Complex Analytic Curves of Minimal Area in Cubes

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1. Introduction

In this paper we prove that, in a cube in $\mathbf{C}^n$, any complex analytic curve passing through the center of this cube has the area which is not less than the area of a complex line containing the center. For the proof, we find the minimal length of real curves, lying on the boundary of the cube and intersecting each real hyperplane passing through the center, in at least m points.

It is well known that the minimal volume of the intersection of the ball in $\mathbf{C}^n$ with center at the origin, and a k-dimensional analytic set passing through the origin, is equal to the volume of a k-dimensional plane with the same property. This statement is not true for an arbitrary symmetric convex domain. The corresponding examples were constructed in [2] and [4].

It is important for some applications to know the minimal value of volume of analytic hypersurface passing through the origin in cubes, since $\mathbf{C}^n$ can be packed by cubes without holes and intersections. Such packing was used for example in [3] to get conditions for the uniqueness of entire functions of exponential type vanishing on some discrete set in $\mathbf{R}^n$. The exact conditions were obtained in this paper only when $n=2$ since in [1] it was proved that, for cubes in $\mathbf{C}^2$, the area minimizing analytic curves containing the cube's center are plane sections.

The proof of the theorem in [1] was based on two facts. First, it was noted that the intersection of a real hyperplane and an analytic curve contains two distinct points of the boundary of the cube if the hyperplane and the curve contain the origin. Second, the area of the curve was expressed through the length of its intersections with cubes of smaller diameter. Therefore, the problem was reduced to a completely real case: to find the minimal length of curves on the boundary of cubes if the intersection of a curve and any hyperplane section passing through the origin contains at least k points. It was proved in [1] that for cubes in $\mathbf{R}^3$ and $\mathbf{R}^4$ real curves of minimal length are intersections of planes, parallel to cube's sides, with the boundary of the cube.

In our paper we prove that the solution of the real problem is the same if the cube has arbitrary dimension and, therefore, we get the exact lower

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estimates for the area of analytic curves in cubes. To prove our basic results we introduce a semi-additive measure on a curve and in Section 3 we calculate the explicit formula for its density and give its upper estimate. This estimate is used in Section 4 to obtain final results.

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2. Notation and Preliminary Results

Let $B^n(r)$ and $S^n(r)$ be a ball and a sphere of radius r in $\mathbf{R}^n$ with center in the origin and $I^n(r)=\{x=(x_1,x_2,\dots,x_n)\in\mathbf{R}^n:|x_i|\le r\}$ be a cube and $K^n(r)$ its boundary. We shall omit r in formulas if it is equal to 1. We denote by $K_i^{\pm}$ the face of the cube where only the ith coordinate is equal to ±1. We suppose that $\mathbf{R}^n$ is endowed with the standard scalar product

$$(x,y)=\sum_{i=1}^{n}(x_i,y_i).$$

If G^n is the Grassmann manifold of all hyperplanes, passing through the origin, then we can define the natural projection $\pi_n: K^n\to G^n$, where $\pi_n(x)$ is a hyperplane orthogonal to the vector x. For simplicity of notation we shall drop the index n. We shall denote the 1-dimensional Hausdorff measure or the length of the curve A by $H_1(A)$ and the k-dimensional volume of P by $\mathrm{Vol}_k P$. If a set Q lies on a plane in $\mathbf{R}^n$ then $m(Q)$ is its Lebesgue measure on this plane. Let us write $d(A)$ for the diameter of a set A. For $Q\subset G$ let $\mu(Q)=m(\pi^{-1}(Q))$. Clearly, $\mu(Q)$ is a measure on G which is equivalent to the invariant one.

We shall denote by $\mathfrak{A}$ the class of sets on K, each of which is a union of a finite number of continuous rectifiable curves. Each of these curves can be regarded as the mapping $\gamma:[0,a]\to K$ with the arc-length parametrization, that is, the length of the arc $\gamma_s:[0,s]\to K$, equal to s. The rectifiable curves as mappings are differentiable almost everywhere at points s such that $\gamma(s)\in K_i^{\pm}$, $i=1,\dots,n$, and at these points $|\gamma'(s)|\le 1$. If $\gamma(s)=(\gamma_1(s),\dots,\gamma_n(s))$ is differentiable at s_0 and $|\gamma'(s_0)|=1$ then

$$\int_{s_0-\epsilon}^{s_0+\epsilon}|\gamma_i'(s)|\,ds=2\epsilon|\gamma_i'(s_0)|+o(\epsilon),\quad i=1,\dots,n.$$

In fact, by Minkowski inequality,

$$4\epsilon^2=\left(\int_{s_0-\epsilon}^{s_0+\epsilon}\sqrt{\sum\gamma_i'^2(s)}\,ds\right)^2\ge\sum\left(\int_{s_0-\epsilon}^{s_0+\epsilon}|\gamma_i'(s)|\,ds\right)^2,$$

and it follows from the differentiability of γ at s_0 that

$$\int_{s_0-\epsilon}^{s_0+\epsilon}|\gamma_i'(s)|\,ds\ge 2\epsilon|\gamma_i'(s_0)|+o(\epsilon).$$

Combining the last two inequalities, we obtain our statement.

Let us define the projection $p: K \to S$ by the formula $p(x) = x/\|x\|$. Evidently, this projection transforms rectifiable curves into rectifiable ones.

The statements of the next lemma are well known.

LEMMA 1. *With the notation above:*

(1) *for any* $A \subset K$, $H_1(A) = 0$ *if and only if* $H_1(p(A)) = 0$;
(2) *if* $A \subset K$ *and* $H_1(A) = 0$, *then the measure of those hyperplanes of* G *that cross* A *is zero*;
(3) *if* D *is a convex domain in* $\mathbf{R}^n$, *containing the origin, with the piecewise smooth boundary, and if* φ *is a continuous function in* $\mathbf{R}^n$ *such that* $\varphi(tx) = |t|\varphi(x)$, *then*

$$(n+1)\int_D \varphi(x)\,dm_x = \int_{\partial D} \varphi(x)(n(x),x)\,dS,$$

where $n(x)$ *is the normal vector to* ∂D *at* x.

Let $P \subset \mathbf{C}^n$ be a complex analytic curve and z a point belonging to P. One can find a neighbourhood $U \subset P$ of z and holomorphic mappings $f_i(\zeta)$, $1 \le i \le m$, of the unit disk $\Delta \subset \mathbf{C}$ into U such that $f_i(0) = z$, $f_i(\Delta) \cap f_j(\Delta) = \{z\}$, $i \ne j$ and $f_i(\zeta) \ne f_i(\xi)$ if $\zeta \ne \xi$. If

$$f_i(\zeta) = (\zeta^{k_{1i}}\varphi_{1i}(\zeta), \ldots, \zeta^{k_{ni}}\varphi_{ni}(\zeta)),$$

where $\varphi_{ji}(0) \ne 0$, and $k_i = \min k_{ji}$, $1 \le j \le n$, then the number

$$\nu_P(z) = \sum k_i$$

is called the multiplicity of P at z.

To prove our basic theorem we need the following inequality.

LEMMA 2. *Let* f_1, f_2 *and* ψ *be real-valued functions on the interval* $[a,b]$ *such that:*

(1) ψ *is nondecreasing*;
(2) $\int_a^b f_1\,dt = \int_a^b f_2\,dt$;
(3) $\int_a^b f_1\psi\,dt = \int_a^b f_2\psi\,dt$; *and*
(4) *the set* $J = \{t \in [a,b] : f_1(t) > f_2(t)\}$ *is connected.*

Then, for any concave function φ,

$$\int_a^b f_1\varphi(\psi)\,dt \ge \int_a^b f_2\varphi(\psi)\,dt.$$

Proof. Let $f = f_1 - f_2$, $I = \psi(J) = [\alpha, \beta]$ and let $l(x)$ be a linear function such that $l(\alpha) = \varphi(\alpha)$ and $l(\beta) = \varphi(\beta)$ (if $\alpha = \beta$, l is the support function to φ at α). Then $\varphi(\psi) - l(\psi) \ge 0$ on J and ≤ 0 out of J. Therefore, $f(\varphi(\psi) - l(\psi)) \ge 0$ and

$$\int_a^b f(\varphi(\psi))\,dt = \int_a^b f(\varphi(\psi) - l(\psi))\,dt \ge 0. \qquad \square$$

The following lemma is a consequence of Lemma 2. We shall use it to estimate the density function.

LEMMA 3. *Let* $g=(g_1,\dots,g_n)\in\mathbf{R}^n$, $\|g\|=1$. *Then*

$$f(g)=\int_K|(x,g)|\,dm_x\le f(e_1^+)=\int_K|x_1|\,dm_x=2^{n-1}(n+1).$$

Proof. Let us note that by statement 3 of Lemma 1

$$\int_K|(x,g)|\,dm_x=(n+1)\int_I|(x,g)|\,dm_x$$

and

$$\int_I|(x,g)|\,dm_x=2\int_0^\infty tS_g(t)\,dt,$$

where $S_g(t)=S_g^n(t)$ is the volume of the polyhedra $Q(t)=\{x\in I:(x,g)=t\}$.

Evidently,

$$\int_0^\infty S_g(t)\,dt=\int_0^\infty S_{e_1^+(t)}\,dt=2^{n-1},$$

$$\int_0^\infty t^2S_g(t)\,dt=\frac{1}{2}\int_0^\infty(x,g)^2\,dm_x=\frac{2^{n-1}}{3}.$$

The function $S_g(t)$ is non-increasing. To see this let us suppose that $g_n\ne 0$. We define the projection $h:\mathbf{R}^n\to\mathbf{R}^{n-1}$ by the formula: $h(x)=h(x_1,\dots,x_n)=(x_1,\dots,x_{n-1})$. Then

$$h(Q(a))=\{x\in I^{n-1}:a-g_n\le(x,h(g))\le a+g_n\}$$

and

(1) $$S_g(a)=(\cos\alpha)^{-1}\int_{\alpha-g_n}^{\alpha+g_n}S_{h(g)}^{n-1}(t)\,dt,$$

where α is the angle between hyperplanes $\{x_n=0\}$ and $\{(x,g)=a\}$. It is evident that our statement is true for $n=2$. For any other n this statement can be derived from (1) by induction.

Now we see that Lemma 3 follows from Lemma 2 if we take $\psi(t)=t^2$, $\varphi(t)=\sqrt{t}$, $f_1(t)=S_{e_1}(t)$, and $f_2(t)=S_g(t)$, and noting that the set where $S_{e_1}(t)>S_g(t)$ is connected. □

3. Calculation and Estimation of the Density Function

Let $\gamma:J=[0,a]\to K$ be a rectifiable curve with the arc-length parametrization. We define the semi-additive measure ν on J as $\nu(A)=\mu(A^*)$, where A^* is the set of all planes crossing $\gamma(A)$. For $x_0=\gamma(t_0)$ we define the density function by the formula

(2) $$f(x_0,\gamma)=\overline{\lim_{\epsilon\to 0}}\frac{\nu(J(\epsilon))}{2\epsilon},$$

where $J(\epsilon)=[t_0-\epsilon,t_0+\epsilon]$.

To write the explicit formula for f we shall use the following notation. If $x \in A$ then $L(x)$ denotes the hyperplane orthogonal to x and $LK(x) = L(x) \cap K$. If $y \in LK(x)$ and $y \in K_i^{\pm}$, $i = 1, \ldots, n$, then $n(y)$ denotes the unit normal vector to $LK(x)$ at y. (The direction of $n(y)$ must be the same for all y when x is fixed.)

The lemma below gives us the explicit formula for the density function.

LEMMA 4. *With the notation above, if $t_0 \in J$, $x_0 = \gamma(t_0) \in K_i^{\pm}$ $(i = 1, \ldots, n)$, γ is differentiable at t_0, and $g = \gamma'(t_0)$, $|g| = 1$, we have*

$$f(x_0, \gamma) \leq \int_{LK(x_0)} \left| \frac{(g, x)}{(n(x), x_0)} \right| dm_x. \tag{3}$$

Equality is attained in (3) *if γ is smooth in a neighbourhood of t_0.*

Let $J(\epsilon) = [t_0 - \epsilon, t_0 + \epsilon]$ and $U(\epsilon) = \gamma(J(\epsilon))$. We shall consider two cases:

(1) the intersection of $LK(x_0)$ with $(n-2)$-dimensional faces of the cube, denoted by $D(x_0)$, has zero measure;

(2) the measure of $D(x_0)$ is not zero.

In the first case we let $LK_\delta(x_0) = LK(x_0) \setminus D_\delta(x_0)$, where $D_\delta(x_0)$ is the δ-neighbourhood of $D(x_0)$. This set lies only on cube's faces, so we can take its normal neighbourhood N of a sufficiently small radius r. When ϵ is small we can define the mapping φ of $LK_\delta(x_0) \times J(\epsilon)$ into $V(\epsilon) = \pi^{-1}(U^*(\epsilon))$, where $U^*(\epsilon)$ is the set of all hyperplanes crossing $U(\epsilon)$, by the formula

$$\varphi(x, t) = x - \frac{(y, x) n(x)}{(n(x), y)} = x - \frac{(y - x_0, x) n(x)}{(n(x), x_0) + (n(x), y - x_0)}, \tag{4}$$

where $y = \gamma(t)$ and $t \in J(\epsilon)$.

Since $(n(x), x_0) \neq 0$, this mapping is well defined and its image belongs to N when ϵ is small. Vectors y and $\varphi(x, t)$ are orthogonal. On the other hand, if $z \in V(\epsilon) \cap N$ and $z = x + \lambda n(x)$ then there is a vector $y \in U(\epsilon)$ such that $y \perp z$ and $\varphi(x, y) = z$. Therefore φ maps $LK_\delta(x_0) \times J(\epsilon)$ onto $V(\epsilon) \cap N$. Since $m(V(\epsilon) \setminus N) \leq C(\delta)\epsilon$, where $C(\delta) \to 0$ as $\delta \to 0$, we have

$$m(V(\epsilon)) \leq \int |\det \varphi'| \, dm_x \, dt + C(\delta)\epsilon \tag{5}$$

(in the last formula φ' is the derivative of φ and the integral is taken over the set $LK_\delta(x_0) \times J(\epsilon)$). Simple calculations show that

$$|\varphi'(x, t)| = \left| \frac{(\gamma'(t), x)}{(x_0, n(x))} \right| + o(\epsilon) |\gamma'(t)|.$$

As mentioned in Section 2,

$$\int_{-\epsilon}^{\epsilon} \left| \frac{(\gamma'(t), x)}{(x_0, n(x))} \right| dt = 2\epsilon \left| \frac{(\gamma'(0), x)}{(x_0, n(x))} \right| + o(\epsilon).$$

So, dividing both sides of (5) by 2ϵ and taking the limit as both ϵ and δ tend to zero, we obtain

(6) $$f(x_0,\gamma)\le\int_{LK(x_0)}\left|\frac{(g,x)}{(n(x),x_0)}\right|dm_x.$$

If γ is smooth in a neighbourhood of t_0 then it is easy to see that φ is one-to-one, provided $(g,x)\neq 0$ and ϵ is sufficiently small. Therefore the equality is attained in (6).

Lemma 5 is also valid in the second case, because $LK(x_0)$ contains an $(n-2)$-dimensional face of the cube that divides $V(\epsilon)$ on two parts lying in adjacent faces. If x belongs to this $(n-2)$-dimensional face then we can choose two vectors $n_1(x)$ and $n_2(x)$ normal to $LK(x_0)$ at x, lying in adjacent faces mentioned above. Volumes of those parts of $V(\epsilon)$ that lie in only one of the faces can be estimated by (5) as before, and therefore (6) will be valid since $(n_1(x),x_0)=(n_2(x),x_0)$.

Let $F(x_0,g)$ be equal to the right-hand side of (6).

LEMMA 5. *If $x\in K_j^\pm$ and $Q_j^\pm=\{x\in K_j^\pm:\sum_{i\neq j}|x_i|\le 1\}$, then $F(x,g)=F(e_j^\pm,g)$ for $x\in Q_j^\pm$ and $F(x,g)<F(e_j^\pm,g)$ for $x\notin Q_j^\pm$.*

Proof. Let $x\in K_1^+$, that is, let $x=(1,x_2,\dots,x_n)$, and let ψ be the orthogonal projection of the hyperplane $L(x)$ onto $L(e_1)$. If $z\in LK(x)\cap K_j^\pm$ then $n(z)=(x-x_je_j)\|x-x_je_j\|^{-1}$ and, therefore, $(n(z),x)=\|x-x_je_j\|$. For $j\neq 1$ we have that $(n(z),e_1)=\|x-x_je_j\|^{-1}$ so, in this case,

(7) $$(n(z),x)(e_1,n(z))=1.$$

If $x\in Q_1^+$ then $L(x)$ does not intersect the set $K_1^+\cup K_1^-$, since for $z=(z_1,\dots,z_n)\in K_1^+\cup K_1^-$

$$(z,x)=\pm 1+\sum_{j=2}^{n}x_jz_j\neq 0.$$

So, in this case ψ maps $LK(x)$ onto $LK(e_1)$, and on $LK(e_1)$ the Euclidean volume form

(8) $$dm_y=(e_1^+,n(z))\,dm_z^*,$$

where dm_z^* is the pull-back under the mapping $z=\psi^{-1}(y)$ of the Euclidean volume form dm_z defined on $LK(x)$. Combining the equality $(y,g)=(z,g)$ with (7) and (8) we see that

$$F(x,g)=\int_{LK(x)}\frac{|(z,g)|}{|(n(z),x)|}\,dm_z=\int_{LK(e_1^+)}|(y,g)|\,dm_y=F(e_1^+,g).$$

If $x\in K_1^+\setminus Q_1^+$, we consider two cases. First, when $z\in LK(x)\cap K_j^\pm$ $(j\neq 1)$, the vector $y=\psi(z)\in LK(e_1)\cap K_j^\pm$ and it follows from (7) and (8) that $dm_z^*=|(n(z),x)|\,dm_y$. Second, when $z\in LK(x)\cap K_1^\pm$ we can see that $dm_z^*=dm_y$ and $(n(z),x)=\|x-e_1\|$, and since $y=z\mp e_1$ we have $|(n(z),y)|=\|x-e_1\|^{-1}$. Therefore,

$$\int_{LK(x_0)}\frac{|(z,g)|}{|(n(z),x)|}\,dm_z=\int_A|(y,g)(n(z),y)|\,dm_y=n\int_B|(y,g)|\,dm_y$$
$$<n\int_{I^{n-1}}|(y,g)|\,dm_y=F(e_1,g),$$

where $A=\psi(LK(x))$ and B is the polyhedra such that $\partial B=A$. □

The next lemma follows immediately from Lemmas 3 and 5.

LEMMA 6. *For the density function $F(x,g)$ we have the inequality*

$$F(x,g) \le n2^{n-2}$$

when $x \in K$ and $|g|=1$. The equality is attained if and only if $g=e_j^{\pm}$, $x_k = \pm 1$, and $\sum_{i \ne k}|x_i| \le 1$.

REMARK. This method can be generalized for sets A of greater dimension in the following way. Let $x \in A$ and $\dim A = k$. Also let U be a neighbourhood of x in A and U^* be a set of all planes P of the dimension $n-k$ such that $U \cap P \ne \emptyset$ and $0 \in P$. Let V^* be the set of all k-tuples of vectors $\{z_1, \dots, z_k\}$, $z_i \in K$, such that the plane generated by these vectors is orthogonal to some plane of U^*. We define the measure μ on the direct product of n copies of K, which is equal to the Fubini product of Lebesgue measures on K. Let us define the density function $F(x,A)$ by the formula

$$f(x,A) = \lim_{d(U)\to 0} \frac{\mu(V^*)}{\mathrm{Vol}_k(U)}.$$

Then the following lemma holds.

LEMMA 7. *If A is smooth in a neighbourhood of x and $G=(g_1,\dots,g_k)$ is an orthonormal polyvector such that $TA(x)$ is generated by G, then*

$$f(x,A) = F(x,G) = \int \left| \det\{(g_i, z_j)\} \prod_{l=1}^{k} (n(z_l), x)^{-1} \right| dm_{z_l},$$

where the integral is taken over the direct product of k copies of $LK(x)$. If $x \in Q_j^{\pm}$ then $F(x,G) = F(e_j^{\pm}, G)$ and if $x \notin Q_j^{\pm}$ then $F(x,G) < F(e_j^{\pm}, G)$.

4. The Estimates for Lengths and Areas

Let $A \subset \mathfrak{U}$ be the union of a finite number of rectifiable curves γ_i $(1 \le i \le N)$ and let l_i be the length of a curve γ_i. We consider each curve γ_i as a mapping $\gamma_i\colon [0, l_i] \to K$ with the arc-length parametrization. Let us introduce the mapping Γ defined as follows: if $s \in (\sum_{i=1}^{k-1} l_i, \sum_{i=1}^{k} l_i)$ then $\Gamma(s) = \gamma_k(s - \sum_{i=1}^{k-1} l_i)$. This mapping is defined everywhere except at a finite number of points. We say that the set A intersects the hyperplane P in m points if there are distinct real numbers $s_1, \dots, s_m$ such that $\Gamma(s_i) \in P$. We shall denote by $\mathfrak{U}_m$ the class of all sets $A \subset \mathfrak{U}$ that intersect almost all hyperplanes in at least m points.

THEOREM 1. *If $A \subset \mathfrak{U}_m$ then $H_1(A) \ge 4m$.*

Proof. Let $A_0 = p(A)$. It is clear that after a rotation the curve A_0 belongs also to $\mathfrak{U}_m$. Therefore, for any $\epsilon > 0$ there is a rotation T_ϵ such that the curve $A_\epsilon = p^{-1}(T_\epsilon(p(A))$ satisfies the following conditions:

(1) A_ϵ intersects $(n-2)$-dimensional faces of the cube in a set D_ϵ of zero measure; and
(2) $|H_1(A) - H_1(A_\epsilon)| < \epsilon$.

It follows from Lemma 1 that the measure of all hyperplanes P such that $P \cap D_\epsilon \neq \emptyset$ is equal to 0.

Hence it is sufficient to prove that $H_1(A_\epsilon) \geq 4m$. If $L \in G$ then let $A(L) = \{s : \Gamma(s) \in L\}$, and for any m-tuple of points $s_i \in A(L)$ we define functions $\rho(s_1, \dots, s_m) = \min|s_i - s_j|$ $(i \neq j)$ and $\psi(L) = \sup \rho(s_1, \dots, s_m)$, where the supremum is taken over m-tuples of points from $A(L)$. Let $G(a) = \{L \in G : \rho(L) > \alpha\}$. Then for any $\delta > 0$ there is $\alpha > 0$ such that $\mu(G(\alpha)) > \mu(G) - \delta$.

Let $D \subset [0, l]$ be the set of all points s such that $x = \Gamma(s)$ does not lie on $(n-2)$-dimensional faces of the cube and the mapping Γ is differentiable in s. By the Vitali covering lemma and Lemma 6, it follows that there are non-intersecting intervals $J_i = [a_i, b_i]$ on $[0, l]$ such that:

(1) the set $D \setminus \bigcup_{i=1}^\infty J_i$ has zero measure;
(2) $|b_i - a_i| < \alpha$;
(3) $\nu(J_i) \leq n2^{n-2}(b_i - a_i) + \delta|b_i - a_i|$.

Due to condition 2, almost all $P \in G(\alpha)$ belong to at least m intervals J_i^*. Hence

$$m\mu(G(\alpha)) \leq \sum \nu(J_i) + \delta l \leq n2^{n-2} \sum (b_i - a_i) + \delta l \leq n2^{n-2}(H_1(A_\epsilon) + \delta).$$

Therefore $H_1(A_\epsilon) \geq m2^{2-n}(\mu(G) - \delta)/n - \delta$. Since $\mu(G) = n2^n$ and δ can be an arbitrary number, it follows that $H_1(A_\epsilon) \geq 4m$. □

THEOREM 2. *Let P be an analytic complex curve in I^n, passing through the origin. Then* $\mathrm{Vol}_2 P \geq 4\nu_P(0)$.

Proof. Let $P_r = P \cap I(r)$ and $A_r = P \cap K(r)$, $0 < r < 1$. It is known [1] that for almost all r the set $A_r \subset \mathfrak{A}$ and

$$\mathrm{Vol}_2 P \geq \int_0^1 H_1(A_r)\, dr.$$

We shall need the following.

LEMMA 8. *The intersection of almost all hyperplanes with A_r contains at least $2\nu_P(0)$ points.*

Let G_1 be a set of all hyperplanes which do not pass through singular points of P except the origin. Since the set of singular points is countable, the complement of G_1 is a set of zero measure. If $L \in G_1$ is given by the equation

$$\mathrm{Re}\, v(z) = 0, \quad v(z) = \sum_{j=1}^n a_j z_j,$$

then we can define the function $f(z) = v(z)|_P$ on P. The intersection of P with L, denoted by D, is the zero-level curve of $\mathrm{Re}\, f$. This curve consists of analytic curves with singularities which are either at the origin or at those points of P where the tangent plane to P belongs to L. It can be proved easily that the measure of the set of all hyperplanes containing points of the second kind is zero. If we remove these hyperplanes from G_1 and denote the remaining set by G_2, then for $L \in G_2$ the set $D \backslash \{0\}$ consists of smooth arcs starting at the origin and ending on the boundary of the cube, because $\operatorname{Im} v$ is strictly monotone on $D \backslash \{0\}$. It follows from the definition of multiplicity of P at the origin that the number of such curves is equal to $2\nu_P(0)$ and that each of these curves intersects $K(r)$ at least once. □

Lemma 8 and Theorem 1 imply that $H_1(A_r) \geq 8\nu_P(0)r$, and therefore

$$\mathrm{Vol}_2 P \geq 8 \int_0^1 \nu_P(0) r \, dr \geq 4\nu_P(0).$$

This completes the proof of Theorem 2. □

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