UTU^* \in \mathcal{K}(\mathcal{H})$.

In fact, a closer examination of T reveals that it is of the form $V \oplus F$, where V is the unilateral weighted shift with weights

$$\{y_1, y_2, \ldots, y_{i_1+1}, y_{i_1+2}, \ldots, y_{i_1+k_1/2}, y_{i_2+1}, y_{i_2+2}, \ldots, y_{i_2+k_2/2}, y_{i_3+1}, \ldots\}$$

and $F = \bigoplus_{n=1}^{\infty} F_n$ is block diagonal, with F_n being the weighted cycle with weights $\{y_{i_n+k_n/2+1}, \ldots, y_{i_{n+1}}\}$. Then

$$W - U(V \oplus F)U^* = (Y - U(V \oplus F)U^*) + (W - Y) \in \mathcal{K}(\mathcal{H})$$

and

$$\|W - U(V \oplus F)U^*\| \le \|Y - U(V \oplus F)U^*\| + \|W - Y\| < \epsilon.$$

Note that V is essentially normal since $y_{i_n+r} = \lambda$ for all $n \ge 1$ and $1 \le r \le k_n/2$. In fact, we have shown that V is a finite rank perturbation of λS. □

Thus we may rewrite Theorem 2.5 and Proposition 2.10 as follows.

2.12. THEOREM. *Let W be a (unilateral or bilateral) weighted shift and let N be a normal operator such that $\sigma_e(N) \supseteq \sigma(W)$. Then $N \oplus W \in (QD)$ if and only if W is the limit of operators of the form $U^*(V \oplus F)U$, where U is unitary, V is essentially normal, and F is block-diagonal.*

Proof. The "only if" implication is clear from the statements of Theorem 2.5, Proposition 2.10, and Proposition 2.11.

The converse follows from the fact that, given that for $n \geq 1$ we can find U_n unitary, V_n essentially normal, and F_n block-diagonal such that

$$\|W - U_n^*(V_n \oplus F_n)U_n\| < 1/n,$$

we then have $N \oplus U_n^*(V_n \oplus F_n)U \in (QD)$ provided that $N \oplus V_n \in (QD)$. But $N \oplus V_n$ is essentially normal with zero index on its semi-Fredholm domain, and thus $N \oplus V_n \cong_a N + K$ (K compact) so $N \oplus V_n \in (QD)$. Since $N \oplus W$ is approximable by quasidiagonals and (QD) is closed, $N \oplus W \in (QD)$.

REMARK. We have implicitly assumed $\sigma(V_n) \subseteq \sigma_e(N)$. If this is not the case, then by the upper semicontinuity of the spectrum we may assume that $\sigma(V_n)$ lies in a small neighbourhood, say $\{\sigma_e(N)\}_{1/n} = \{z \mid \operatorname{dist}(z, \sigma_e(N)) < 1/n\}$. By replacing N by a normal N_n with $\sigma_e(N_n)$ containing this neighbourhood, the result follows.

2.13. We note that the converse implication above has nothing to do with W being a weighted shift, and hence works just as well for any $T \in \mathcal{B}(\mathcal{H})$ that can be approximated by operators of the form $U^*(V \oplus F)U$ as above. It seems reasonable to ask whether this condition is also necessary.

QUESTION. Let $T \in \mathcal{B}(\mathcal{H})$ and let N be a normal operator such that $\sigma(T) \subseteq \sigma_e(N)$. Does $N \oplus T \in (QD)$ imply that T is approximable by operators of the form $U^*(V \oplus F)U$, U unitary, V essentially normal, and F block-diagonal? If so, can we always make the difference $T - U^*(V \oplus F)U$ compact? It is not too difficult to verify that this is the case when T is the direct sum of weighted shifts (unilateral or bilateral).

3. The Example

3.0. As mentioned above, one of the motivations for the previous results was to verify the validity of the following set of examples for each $n \geq 1$ of a biquasitriangular operator T_n such that T_n^k is quasidiagonal if and only if k is a multiple of n.

3.1. Temporarily fix $n > 1$, the case $n = 1$ being trivial. We shall suppress the subscript n in what follows.

Let $\{p_i\}_{i=1}^{\infty}$ be an enumeration of the prime numbers. Let W be the periodic unilateral weighted shift with weights $\{\omega_i\}_{i=1}^{\infty}$, where $\omega_i = 1/p_i$ for $1 \leq i \leq n$ and $\omega_{i+n} = \omega_i$ for $i \geq 1$. let N be a normal operator with $\sigma(N) = \{z \mid |z| \leq 1\} \supseteq \sigma(W)$. For $1 \leq k < n$ we have $W^k \cong \bigoplus_{i=1}^{k} V_i$, where V_i is a weighted shift with weights

$$\{\omega_i\omega_{i+1}\cdots\omega_{i+(k-1)}, \omega_{i+k}\omega_{i+k+1}\cdots\omega_{i+2k-1}, \omega_{i+2k}\omega_{i+2k+1}\cdots\omega_{i+3k-1}, \cdots\}.$$

It is not hard to verify that V_i is periodic with period at most n and that the first two weights of V_i are different for each i. As such, $V_1, V_2, V_3, \ldots, V_k$ and N satisfy the conditions of Corollary 2.7.

It follows that $(N\oplus W)^k = N^k \oplus W^k \cong_a N\oplus(\bigoplus_{i=1}^k V_i)$ is not quasidiagonal for $1 \le k < n$. However,

$$(N\oplus W)^n = N^n \oplus W^n \cong_a N\oplus[(\omega_1\omega_2\cdots\omega_n)S^n]$$
$$\cong_a N^{(n)} \oplus (\omega_1\cdots\omega_n)S^{(n)}.$$

Since each $N\oplus(\omega_1\cdots\omega_n)S$ is quasidiagonal (by Theorem 2.5, for example) we conclude that $(N\oplus W)^n \in (QD)$.

Clearly, $T = N\oplus W$ is biquasitriangular and satisfies the conditions for $1 \le k \le n$. The case $k > n$ is handled similarly. □

3.2. Assume $p_1 = 2$ and $p_2 = 3$ and let T_2 be as above, namely: $T_2 = N\oplus W$, where W is the periodic unilateral weighted shift with weights $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \ldots\}$ and N is normal with $\sigma(N) = \{z \mid |z| \le 1\}$. Then $\|W\| = \frac{1}{2} < 1$, and so it follows from the von Neumann inequality that $\sigma(N)$ is a spectral set for W. Moreover, $\sigma(N)$ is connected and contains the origin.

As we have seen above, $T_2 = N\oplus W \notin (QD)$ and so, *a fortiori*, $N\oplus W$ is not the limit of block-diagonal nilpotents either. Thus T_2 is a counterexample to Conjecture 2 of [13] (cf. §2.6).

Indeed, if $r > 0$ and if M and N_r are normal operators with $\sigma(M) \subseteq \sigma(N_r) = D_r$, then

$$\operatorname{dist}[M\oplus W, (QD)] \ge \operatorname{dist}[N_r\oplus W, (QD)] = \epsilon_r > 0$$

(cf. the proof of Theorem 2.5). In particular, $M\oplus W \notin (QD)$ for any normal operator M.

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