Discrete Quasiconformal Groups with Small Dilatation

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1. Introduction

In this paper we consider discrete uniformly quasiconformal groups which act on $\bar{\mathbf{R}}^n$, $n \geq 3$. We provide examples of such groups which have small dilatation, yet are not quasiconformally conjugate to conformal (i.e., Möbius groups) on $\bar{\mathbf{R}}^n$. Examples of both elementary discrete groups (limit set of at most two points) and non-elementary (an uncountable, perfect limit set) discrete groups are furnished.

A natural way to construct a quasiconformal group acting on $\bar{\mathbf{R}}^n$ is to conjugate a conformal group by a quasiconformal mapping. Indeed, Gehring and Palka [5] first raised the question whether every uniformly quasiconformal group might not be of this form. This question was answered in the affirmative by Sullivan [10] and Tukia [11] for groups acting on subsets of $\bar{\mathbf{R}}^2$. Hinkkanen [6; 7] has shown that if G is a quasisymmetric (i.e., a 1-dimensional quasiconformal) group G acting on $\mathbf{R}$, then there is a quasisymmetric function f such that $f^{-1} \circ G \circ f$ is a group of linear functions. Later, Tukia [12] constructed for each $n \geq 3$ a quasiconformal group which is not isomorphic to, and hence not quasiconformally conjugate to, a Möbius group. Methods used by Tukia were later modified by Martin [8] to yield discrete quasiconformal groups which are not quasiconformally conjugate to a Möbius group. Further examples were provided by Gehring and Martin [3] as well as by Freedman and Skora [2]. As each of these groups possesses a large dilatation, we asked whether a uniformly quasiconformal group with sufficiently small dilatation must be quasiconformally conjugate to a conformal group. The answer is no; by a modification of Tukia's methods we showed in [9] that for each $n \geq 3$ and $K > 1$ there is a K-quasiconformal group acting on $\bar{\mathbf{R}}^n$ which is not the quasiconformal conjugate of a Möbius group. Our examples, like Tukia's, were not discrete, so the question remained whether such examples existed in the category of discrete quasiconformal groups. While the methods used by Martin to extract elementary discrete subgroups from Tukia's group are entirely applicable to our groups of small dilatation, the modifications he used to obtain non-elementary discrete

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groups result in a growth of dilatation. Our main result here settles the question: We demonstrate that it is possible to construct, for each $n \geq 3$ and $K > 1$, a non-elementary discrete K-quasiconformal group which is not quasiconformally conjugate to a Möbius group. Thus we establish, in the case of uniformly quasiconformal groups, that the size of the dilatation alone fails to determine whether the group is quasiconformally conjugate to a Möbius group.

2. Notation and Definitions

For $n \geq 3$ we let $\mathbf{R}^n$ denote Euclidean n-space; $\bar{\mathbf{R}}^n$ is its one-point compactification, $\bar{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. We let $(e_1, e_2, \ldots, e_n)$ denote the standard basis of $\mathbf{R}^n$. We write $\mathbf{R}^2$ and $\mathbf{C}$ interchangeably, depending on whether complex notation is more convenient. We let Ω_0 denote the upper half-plane, $\Omega_0 = \{z \in \mathbf{C}: \operatorname{Im} z > 0\}$; the lower half-plane is Ω_1. The notation $B^n(x, r)$ indicates the open ball $\{y \in \mathbf{R}^n: |x-y| < r\}$, and we abbreviate $B^n(r) = B^n(0, r)$. Similarly, $S^{n-1}(x, r) = \{y \in \mathbf{R}^n: |x-y| = r\}$, while $S^{n-1}(r) = S^{n-1}(0, r)$. By $A(a,b)$ we mean the closed annulus $\{y \in \mathbf{R}^n: a \leq |y| \leq b\}$, where $0 < a < b < \infty$. For a set $A \subset \bar{\mathbf{R}}^n$, ∂A denotes the boundary of A and $\bar{A}$ denotes the closure of A, both taken in $\bar{\mathbf{R}}^n$. We use $\operatorname{dist}(A, B)$ to denote the Euclidean distance between A and B in $\mathbf{R}^n$, and write $\operatorname{dia}(A)$ to indicate the diameter of $A \subset \mathbf{R}^n$.

Let X and Y be metric spaces. An embedding $f: X \to Y$ is called *L-bilipschitz* if there is an $L \geq 1$ such that

$$\frac{|x-y|}{L} \leq |f(x) - f(y)| \leq L|x-y| \quad \text{for all } x, y \in X,$$

where we use $|x-y|$ to denote the distance from x to y in an arbitrary metric space. Let $\eta: [0, \infty) \to [0, \infty)$ be a homeomorphism. An embedding $f: X \to Y$ is *η-quasisymmetric* (η-QS) if

$$|f(a) - f(c)| \leq \eta(t) |f(b) - f(c)|$$

whenever $a, b, c \in X$ satisfy $|a-c| \leq t|b-c|$. An embedding is *weakly H-quasisymmetric* (*weakly H*-QS), where $H \geq 1$ is a constant, if it is merely true that we have $|f(a) - f(c)| \leq H|f(b) - f(c)|$ whenever $|a-c| \leq |b-c|$. If we wish to be less specific, we say that f is *quasisymmetric,* or *weakly quasisymmetric,* respectively. As in [15], if $s > 0$ we say that f is *s-quasisymmetric* if f is quasisymmetric and enjoys the following property: if $t \leq 1/s$ and $a, b, c \in X$ with $|a-c| \leq t|b-c|$, then $|f(a) - f(c)| \leq (t+s)|f(b) - f(c)|$. This implies that f is η-QS for some η in a neighborhood of the identity in the space of homeomorphisms $\eta: [0, \infty) \to [0, \infty)$; namely,

$$N(\mathrm{id}, s) = \{\eta: |\eta(t) - t| \leq s \text{ for } 0 \leq t \leq 1/s\}.$$

In fact, this was an earlier definition given in [14] for s-QS mappings. We denote both the identity mapping of a space to itself and the inclusion mapping of X into Y by id. We say f is a *similarity* if f is 0-QS, that is, if f is

η-QS with $\eta \doteq \text{id}$. Every L-bilipschitz embedding is s-QS with s given by $s = (L^2 - 1)^{1/2}$.

If $U \subset \bar{\mathbf{R}}^n$ is open and G is a group of self-homeomorphisms of U with the feature that each $g \in G$ is K-QC, for some fixed K, we say that G is a *K-quasiconformal* (K-QC) *group*. A *uniformly quasiconformal group* is one which is K-QC for some K. If each $g \in G$ is the restriction of a Möbius transformation to U, we call G a *Möbius group*. If each $g \in G$ is L-bilipschitz, we call G an *L-Lipschitz group*.

Let G be a group of self-homeomorphisms of U. We call G a *discrete group* on U if G contains no infinite sequence of distinct elements which converges uniformly on compact subsets of U to an element of G. We say the group G is *discontinuous* at a point $x \in U$ if there is a neighborhood V of x in U so that $g(V) \cap V = \emptyset$ for all but finitely many $g \in G$. We denote by $O(G)$ the set of all points x in U at which G is discontinuous and call $L(G) = \bar{U} \setminus O(G)$ the *limit set* of G. We say G is a *discontinuous group* if $O(G)$ is non-empty. If G is a discontinuous group, then G is discrete [1].

Suppose that U is a simply connected region in $\mathbf{C}$ which is not the whole plane. The *hyperbolic metric* h_U in U is defined by

$$h_U(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \lambda_U(z)\, |dz|,$$

where λ_U is the *Poincaré density* in U, and the infimum extends over all rectifiable paths γ joining z_1 and z_2 in U. The density λ_U is given by

$$\lambda_U(f(z))|f'(z)| = \frac{1}{\operatorname{Im} z},$$

where f is any conformal mapping of Ω_0 onto U.

If U is a proper subdomain of $\mathbf{R}^n$ then the *quasihyperbolic metric* k_U of U is defined using the density ρ_U, where $\rho_U(x) = 1/\text{dist}(x, \partial U)$. For $x, y \in U$ we let

$$k_U(x, y) = \inf_{\gamma} \int_{\gamma} \rho_U(\zeta)\, |d\zeta|$$

with the infimum being extended over all rectifiable paths γ joining x and y in U. In case $U = \Omega_0$ or Ω_1, we have $k_U = h_U$.

Let D be a proper subdomain of $\mathbf{R}^n$. An embedding $f: D \to \mathbf{R}^n$ is called *L-quasihyperbolic* (L-QH) if $f(D) \neq \mathbf{R}^n$ and if f is L-bilipschitz with respect to the quasihyperbolic metrics of D and $f(D)$.

3. Preliminary Results

In order to produce the uniformly quasiconformal discrete groups we desire, we combine the methods of construction of Martin [8], which yield discrete groups, and of ourselves [9], which allow control of the dilatation of the groups. This is a straightforward procedure for the construction of the elementary groups, but some different results are needed to produce the non-elementary groups. We establish the needed results, similar to [8, Thms. 2.6–2.7], in this section.

In what follows, $0<a<b<c<d<\infty$.

LEMMA 1. *Let $f: A(a,b)\cup A(c,d)\to \mathbf{C}$ be 1-bilipschitz with $f|_{A(a,b)}=\mathrm{id}$. Then $f=\mathrm{id}$.*

Proof. Given any $z\in A(c,d)$, choose $t\in(0,1)$ so that $w=tz\in S^1(b)$. Since $|w|+|w-z|=|z|$, we see that $|w|+|f(w)-f(z)|=|z|$. Then $|w-f(z)|=|z|-|w|$, which implies that $f(z)\in S^1(w,|z|-|w|)$. Since $|f(z)|=|z|$, $f(z)\in S^1(|z|)$ as well. Therefore, $f(z)\in S^1(w,|z|-|w|)\cap S^1(|z|)=\{z\}$. We see that $f=\mathrm{id}$. □

Now we give a slightly modified version of the so-called annulus theorem of [14]. In the following, "near A" means "in a neighborhood of A".

THEOREM 2. *Let $f: A(a,b)\cup A(c,d)\to\mathbf{C}$ be L-bilipschitz with*

$$\frac{\mathrm{dia}(fS^1(d))}{\mathrm{dia}(fS^1(a))}\le M^*,$$

and suppose that $f|_{A(a,b)}$ is the restriction of an isometry of $\mathbf{C}$. Then there is an L^*-bilipschitz embedding $g: A(a,d)\to\mathbf{C}$ such that $g=f$ near $\partial A(a,d)$. In addition, L^* may be selected to satisfy $L^*\to 1$ as $L\to 1$.

Proof. By composition with an isometry of $\mathbf{C}$ we may assume that $f|_{A(a,b)}=\mathrm{id}$. By [14, §5.9] there is an L_2-bilipschitz embedding $\phi(f): A(a,d)\to\mathbf{C}$ that agrees with f near $\partial A(a,d)$. Let $\hat\phi(f)=\phi(f)\circ\phi(\mathrm{id})^{-1}$. Then $\hat\phi(f)$ is $L^*=L_2^2$-bilipschitz, $\hat\phi(f)=\phi(f)$ near $\partial A(a,d)$, and $\hat\phi(\mathrm{id})=\mathrm{id}$. Since ϕ is continuous in the compact-open topology, so is $\hat\phi$. We need only show that $L^*\to 1$ as $L\to 1$. Suppose that $L^*\not\to 1$ as $L\to 1$. Then there is $\delta>0$, points z_n, w_n, and maps f_n which are L_n-bilipschitz embeddings with $L_n\to 1$ as $n\to\infty$, so that either

$$(*)\qquad \frac{|g_n(z_n)-g_n(w_n)|}{|z_n-w_n|}>1+\delta \quad\text{for all } n,$$

or

$$(**)\qquad \frac{|g_n(z_n)-g_n(w_n)|}{|z_n-w_n|}<1-\delta \quad\text{for all } n,$$

where $g_n=\hat\phi(f_n)$. Suppose first that (∗) holds. Then, by passing to subsequences, $z_n\to z$, $w_n\to w$, $f_n\to f$, and $g_n\to g=\hat\phi(f)$, with the mappings converging uniformly on compact subsets of $A(a,b)\cup A(c,d)$ and $A(a,d)$, respectively [13, Thms. 3.4–3.7]. Then f is 1-bilipschitz on $A(a,b)\cup A(c,d)$ with $f|_{A(a,b)}=\mathrm{id}$. By Lemma 1, $f=\mathrm{id}$, whence $g=\mathrm{id}$. Taking limits we see that (∗) implies that $|g(z)-g(w)|/|z-w|>1+\delta$. This clearly contradicts $g=\mathrm{id}$. A similar argument gives a contradiction if (∗∗) is assumed to hold. Thus it follows that $L^*\to 1$ as $L\to 1$. □

We shall show that an L-bilipschitz embedding with small L may be replaced by an isometry in a certain small region without altering L very much.

THEOREM 3. *Let $L_1 \geq 1$ and let $B = \bar{B}^2(d)$, where $d > 0$. There exist constants $L_2 \geq 1$ and $\delta > 0$ so that, for each L_1-bilipschitz embedding $f: B \to \mathbf{C}$ with $f(0) = 0$, there is an L_2-bilipschitz embedding $g: B \to \mathbf{C}$ such that*

(i) *$g = f$ near $S^1(d)$, and*
(ii) *$g|_{B^2(\delta)}$ is the restriction of an isometry of $\mathbf{C}$.*

Furthermore, we may choose L_2 so that $L_2 \to 1$ as $L_1 \to 1$.

Proof. Let $a = d/10L_1$, $b = d/8L_1$, and $c = 8d/10$. Now f is s-QS, where $s = (L_1^2 - 1)^{1/2}$. From [16, Thm. 3.1], there is an isometry $h: \mathbf{C} \to \mathbf{C}$ so that $\|h - f\|_B \leq \mathcal{H}(s)\,\mathrm{dia}(B)$, where $\mathcal{H}(s)$ is increasing in s and $\mathcal{H}(s) \to 0$ as $s \to 0$ or as $L_1 \to 1$. (By $\|h - f\|_B$ we mean $\sup\{|h(x) - f(x)| : x \in B\}$.)

Define f_1 on $A(a,b) \cup A(c,d)$ by

$$f_1(z) = \begin{cases} h(z), & z \in A(a,b), \\ f(z), & z \in A(c,d). \end{cases}$$

We claim that f_1 is L_1^*-bilipschitz, where $L_1^* \to 1$ as $L_1 \to 1$. To see this, we must check $|f_1(z) - f_1(w)|$ for $z \in A(a,b)$ and $w \in A(c,d)$; otherwise, there is nothing to prove. Thus

$$\begin{aligned} |f_1(z) - f_1(w)| &= |h(z) - f(w)| \\ &\leq |h(z) - h(w)| + |h(w) - f(w)| \\ &\leq |z - w| + 2d\mathcal{H}(s). \end{aligned}$$

In consequence,

$$\frac{|f_1(z) - f_1(w)|}{|z - w|} \leq 1 + \frac{2d\mathcal{H}(s)}{|z - w|}. \tag{3.1}$$

Since $d/2 \leq |c - b| \leq |z - w| \leq d + b \leq 2d$, we have

$$\frac{1}{2d} \leq \frac{1}{|z - w|} \leq \frac{2}{d}. \tag{3.2}$$

Using this in (3.1) gives us

$$\frac{|f_1(z) - f_1(w)|}{|z - w|} \leq 1 + 4\mathcal{H}(s). \tag{3.3}$$

By the triangle inequality, $|h(z) - f(w)| + |f(w) - h(w)| \geq |h(z) - h(w)|$; or, since h is an isometry, $|h(z) - f(w)| \geq |z - w| - |f(w) - h(w)|$. Write

$$\frac{|f_1(z) - f_1(w)|}{|z - w|} \geq 1 - \frac{|f(w) - h(w)|}{|z - w|}.$$

Now, from the above and (3.2) we obtain

$$\frac{|f_1(z) - f_1(w)|}{|z - w|} \geq 1 - 4\mathcal{H}(s). \tag{3.4}$$

Choose $L_1^* = \max\{1+4\mathcal{K}(s), \{1-4\mathcal{K}(s)\}^{-1}\}$. Note that $L_1^* \to 1$ as $s \to 0$, that is, as $L_1 \to 1$. From (3.3) and (3.4) the claim follows. We now appeal to Theorem 2 to obtain $g: \bar{B}^2(d) \to \mathbf{C}$, so that g is L_2-bilipschitz and

(i) $g = f_1 = f$ near $S^1(d)$, and

(ii) $g|_{S^1(a)} = f_1|_{S^1(a)}$ is the restriction of an isometry of $\mathbf{C}$.

Since $L_2 \to 1$ as $L_1^* \to 1$, we note that $L_2 \to 1$ as $L_1 \to 1$. Set $\delta = a$ and extend g to $B^2(\delta)$ by using the isometry, and we are done. □

4. Construction of the Groups

We give a brief outline of the constructions in [9, §§4, 5] and [8, §3], and then observe that together they allow construction of elementary discrete quasiconformal groups with small dilatation which are not quasiconformally conjugate to Möbius groups. The results from Section 3 are then incorporated to construct the non-elementary groups.

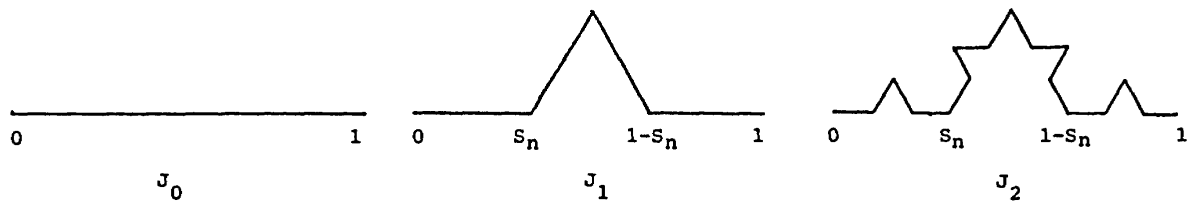

Figure 1

Fix $n \geq 1$. Define $s_n = \frac{1}{4}(1+3^{-n})$. We construct arcs $J_0, J_1, J_2, \ldots$ as shown in Figure 1 and take J^* as the limit of these arcs. Note that each arc, and hence the limit arc J^*, depends on the parameter n. (See [9] for details.) Also, $s_n J^*$ is a subarc of J^*, while J^* is a subarc of $(1/s_n)J^*$. There is a natural map $f: [0,1] \to J^*$, where f depends on n and satisfies $f(4^j x) = (1/s_n)^j f(x)$ for $j \geq 0$ and $0 \leq x \leq 4^{-j}$. Now we define

$$J = \bigcup_{j \geq 0} \left(\frac{1}{s_n}\right)^j (J^* \cup (-J^*)),$$

and extend f to $\mathbf{R}$ as follows:

$$f(\pm 4^j x) = \pm\left(\frac{1}{s_n}\right)^j f(x) \quad \text{for } x \in [0,1].$$

We retain the name f for the extended map. Note that $f(\mathbf{R}) = J$ and that f is normalized; that is, $f(0) = 0$ and $f(1) = 1$. We remark that the choice $n = 1$ yields Tukia's original construction of [11]. Set $\alpha_n = \log(s_n)/\log(\frac{1}{4})$. Then for ϵ in $(0,1)$ there is an $N \geq 1$ such that for all $n > N$ the map $f: \mathbf{R} \to \mathbf{C}$ associated with n satisfies

$$1-\epsilon \leq \frac{|f(x)-f(y)|}{|x-y|^{\alpha}n} \leq \frac{1}{1-\epsilon} \quad \text{for all } x, y \in \mathbf{R}.$$

Consequently, f is weakly $(1/(1-\epsilon)^2)$-QS.

Changing the parameter from n to H as in [9, Lemma 6] we observe that, given $H>1$, there exists a normalized weakly H-QS embedding $f: \mathbf{R} \to \mathbf{C}$ satisfying

$$\frac{1}{H} \leq \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \leq H \quad \text{for all } x, y \in \mathbf{R}, \tag{4.1}$$

where $\alpha = \alpha(H) \in (0,1)$ can be chosen to satisfy $\alpha \to 1$ as $H \to 1$. By [9, §4] we see that f can be extended to a K-quasiconformal homeomorphism F_f: $\mathbf{C} \to \mathbf{C}$ with the properties:

(i) there is an exponent $\alpha \in (0,1)$ and a constant $M \geq 1$ such that

$$\frac{|v|^{\alpha}}{M} \leq \operatorname{dist}(F(u+iv), J) \leq M|v|^{\alpha} \quad \text{for all } u, v \in \mathbf{R}; \tag{4.2}$$

(ii) there is a constant $L_1 \geq 1$ such that, for $z, w \in \Omega_j$ with $C_j = F_f(\Omega_j)$,

$$\frac{k_{\Omega_j}(z,w)}{L_1} \leq k_{C_j}(F_f(z), F_f(w)) \leq L_1 k_{\Omega_j}(z,w) \quad \text{for } j=0 \text{ or } 1. \tag{4.3}$$

Further, each of K, M, L_1, and α depend on H and may be chosen so that K, M, L_1, $\alpha \to 1$ as $H \to 1$. If we extend F_f to $\bar{\mathbf{C}}$ by setting $F_f(\infty) = \infty$ then the results which follow hold on $\bar{\mathbf{R}}^n$. For convenience, however, we usually state them on $\mathbf{R}^n$. For $n \geq 3$ and $x \in \mathbf{R}^n$ write $x = (z, y) \in \mathbf{R}^2 \times \mathbf{R}^{n-2}$. Define $F(x) = (F_f(z), y)$. Let $\hat{G} = \{x \mapsto x + a \mid a = (a_1, 0, a_3, \ldots, a_n),\ a_i \in \mathbf{R}\}$. Set $G = F \circ \hat{G} \circ F^{-1}$. Notice that G varies with H. We note that Tukia's group G_0 of [11] is a $G(H)$ for some large H.

We next present the desired examples of elementary discrete quasiconformal groups mentioned in the introduction.

THEOREM 4. *For each $n \geq 3$ and $K > 1$, there is an elementary K-quasiconformal discrete group $G' = G'(H)$ acting on $\bar{\mathbf{R}}^n$ such that for no quasiconformal $h: \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}^n$ is $h \circ G' \circ h^{-1}$ a Möbius group.*

Proof. Let t_{e_j} be the mapping of $\bar{\mathbf{R}}^n$ given by $t(x) = x + e_j$. Let T' be the group generated by $\{t_{e_j}: j = 1, 3, 4, \ldots, n\}$. Let $G'(H)$ be the group $G' = F \circ T' \circ F^{-1}$. Then G' is a discrete subgroup of rank $n-1$ in G. By [9, Thm. 13], G and hence G' are K-quasiconformal, where $K \to 1$ as $H \to 1$. By [8, Thm. 3.8], G' is not quasiconformally conjugate to a Möbius group. All the mappings in $G' \setminus \{\mathrm{id}\}$ fix only ∞, and the order of each $g \in G' \setminus \{\mathrm{id}\}$ is infinite. Therefore, [3, Thm. 5.10] tells us that the limit set $L(G') = \{\infty\}$, and consequently G' is elementary. We remark that any discrete subgroup of rank $n-1$ of G could be used instead of G'. □

As in [8], we now wish to alter F_f so that it is conformal in a neighborhood (whose size depends only on H) of each point of $\{m+i: m\in \mathbf{Z}\}$, yet remains unchanged outside a slightly larger neighborhood. Moreover, we require that the dilatation of the resulting mapping remain controlled.

THEOREM 5. *For $1<H<5/4$ there are constants $\delta=\delta(H)>0$ and $L'=L'(H)\geq 1$ and a mapping $F^*: \mathbf{C}\to\mathbf{C}$ with the following properties:*

(i) *F^* is L'-QH in Ω_j, $j=0$ or 1;*

(ii) *$F^*|_{\Omega_0\setminus\bigcup_{m\in\mathbf{Z}}B^2(m+i,H-1)}=F_f|_{\Omega_0\setminus\bigcup_{m\in\mathbf{Z}}B^2(m+i,H-1)}$; and*

(iii) *for every $m\in\mathbf{Z}$, $F^*|_{B^2(m+i,\delta)}$ is the restriction of an isometry of $\mathbf{C}$.*

Moreover, we may select L' so that $L'\to 1$ as $H\to 1$.

Proof. For any $m\in\mathbf{Z}$, let $z_m=m+i$ and let $B_m=B^2(z_m,H-1)$. It suffices to find $\delta>0$ and $F_m:\Omega_0\to\mathbf{C}$ such that:

(i) F_m is L'-quasihyperbolic in Ω_0 for some L' depending on H,

(ii) $F_m|_{\Omega_0\setminus B_m}=F_f|_{\Omega_0\setminus B_m}$, and

(iii) $F_m|_{B^2(z_m,\delta)}$ is the restriction of an isometry of $\mathbf{C}$,

where $L'=L'(H)$ may be chosen to satisfy $L'\to 1$ and $H\to 1$.

We may then find F_m for each $m\in\mathbf{Z}$ and replace F_f with F_m in each B_m to obtain F^*. Then

$$F^*(\Omega_j)=F_f(\Omega_j)=C_j,$$

$F^*=F_f$ outside the set $\bigcup_{m\in\mathbf{Z}}B^2(m+i,H-1)$, and F^* is locally L'-QH in Ω_0 (and in Ω_1 since $F^*=F_f$ in $\bar{\Omega}_1$) with $L'\to 1$ and $H\to 1$. Then, by [14, Lemma 6.21], $F^*: \mathbf{C}\to\mathbf{C}$ is L'-QH in Q_j, $j=0$ or 1.

We now consider $F_f|_{\Omega_0}$. By (4.3), F_f is L_1-QH in Ω_0 with $L_1\to 1$ as $H\to 1$. Let $\Omega_0^*=\{z-i: z\in\Omega_0\}$ and define $G_m:\Omega_0^*\to\mathbf{C}$ by

$$G_m(z)=F_f(z+z_m)-F_f(z_m).$$

Then $G_m(0)=0$ and G_m is L_1-QH in Ω_0^*. It suffices to find $\hat{G}_m:\Omega_0^*\to\mathbf{C}$ with the properties

(i′) $\hat{G}_m$ is L_3-QH in Ω_0^*,

(ii′) $\hat{G}_m|_{\Omega_0^*\setminus B^2(H-1)}=G_m|_{\Omega_0^*\setminus B^2(H-1)}$, and

(iii′) $\hat{G}_m|_{B^2(\delta)}$ is the restriction of an isometry of $\mathbf{C}$,

for then we may set $F_m(z)=\hat{G}_m(z-z_m)+\hat{G}_m(z_m)$. Given a set $D\subset\Omega_0^*$, let

$$r_{\Omega_0^*}(D)=\frac{\operatorname{dia}(D)}{\operatorname{dist}(D,\partial\Omega_0^*)}.$$

Let $A=\bar{B}^2(H-1)$. Then

$$r_{\Omega_0^*}(A)=\frac{2(H-1)}{2-H}=\mu,$$

where $\mu=\mu(H)\to 0$ as $H\to 1$. By [14, Thm. 6.5] there is $c_1(\mu)$ such that

$$\frac{|a-b|}{c(2-H)}\leq k_{\Omega_0^*}(a,b)\leq c_1\frac{|a-b|}{2-H} \tag{4.4}$$

for all $a, b \in A$ with $a \neq b$. Also, $c_1 \to 1$ as $\mu \to 0$. We observe next that

$$\operatorname{dist}(0, \partial G_m(\Omega_0^*)) = \operatorname{dist}(z_m, J);$$

then an application of (4.2) yields $M^{-1} \leq \operatorname{dist}(z_m, J) \leq M$. Let

$$t = HM\left[\exp\left(L_1 c_1 \frac{H-1}{2-H}\right) - 1\right],$$

and let $B = B^2(t)$. We claim that $G_m(A) \subset B$. Suppose $z \in S^1(H-1)$. By (4.4) we get $k_{\Omega_0^*}(0,z) \leq c_1[(H-1)/(2-H)]$. Since G_m is L_1-QH, the above gives

$$k_{G_m(\Omega_0^*)}(0, G_m(z)) \leq L_1 k_{\Omega_0^*}(0,z) \leq c_1 L_1 \frac{H-1}{2-H}.$$

From [4, Thm. 1.2] we know that

$$k_{G_m(\Omega_0^*)}(0, G_m(z)) \geq \log\left(1 + \frac{|G_m(z)|}{M}\right).$$

Thus $|G_m(z)| \leq M[\exp(L_1 c_1\{H-1\}/(2-H)) - 1] = t/H < t$, and we conclude that $G_m(A) \subset B$. Observe that

$$r_{G_m(\Omega_0^*)}(B) \leq \frac{t}{M^{-1}-t} = \mu^*,$$

and that $\mu^* = \mu^*(H)$ satisfies $\mu^* \to 0$ as $H \to 1$. Thus we find $c_2(\mu^*)$ such that

$$\frac{|a-b|}{c_2(M-t)} \leq k_{G_m(\Omega_0^*)}(a,b) \leq c_2 \frac{|a-b|}{M^{-1}-t} \tag{4.5}$$

for all $a \neq b \in B$. Consider $G_m|_A$. For $a, b \in A$ we have $G_m(a)$ and $G_m(b)$ in B. If we combine (4.5) with the fact that G_m is L_1-QH, we discover

$$\frac{M^{-1}-t}{c_2 L_1} k_{\Omega_0^*}(a,b) \leq |G_m(a) - G_m(b)| \leq c_2 L_1 (M-t) k_{\Omega_0^*}(a,b).$$

We may use (4.4) in the above to see that $G_m|_A$ is L_2-bilipschitz, where

$$L_2 = \max\left\{c_1 c_2 L_1 \frac{M-t}{2-H}, c_1 c_2 L_1 \frac{2-H}{M^{-1}-t}\right\}.$$

Note that A and B depend on the constant H but not on z_m. Now apply Theorem 3 with $d = H-1$ to find L_3, $\delta > 0$, and $\hat{G}_m : A \to \mathbf{C}$ so that $\hat{G}_m = G_m$ near $S^1(H-1)$ and so that $\hat{G}_m|_{B^2(\delta)}$ is an isometry. Recall that $L_3 \to 1$ as $L_2 \to 1$, that is, as $H \to 1$. It remains only to show that $\hat{G}_m$ is L'-QH in Ω_0^*. Because $\hat{G}_m = G_m$ in $\Omega_0^* \backslash B^2(H-1)$, we may take $L' = L_1$ there. In $B^2(H-1)$, use (4.4) and (4.5) together with the fact that $\hat{G}_m$ is L_3-bilipschitz to write

$$\begin{aligned}\frac{2-H}{c_1 c_2 L_3 (M-t)} k_{\Omega_0^*}(a,b) &\leq k_{\hat{G}_m(\Omega_0^*)}(\hat{G}_m(a), \hat{G}_m(b)) \\ &\leq c_1 c_2 L_3 \frac{2-H}{M^{-1}-t} k_{\Omega_0^*}(a,b).\end{aligned}$$

So select

$$L' = \max\left\{L_1, \left(\frac{2-H}{c_1c_2L_3(M-t)}\right)^{-1}, c_1c_2L_3\frac{2-H}{M^{-1}-t}\right\}$$

to see that $\hat{G}_m$ is locally L'-QH with $L' \to 1$ as $H \to 1$. Finally, [14, Lemma 6.21] gives that $\hat{G}_m$ is L'-QH as desired. □

REMARK. Since F^* is L'-QH for $z \in \mathbf{C} \setminus \mathbf{R}$ the dilatation of F^* is bounded by $(L')^2$, hence F^* is $K(H)$-QC where $K \to 1$ as $H \to 1$. Following the proof of [9, Lemma 7], we see that F^* also satisfies (4.2) and (4.3).

Now we have the necessary tools to construct the non-elementary discrete groups we desire. We follow [8, §4]. For $x = (z, y) \in \mathbf{R}^2 \times \mathbf{R}^{n-2}$ we set $F_2(x) = (F^*(z), y)$. With $\hat{G}$ and T' defined as earlier, let $G^* = G^*(H) = F_2 \circ \hat{G} \circ F_2^{-1}$ and write $G_1(H) = F_2 \circ T' \circ F_2^{-1}$. With δ from Theorem 5, let

$$B = F_2(B^n((0, 1, 0, \ldots, 0), \delta)).$$

Let Q be a Schottky group generated by reflections in spheres contained in B. Finally, let $G_2(H) = \langle G_1, Q \rangle$, the free group generated by G_1 and Q. Our main result is next.

THEOREM 6. *For each $n \geq 3$ and $K > 1$, there is a K-QC non-elementary discrete group G_2 acting on $\bar{\mathbf{R}}^n$ which is not the quasiconformal conjugate of any Möbius group.*

Proof. By the same proof as in [9, Thm. 13], G^* is $K = K(H)$-quasiconformal, and for no quasiconformal $h: \mathbf{R}^n \to \mathbf{R}^n$ is $h \circ G^* \circ h^{-1}$ a Möbius group. Further, we may select K so that $K \to 1$ as $H \to 1$. Hence both G_1 and Q are K-quasiconformal. By [8, Thm. 4.3 and remarks], $G_2(H)$ is a K-QC group which is discrete and non-elementary. Finally, if $h \circ G_2 \circ h^{-1}$ were a Möbius group for some quasiconformal h, then $h \circ G_1 \circ h^{-1}$ would be a Möbius group as well, because $G_1 \subset G_2$. However, by the same proof as in [8, Thm. 3.8], this is impossible. □

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