Complemented Ideals in Weighted Algebras of Analytic Functions on the Disc

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Introduction

Weighted algebras of analytic functions introduced in a more general setting by Hörmander [2] have been investigated for many years with different aims. During the last years in the case of entire functions, several authors have considered the question of whether closed ideals are complemented subspaces of the algebra regarded as a locally convex space (see e.g. Taylor [14]; Meise [7]; Meise and Taylor [9], [10]; Meise and Vogt [11]; Meise, Momm, and Taylor [8]). We deal with the case of analytic functions on the disc and give a criterion to decide whether an arbitrary given ideal is complemented.

Let $(\psi_k)_{k\in\mathbf{N}}$ be an increasing sequence of continuous, nonnegative, strictly increasing, unbounded functions on $x\geq 1$. By $A_{\mathbf{P}}$, we denote the locally convex space of all analytic functions f on the open unit disc satisfying some estimate

$$|f(z)|\leq C\exp\psi_k\left(\frac{1}{1-|z|}\right),\quad |z|<1.$$

We require conditions on the weight functions to guarantee that $A_{\mathbf{P}}$ becomes an algebra of analytic functions in which division is possible. We characterize the complemented closed ideals in $A_{\mathbf{P}}$ by their distribution of zeros: *A nonzero closed ideal I is complemented in $A_{\mathbf{P}}$ if and only if there exists $m\in\mathbf{N}$ such that for each $k\in\mathbf{N}$ there is $n\in\mathbf{N}$ with*

$$\psi_k\left(\frac{1}{1-|a|}\right)\psi_n^{-1}\left(\psi_k\left(\frac{1}{1-|a|}\right)\right)\leq\frac{1}{1-|a|}\psi_m\left(\frac{1}{1-|a|}\right)$$

for almost all zeros a of I.

We also obtain a corresponding result for another (dual) type of weighted algebras which can be defined similarly to $A_{\mathbf{P}}$.

Via duality theory, our results can be regarded as results on the existence of continuous linear right inverses for Toeplitz operators, or (more generally) on the complementation of shift invariant subspaces in certain locally convex spaces of analytic functions on the disc which are smooth up to the boundary (see Momm [12]).

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If I is a nonzero closed ideal, note that I is complemented in $A_{\mathbf{P}}$ if and only if the quotient map $\pi: A_{\mathbf{P}} \to A_{\mathbf{P}}/I$ has a continuous linear right inverse. The crucial point of our investigation is a decomposition of the quotient $A_{\mathbf{P}}/I$ in finite dimensional Banach algebras E_l, $l \in \mathbf{N}$. If R is a continuous linear right inverse for π, by applying R to the identities of E_l we obtain a sequence of certain subharmonic functions on $|z|<1$ which, by the continuity of R, satisfy a set of growth estimates. This necessary condition for I to be complemented is equivalent to the announced one. In fact, using an idea of Taylor [14, Thm. 5.2] and Meise, Momm, and Taylor [8, 3.3], we can prove that it is sufficient, too: First, on each part E_l of the finite-dimensional decomposition of $A_{\mathbf{P}}/I$ there are right inverses R_l for the quotient map π such that the operators $(\dim E_l)^{-1} R_l$, $l \in \mathbf{N}$, are equicontinuous. From the subharmonic functions of the hypothesis, using Hörmander's L^2 technique to get solutions for the $\bar\partial$ equation, we construct analytic functions g_l, $l \in \mathbf{N}$, satisfying corresponding growth estimates and which are preimages of the identities of E_l with respect to π. Then a continuous linear right inverse R for π can be defined by

$$R: A_{\mathbf{P}}/I \to A_{\mathbf{P}}, \quad R\left(\sum_{l=1}^{\infty} x_l\right) := \sum_{l=1}^{\infty} g_l R_l(x_l).$$

Proving our main theorem, by the way, we obtain division and localization results for the algebras described above.

1. (DFN) Algebras

By $\mathbf{D}$, we denote the open unit disc $\{z \in \mathbf{C} \mid |z|<1\}$.

1.1. DEFINITION. Let $\Psi = (\psi_k)_{k\in\mathbf{N}}$ be a sequence of continuous, nonnegative, strictly increasing, unbounded functions on $x \ge 1$. Ψ will be called a *weight system* if for each $k \in \mathbf{N}$ there are $n \in \mathbf{N}$ and $x_0 \ge 1$ such that, for all $x \ge x_0$,

(α) $\psi_k(x) \le \psi_{k+1}(x)$, $\quad(\beta)$ $2\psi_k(x) \le \psi_n(x)$,

(γ) $\psi_k(2x) \le \psi_n(x)$, $\quad(\delta)$ $x(\int_1^x \sqrt{\psi_k(t)/t^3}\, dt)^2 \le \psi_n(x)$.

Two weight systems $\Psi = (\psi_k)_{k\in\mathbf{N}}$ and $\Phi = (\phi_k)_{k\in\mathbf{N}}$ will be called *equivalent* if for each $k \in \mathbf{N}$ there are $n \in \mathbf{N}$ and $x_0 \ge 1$ such that $\psi_k(x) \le \phi_n(x)$ and $\phi_k(x) \le \psi_n(x)$ for all $x \ge x_0$.

1.2. LEMMA. *Let Ψ be as in Definition 1.1. There is an equivalent weight system $\Phi = (\phi_k)_{k\in\mathbf{N}}$ of convex functions ϕ_k such that $x \mapsto \phi_k(x)/x$ is increasing.*

Proof. We put

$$\tilde\phi_k(x) := x\left(\int_1^x \sqrt{\frac{\psi_k(t)}{t^3}}\, dt\right)^2, \quad x \ge 1, \ k \in \mathbf{N}.$$

Since

$$4(\sqrt{2}-1)^2\psi_k\left(\frac{x}{2}\right)\le x\left(\int_{x/2}^{x}\sqrt{\frac{\psi_k(t)}{t^3}}\,dt\right)^2,$$

using 1.1(β), (γ), and (δ), we get that $\tilde{\Phi}=(\tilde{\phi}_k)_{k\in\mathbf{N}}$ is an equivalent weight system. If we repeat this procedure with Ψ replaced by $\tilde{\Phi}$, we again get an equivalent weight system $\Phi=(\phi_k)_{k\in\mathbf{N}}$. Since $x\mapsto\tilde{\phi}_k(x)/x$ increases, differentiation shows that the functions ϕ_k are convex. □

1.3. LEMMA. *Let ψ be a continuous, nonnegative, strictly increasing, unbounded function on $x\ge 1$ with $\psi(2x)=O(\psi(x))$. Then $(k\psi)_{k\in\mathbf{N}}$ is a weight system if and only if there exists $C>1$ such that*

$$\liminf_{x\to\infty}\frac{\psi(Cx)}{C\psi(x)}>1.$$

Proof. To prove sufficiency, the only thing to show is 1.1(δ). If x is the lth power of C, then this can easily be done by dividing the range of integration into intervals $[C^{n-1},C^n]$, $0\le n\le l$, and estimating straight forward using the following consequence of the hypothesis $l-n$ times: There are $\epsilon>0$ and $x_0\ge 1$ with

$$\psi(x)\le C^{-(1+\epsilon)}\psi(Cx),\quad x\ge x_0.$$

To get the necessity, note that 1.1(δ) is equivalent to

$$\int_1^x\frac{\phi(t)}{t}\,dt=O(\phi(x)),\quad \phi(x):=\sqrt{\frac{\psi(x)}{x}},\quad x\ge 1.$$

By 1.2, we may assume that ϕ is increasing. As in 1.2, we get that there are $x_0\ge 1$ and $K\ge 1$ such that

$$\frac{1}{K}\phi(x)\le\int_1^x\frac{\phi(t)}{t}\,dt\le K\phi(x),\quad x\ge x_0.$$

Choose $C>\exp(K^3-K)$. Then for $x\ge x_0$, we get

$$K^2\frac{\phi(Cx)}{\phi(x)}\ge\frac{\int_1^{Cx}\phi(t)t^{-1}\,dt}{\int_1^x\phi(t)t^{-1}\,dt}=1+\frac{\int_x^{Cx}\phi(t)t^{-1}\,dt}{\int_1^x\phi(t)t^{-1}\,dt}\ge 1+\frac{\phi(x)\log C}{K\phi(x)}\ge K^2.$$

Hence

$$\liminf_{x\to\infty}\frac{\psi(Cx)}{C\psi(x)}=\left(\liminf_{x\to\infty}\frac{\phi(Cx)}{\phi(x)}\right)^2>1.$$ □

1.4. DEFINITION. Let ψ be a weight system as in Definition 1.1, and let $A(\mathbf{D})$ denote the algebra of all analytic functions on $\mathbf{D}$. We put

$$\mathbf{P}=(p_k)_{k\in\mathbf{N}},\quad p_k(z):=\psi_k\left(\frac{1}{1-|z|}\right),\quad z\in\mathbf{D},\ k\in\mathbf{N},$$

and define

$$A_{\mathbf{P}} = \left\{ f \in A(\mathbf{D}) \,\middle|\, \|f\|_k = \sup_{z \in \mathbf{D}} |f(z)| e^{-p_k(z)} < \infty \text{ for some } k \in \mathbf{N} \right\}.$$

Endowed with the topology induced by $(\|\ \|_k)_{k \in \mathbf{N}}$, $A_{\mathbf{P}}$ is a (DFN) algebra; that is, $A_{\mathbf{P}}$ is the strong dual of a nuclear Fréchet space. In view of Lemma 1.2, by Hörmander [3, Thm. 1.6.7] we may assume that the functions p_k, $k \in \mathbf{N}$, are subharmonic.

1.5. EXAMPLES. $\Psi = (\psi_k)_{k \in \mathbf{N}}$ is a weight system if:

(a) $\psi_k(x) = kx^\rho$, $\rho > 1$;
(b) $\psi_k(x) = x^{\rho_k}$, $\rho_k > 1$, $(\rho_k)_{k \in \mathbf{N}}$ strictly increasing;
(c) $\psi_k(x) = x(\log x)^k$.

CONVENTION. In the sequel, Ψ and $\mathbf{P}$ will always be as in Definition 1.4.

1.6. PROPOSITION. *For each $F \in A_{\mathbf{P}} \backslash \{0\}$ there are Jordan curves Γ_l in $1 - 2^{-l} < |z| < 1 - 2^{-l-1}$, $l \in \mathbf{N}$, around the origin such that there are $k \in \mathbf{N}$ and $C > 0$ with*

$$\log|F(z)| \geq -p_k(z) - C, \quad z \in \Gamma_l, \ l \in \mathbf{N}.$$

Proof. Apply the Corollary in Momm [13] with $\epsilon = \frac{1}{2}$ to the function $cF(z)/z^n$, $c \neq 0$ and $n \in \mathbf{N}_0$ appropriate, and use 1.1(β), (γ), and (δ). □

From Matsaev and Mogul'skii [6] or Proposition 1.6, we have the next proposition.

1.7. PROPOSITION. *In $A_{\mathbf{P}}$ division is possible; that is, if f and g are in $A_{\mathbf{P}}$ and f/g is analytic on $\mathbf{D}$, then f/g is in $A_{\mathbf{P}}$. In particular, each principal ideal in $A_{\mathbf{P}}$ is closed.*

1.8. REMARK. If $\Psi \doteq (k\psi)_{k \in \mathbf{N}}$, then each closed ideal in $A_{\mathbf{P}}$ is principal (see Lemma 1.3 and Momm [12, 3.13]).

1.9. PROPOSITION. *Each closed ideal in $A_{\mathbf{P}}$ is localized and generated by two functions; that is, there are $F_1, F_2 \in A_{\mathbf{P}}$ such that $I = I_{\mathrm{loc}}(F_1, F_2)$, where $I_{\mathrm{loc}}(F_1, F_2)$ consists of all functions $f \in A_{\mathbf{P}}$ which vanish at all common zeros of F_1 and F_2 (with respect to multiplicities).*

Proof. By Proposition 1.6, this follows from Kelleher and Taylor [4, 4.6] and the "jiggling of zeros" argument indicated in Berenstein and Taylor [1, p. 120].

1.10. REMARK. With the notion of the associated functions

$$\omega_k(t) := \inf_{0<r<1} (p_k(r) - t \log r), \quad t \geq 0, \ k \in \mathbf{N},$$

we get

$$A_{\mathbf{P}}=\left\{f=\sum_{j\in\mathbf{N}_0}x_jz^j \,\middle|\, |f|_k=\sup_{j\in\mathbf{N}_0}|x_j|e^{-\omega_k(j)}<\infty \text{ for some } k\in\mathbf{N}\right\},$$

as (DFN) spaces. Furthermore, for each $k\in\mathbf{N}$, ω_k is a continuous, strictly increasing, unbounded function.

Proof. Use straightforward estimation of the integral representation for the coefficients of the power series. To prove the unboundedness of ω_k, for given $L>0$ choose $0<r_0<1$ with $p_k(r_0)\geq L$ and choose $t_0\geq L/\log(1/r_0)$. Then $\omega_k(t_0)\geq L$. □

CONVENTION. In the sequel let $(\omega_k)_{k\in\mathbf{N}}$ be as in Remark 1.10.

1.11. LEMMA. *For each $k\in\mathbf{N}$ there exists $t_0\geq 0$ such that, for all $t\geq t_0$,*

$$\omega_k^{-1}(t)\leq t\psi_k^{-1}(t)\leq 2\omega_k^{-1}(2t).$$

Proof. We omit the index k. Considering the cases $r\leq s$ and $s\leq r$ separately, we get

$$p(r)\leq p(s)+p(r)\frac{\log(1/s)}{\log(1/r)},\quad 0<r,s<1.$$

Hence, for each $0<r<1$ we have

$$p(r)\leq\omega\left(\frac{p(r)}{\log(1/r)}\right)=\inf_{0<s<1}\left(p(s)+\frac{p(r)}{\log(1/r)}\log(1/s)\right)\leq 2p(r).$$

Thus, for $t>p(0)$ we obtain

$$\omega^{-1}(t)\leq\frac{t}{\log(1/p^{-1}(t))}\leq\omega^{-1}(2t).$$

Since $1/p^{-1}(t)=1+1/(\psi^{-1}(t)-1)$, this implies the assertion. □

2. Characterization of Complemented Ideals

Following the proof of Meise [7, 3.7], we get the following description of the quotients of $A_{\mathbf{P}}$ modulo closed ideals which is crucial for our investigations (see also Berenstein and Taylor [1, Thm. 7]).

2.1. REPRESENTATION OF $A_{\mathbf{P}}/I$. Let I be a nonzero closed ideal in $A_{\mathbf{P}}$. According to Proposition 1.9, we choose $F_1, F_2\in A_{\mathbf{P}}\setminus\{0\}$ with $I=I_{\text{loc}}(F_1,F_2)$. Let $V(I)$ denote the zero set of I, that is, the set of all joint zeros of F_1 and F_2. Let $(\Gamma_l)_{l\in\mathbf{N}}$ be the Jordan curves which we get applying Proposition 1.6 to the function F_1. Put $\Gamma_0:=\emptyset$ and define

$$S_l:=\operatorname{int}\Gamma_{l+1}\setminus\overline{\operatorname{int}\Gamma_l},\ l\in\mathbf{N}_0\quad\text{and}\quad \mathcal{S}:=\bigcup_{l\in\mathbf{N}_0}S_l=\mathbf{D}\setminus\bigcup_{l\in\mathbf{N}_0}\Gamma_l.$$

For $l\in\mathbf{N}_0$, by A_l we denote the Banach algebra of all bounded analytic functions on S_l. By I_l, we denote the closed ideal in A_l which has the same zeros in S_l (with respect to multiplicities) as the ideal I. Now, let E_l be the quotient $E_l = A_l/I_l$ endowed with the quotient norm $|\ |_l$.

We consider the following (LB) space which, with componentwise multiplication, is an algebra with identity element:

$$\left\{x \in \prod_{l \in \mathbf{N}_0} E_l \,\middle|\, \|\!|x|\!\|_k = \sup_{l \in \mathbf{N}_0} |x_l|_l \exp(-\psi_k(2^l)) < \infty \text{ for some } k \in \mathbf{N}\right\},$$

which, by 1.1(γ), equals the (LF) space

$$A_{\mathbf{P}}(\mathcal{S})/I := \left\{x \in \prod_{l \in \mathbf{N}_0} E_l \,\middle|\, \|x\|_k = \sup_{l \in \mathbf{N}_0} |x_l|_{lk} < \infty \text{ for some } k \in \mathbf{N}\right\},$$

$$|x_l|_{lk} := \inf_{\xi \in x_l} \sup_{z \in S_l} |\xi(z)| \exp(-p_k(z)), \quad x_l \in E_l, \ l \in \mathbf{N}_0, \ k \in \mathbf{N}.$$

Now, consider the map

$$\rho: A_{\mathbf{P}} \to A_{\mathbf{P}}(\mathcal{S})/I, \quad \rho(f) = (\rho_l(f))_{l \in \mathbf{N}_0} = (f \,|\, S_l + I_l)_{l \in \mathbf{N}_0}.$$

Obviously, ρ is a continuous homomorphism of algebras with identity, and $\ker \rho = I$. We show that ρ is onto.

Let $x \in A_{\mathbf{P}}(\mathcal{S})/I$, $x \neq 0$. Then there are $k \in \mathbf{N}$ and functions $\xi_l \in x_l$, $l \in \mathbf{N}_0$, with

$$\sup_{l \in \mathbf{N}_0} \sup_{z \in S_l} |\xi_l(z)| \exp(-p_k(z)) \leq 2\|x\|_k < \infty.$$

Define an analytic function $\xi(z) := \xi_l(z)$, $z \in S_l$, $l \in \mathbf{N}_0$. Then

$$|\xi(z)| \leq 2\|x\|_k \exp p_k(z), \quad z \in \mathcal{S}.$$

More generally, for further application let u be any subharmonic function on $\mathbf{D}$ with

$$|\xi(z)| \leq \exp u(z), \quad z \in \mathcal{S}.$$

We have to repeat the proof of the semi-local interpolation theorem of Berenstein and Taylor [1, p. 120]. Choose k_1 and $C_1 \geq 1$ with

$$|F_i(z)| \leq C_1 \exp p_{k_1}(z), \quad z \in \mathbf{D}, \ i = 1, 2.$$

As in [1], we get a neighborhood $\tilde{\mathcal{S}} \subset \mathcal{S}$ of $V(I)$ such that there are k_2 and $C_2 \geq 1$ with

$$|F(z)| := \sqrt{|F_1(z)|^2 + |F_2(z)|^2} \geq (C_2 \exp p_{k_2}(z))^{-1}, \quad z \in \mathbf{D} \setminus \tilde{\mathcal{S}};$$

$$\operatorname{dist}(z, \mathbf{D} \setminus \mathcal{S}) \geq (C_2 \exp p_{k_2}(z))^{-1}, \quad z \in \tilde{\mathcal{S}}.$$

Hence, as in [1], there exists $\chi \in C^\infty(\mathbf{D})$ with $0 \leq \chi \leq 1$, $\chi \,|\, \tilde{\mathcal{S}} \equiv 1$, and $\operatorname{supp} \chi \subset \mathcal{S}$ such that there are k_3 and $C_3 \geq 1$ with

$$|\bar{\partial}\chi(z)| \leq C_3 \exp p_{k_3}(z), \quad z \in \mathbf{D}.$$

If we put

$$v_i := -\frac{\overline{F_i}\,\bar{\partial}(\chi\xi)}{|F|^2}, \quad i = 1, 2,$$

then $v_i \in C^\infty(\mathbf{D})$, $i=1, 2$, and

$$\left(\int_{\mathbf{D}} (|v_i| \exp(-p_{k_1} - 2p_{k_2} - p_{k_3} - u))^2 \, dm_2\right)^{1/2} \leq \sqrt{\pi} C_1 C_2^2 C_3, \quad i=1,2.$$

By Hörmander's theorem (see [1, Thm. 1]), we obtain functions $u_i \in C^\infty(\mathbf{D})$ with $\bar{\partial} u_i = v_i$ and

$$\left(\int_{\mathbf{D}} (|u_i| \exp(-p_{k_1} - 2p_{k_2} - p_{k_3} - u))^2 \, dm_2\right)^{1/2} \leq \sqrt{2\pi} C_1 C_2^2 C_3, \quad i=1,2.$$

Defining

$$g := \chi\xi + u_1 F_1 + u_2 F_2,$$

we have $\bar{\partial} g \equiv 0$, hence $g \in A(\mathbf{D})$. With $p := 2p_{k_1} + 2p_{k_2} + p_{k_3}$, we get

$$\left(\int_{\mathbf{D}} |g|^2 \exp(-2(p+u)) \, dm_2\right)^{1/2} \leq 3\sqrt{2\pi} C_1^2 C_2^2 C_3.$$

Using the averaging property of analytic functions we get (with $r(z) := \frac{1}{2}(1-|z|)$)

$$|g(z)| = (\pi r(z)^2)^{-1} \left| \int_{|z-w| \leq r(z)} g \, dm_2 \right| \leq (\sqrt{\pi} r(z))^{-1} \left(\int_{|z-w| \leq r(z)} |g|^2 \, dm_2 \right)^{1/2}$$

$$\leq 6\sqrt{2} C_1^2 C_2^2 C_3 \exp\left(\sup_{|z-w| \leq r(z)} p(w) + \log \frac{1}{1-|z|} + \sup_{|z-w| \leq r(z)} u(w) \right), \quad z \in \mathbf{D}.$$

By 1.1(β), (γ), and (δ), there are $k_4 \in \mathbf{N}$ and $C_4 \geq 1$ with

$$|g(z)| \leq C_4 \exp\left(p_{k_4}(z) + \sup_{|z-w| \leq r(z)} u(w) \right), \quad z \in \mathbf{D}.$$

Applying this result with $u = p_k + \log(2\|x\|_k)$, we obtain that $g \in A_{\mathbf{P}}$. Since $\chi \equiv 1$ in a neighbourhood of $V(I)$, by the definition of g we get $\rho(g) = x$. Hence ρ is onto. By the open mapping theorem for (LF) spaces, ρ is an open mapping and

$$A_{\mathbf{P}}/I \cong A_{\mathbf{P}}(\mathrm{S})/I.$$

As in [1, Lemma 4], in addition we have that there are $k_5 \in \mathbf{N}$ and $C_5 > 0$ such that

$$\dim E_l = \sum_{a \in V(I) \cap S_l} n(a) \leq C_5 \exp \psi_{k_5}(2^l), \quad l \in \mathbf{N}_0,$$

where $n(a)$ denotes the multiplicity of a.

2.2. REMARK. If in the situation of 2.1 the ideal I has the zeros $(a_j)_{j \in \mathbf{N}}$ (counted with respect to multiplicities), then

$$A_{\mathbf{P}}/I \cong \left\{ x \in \mathbf{C}^{\mathbf{N}} \,\middle|\, \|x\|_k = \sup_{j \in \mathbf{N}} |x_j| \exp(-p_k(a_j)) < \infty \text{ for some } k \in \mathbf{N} \right\},$$

as (DFN) spaces (see Meise [7, 3.7]).

2.3. LEMMA. *Let $l \in \mathbf{N}_0$ and let u be a radial subharmonic function on $\mathbf{D}$. If $u(z) \geq 0$ on $|z| = 1 - 2^{-l}$, then, with the notation of 2.1, there exists $g \in A_{\mathbf{P}}$ such that $\rho_l(g)$ is the identity element of E_l, and there are $k \in \mathbf{N}$ and $C > 0$ (not depending on l or u) with*

$$\log|g(z)| \leq p_k(z) + \sup_{|w-z| \leq (1/2)(1-|z|)} u(w) + C, \quad z \in \mathbf{D}.$$

Proof. In 2.1, consider the function

$$\xi(z) = \begin{cases} 1 & \text{if } z \in S_l, \\ 0 & \text{if } z \in \bigcup_{j \neq l} S_j. \end{cases}$$

By the hypothesis, we have $u(z) \geq 0$ on S_l, hence $|\xi(z)| \leq \exp u(z)$ with $z \in \bigcup_{j \in \mathbf{N}_0} S_j$. Checking the proof of 2.1, we get the assertion. □

2.4. LEMMA. *Let $(a_j)_{j \in \mathbf{N}}$ be a sequence in $\mathbf{D}$ with $\lim_{j \to \infty} |a_j| = 1$. Then the following assertions are equivalent:*

(i) *There exists $m \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there are $n \in \mathbf{N}$ and $j_0 \in \mathbf{N}$ with*

$$\omega_n^{-1}(p_k(a_j)) \leq \frac{p_m(a_j)}{1 - |a_j|}, \quad j \in \mathbf{N}, \ j \geq j_0.$$

(ii) *There exists a sequence $(d_j)_{j \in \mathbf{N}}$ of nonnegative numbers such that for each $k \in \mathbf{N}$ there are $n \in \mathbf{N}$ and $j_0 \in \mathbf{N}$ with*

$$\exp p_k(a_j) \leq |a_j|^{d_j} \exp \omega_n(d_j), \quad j \in \mathbf{N}, \ j \geq j_0.$$

Proof. By Definition 1.1(β), (i) is equivalent to: There exists $m \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there are $n \in \mathbf{N}$ and $j_0 \in \mathbf{N}$ with

$$\omega_n^{-1}(p_k(a_j)) \leq \frac{p_m(a_j)}{\log(1/|a_j|)}, \quad j \in \mathbf{N}, \ j \geq j_0. \tag{1}$$

(i)⇒(ii): Let m be chosen according to (1). For $j \in \mathbf{N}$, put

$$d_j := \frac{p_m(a_j)}{\log(1/|a_j|)} \text{ if } a_j \neq 0 \quad \text{and} \quad d_j := 0 \text{ if } a_j = 0.$$

Now, let $k \in \mathbf{N}$ be given. Choose $\tilde{k} \in \mathbf{N}$ with

$$p_m(a_j) + p_k(a_j) \leq p_{\tilde{k}}(a_j)$$

for almost all $j \in \mathbf{N}$. From (1) we get $n \in \mathbf{N}$ such that, for almost all $j \in \mathbf{N}$,

$$\omega_n^{-1}(p_{\tilde{k}}(a_j)) \leq d_j,$$

hence

$$p_{\tilde{k}}(a_j) \leq \omega_n(d_j) = \inf_{0<r<1} (p_n(r) - d_j \log r);$$

that is,

$$d_j \log r \leq p_n(r) - p_{\tilde{k}}(a_j), \quad 0 < r < 1.$$

By the definition of d_j, we conclude that

$$(r/|a_j|)^{d_j} \le \exp(p_n(r) - p_{\bar{k}}(a_j) + p_m(a_j)) \\ \le \exp(p_n(r) - p_k(a_j)), \qquad 0<r<1,$$

for almost all $j \in \mathbf{N}$. From this we get (ii).

(ii)$\Rightarrow$(i): Since $|a_j|<1$, by (ii) there is $n \in \mathbf{N}$ for each $k \in \mathbf{N}$ such that, for almost all $j \in \mathbf{N}$,

(2) $$\omega_n^{-1}(p_k(a_j)) \le d_j,$$

and, since $p_k(a_j) \ge 0$,

$$d_j \log(1/|a_j|) \le \omega_n(d_j) \le p_n(\sqrt{|a_j|}) - d_j \log\sqrt{|a_j|}$$

and hence

$$d_j \le \frac{2p_n(\sqrt{|a_j|})}{\log(1/|a_j|)}.$$

By Definition 1.1(β) and (γ), there exists $m \in \mathbf{N}$ with

(3) $$d_j \le \frac{p_m(|a_j|)}{\log(1/|a_j|)}$$

for almost all $j \in \mathbf{N}$. Now (2) and (3) yield (1), so the proof is finished. □

REMARK. If $(a_j)_{j\in\mathbf{N}}$ are the zeros of an ideal in $A_\mathbf{P}$, then by applying Jensen's formula one can prove that $(d_j)_{j\in\mathbf{N}}$ can be chosen as a subsequence of $(j)_{j\in\mathbf{N}}$ (see Momm [12, 4.4]).

2.5. PROPOSITION. *Let I be a nonzero closed ideal in $A_\mathbf{P}$ with infinitely many zeros $(a_j)_{j\in\mathbf{N}}$. If I is complemented, then there exists $m \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there are $n \in \mathbf{N}$ and $j_0 \in \mathbf{N}$ with*

$$\omega_n^{-1}(p_k(a_j)) \le \frac{p_m(a_j)}{1-|a_j|}, \quad j \in \mathbf{N},\ j \ge j_0.$$

Proof. We use the notation of 2.1. Let R be a continuous linear right inverse for the quotient map $A_\mathbf{P} \to A_\mathbf{P}/I$. For $j \in \mathbf{N}$, choose $l \in \mathbf{N}_0$ with $a_j \in N_l = V(I) \cap S_l$. Let 1_l denote the identity element of E_l and define

$$u_j(z) := \max_{|w|=|z|} \log|R[1_l](w)|, \quad z \in \mathbf{D}.$$

Then from the continuity of R we get (see Meise and Taylor [10, 2.13]): For each $k \in \mathbf{N}$ there are $n \in \mathbf{N}$ and $j_0 \in \mathbf{N}$ with

(1) $$u_j(z) \le p_n(z) - p_k(a_j), \quad z \in \mathbf{D},\ j \in \mathbf{N},\ j \ge j_0.$$

Since u_j is subharmonic, $x \mapsto u_j(e^x)$ is a convex and increasing function of $x<0$. Thus, since $u_j(a_j) \ge 0$, there exists $d_j \ge 0$ with

$$d_j \log(r/|a_j|) \le u_j(r), \quad 0<r<1,\ j \in \mathbf{N}.$$

Then, from (1) and Lemma 2.4, we get the assertion. □

2.6. PROPOSITION. *Let I be a nonzero closed ideal in $A_{\mathbf{P}}$ with infinitely many zeros which satisfies the following condition: There exists $m \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there is $n \in \mathbf{N}$ with*

$$\omega_n^{-1}(p_k(a)) \leq \frac{p_m(a)}{1-|a|}$$

for almost all zeros a of I. Then I is complemented in $A_{\mathbf{P}}$.

Proof. Let the ideal I be given. Let π be the quotient map $A_{\mathbf{P}} \to A_{\mathbf{P}}/I$. We shall construct a continuous linear right inverse for π using an idea of proof from Taylor [14, Thm. 5.2] and Meise, Momm, and Taylor [8, 3.3].

We will use the notation of 2.1. In fact, we will construct a continuous linear right inverse for $\rho = (\rho_l)_{l \in \mathbf{N}_0}$. Occasionally, we will identify the algebras E_l with subalgebras of $A_{\mathbf{P}}/I$. First, choose right inverses R_l for ρ_l such that there exist $k_1 \in \mathbf{N}$ and $C_1 > 0$ with

$$\text{(1)} \qquad \|R_l[x_l]\|_{k_1} \leq C_1 \dim E_l |x_l|_l, \quad x_l \in E_l, \ l \in \mathbf{N}_0.$$

This can be done choosing an Auerbach basis $(e_{l,i})_{1 \leq i \leq \dim E_l}$ in $(E_l, |\ |_l)$, $l \in \mathbf{N}_0$ (see Meise [7, 1.3]). Since $\{e_{l,i} \mid l, i\}$ is bounded in $A_{\mathbf{P}}/I$, there are functions $f_{l,i} \in A_{\mathbf{P}}$ with $\pi(f_{l,i}) = e_{l,i}$ such that $\{f_{l,i} \mid l, i\}$ is bounded in $A_{\mathbf{P}}$. Then, put $R_l(\sum_i \lambda_i e_{l,i}) := \sum_i \lambda_i f_{l,i}$, $l \in \mathbf{N}_0$. Another way to get such operators is to use interpolation formulas like Berenstein and Taylor [1, formula (44), p. 132].

Now we construct "good" preimages g_l of the identity elements 1_l of E_l, $l \in \mathbf{N}_0$, with respect to π: If $E_l = 0$, put $g_l :\equiv 0$. If $l \in \mathbf{N}_0$ with $E_l \neq 0$ then $N_l := V(I) \cap S_l \neq \emptyset$. We define $r_l := 1 - 2^{-l}$. Since N_l is contained in $r_l \leq |z| \leq r_l + \frac{1}{4}(1 - r_l)$, by Definition 1.1($\gamma$), the hypothesis is also valid with the zeros of I replaced by $\{r_l \mid E_l \neq 0\}$. Hence, from Lemma 2.4 we get that there are nonnegative numbers $(d_l)_{E_l \neq 0}$ such that for each $k \in \mathbf{N}$ there are $n \in \mathbf{N}$ and $l_0 \in \mathbf{N}$ with

$$\text{(2)} \qquad u_l(z) := d_l \log(|z|/r_l) \leq p_n(z) - p_k(r_l), \quad z \in \mathbf{D}, \ l \geq l_0, \ E_l \neq 0.$$

Since $u_l(z) = 0$ on $|z| = 1 - 2^{-l}$, for $l \in \mathbf{N}_0$ with $E_l \neq 0$, we can apply Lemma 2.3 to get functions g_l such that $\rho(g_l)$ is the identity element of E_l.

By (2), 1.2(γ), and 2.3, the functions g_l satisfy the following condition: For each $k \in \mathbf{N}$ there are $n \in \mathbf{N}$ and $C > 0$ with

$$\text{(3)} \qquad \log|g_l(z)| \leq p_n(z) - p_k(r_l) + C, \quad z \in \mathbf{D}, \ l \in \mathbf{N}_0.$$

Now, identifying $A_{\mathbf{P}}/I$ with $A_{\mathbf{P}}(\mathcal{S})/I$ and using the nuclearity of $A_{\mathbf{P}}(\mathcal{S})/I$ (or just applying Jensen's formula to get $j = O(p_{k_2}(a_j))$ for an appropriate $k_2 \in \mathbf{N}$), by standard arguments, from (1) and (3) we get that a continuous linear right inverse R for π is given by

$$R: A_{\mathbf{P}}/I \to A_{\mathbf{P}}, \qquad \sum_{l=0}^{\infty} x_l \mapsto \sum_{l=0}^{\infty} g_l R_l[x_l]. \qquad \square$$

2.7. THEOREM. *Let Ψ be a weight system and let $\mathbf{P}$ be as in Definition 1.4. A nonzero closed ideal I is complemented in $A_{\mathbf{P}}$ if and only if its zeros satisfy* *the following condition: There exists $m\in\mathbf{N}$ such that for each $k\in\mathbf{N}$ there is $n\in\mathbf{N}$ with*

$$\psi_k\left(\frac{1}{1-|a|}\right)\psi_n^{-1}\left(\psi_k\left(\frac{1}{1-|a|}\right)\right)\le\frac{1}{1-|a|}\psi_m\left(\frac{1}{1-|a|}\right)$$

for almost all zeros a of I.

Proof. For a closed ideal with finitely many zeros, the codimension is finite and hence the ideal is complemented. For I having infinitely many zeros, from Propositions 2.5 and 2.6 we get that I is complemented if and only if there exists $m\in\mathbf{N}$ such that, for each $k\in\mathbf{N}$, there is $n\in\mathbf{N}$ with

$$\omega_n^{-1}(p_k(a))\le\frac{p_m(a)}{1-|a|}$$

for almost all zeros a of I. By Lemma 1.11 and Definition 1.1(β) and (γ), this is equivalent to the above condition. □

2.8. COROLLARY. *Let $\Psi=(k\psi)_{k\in\mathbf{N}}$ be as in Lemma 1.3. Then each closed ideal is complemented in $A_{\mathbf{P}}$.*

Proof. Using 1.1(γ), it is easy to verify the criterion of Theorem 2.7. Just put $m=1$ and, for given $k\in\mathbf{N}$, choose $n\in\mathbf{N}$ with $\psi(kx)\le(n/k)\psi(x)$ for sufficiently large $x\ge1$. □

2.9. EXAMPLE. Let $\rho>1$ and define $\Psi=(\psi_k)_{k\in\mathbf{N}}$ as follows:

$$\psi_k(x)=x^\rho f_k(x),\quad x\ge1,$$

where

$$f_k(x)=\alpha(x)^k\beta(x)^{1-1/k},\quad x\ge1,$$

with continuous, increasing, unbounded functions α and β on $x\ge1$ with values in $x\ge1$ which have the properties

$$\limsup_{x\to\infty}\frac{\alpha(x^2)}{\alpha(x)}<\infty\quad\text{and}\quad\limsup_{x\to\infty}\frac{\beta(x^2)}{\beta(x)}<\infty.$$

Then Ψ is a weight system. We put $q(x):=\log a(x)/\log\beta(x)$, $x\ge1$, and consider three cases.

1. $\liminf_{x\to\infty}q(x)>0$. Then each closed ideal is complemented in $A_{\mathbf{P}}$.
2. $\lim_{x\to\infty}q(x)=0$. Then only those closed ideals are complemented in $A_{\mathbf{P}}$ which are generated by a polynomial, that is, only the closed ideals with finitely many zeros and the null ideal.
3. $0=\liminf_{x\to\infty}q(x)<\limsup_{x\to\infty}q(x)$. Then there are noncomplemented closed ideals as well as complemented nonzero ideals with infinitely many zeros.

Proof. Condition 1.1(γ) can be verified as in the proof of Lemma 1.3, since $\liminf_{x\to\infty}\psi_k(2x)/2\psi_k(x)\ge2^{\rho-1}>1$, $k\in\mathbf{N}$. Hence Ψ is a weight system.

If $(a_j)_{j\in\mathbf{N}}$ are the zeros of a closed ideal I of $A_{\mathbf{P}}$, put $x_j:=1/(1-|a_j|)$, $j\in\mathbf{N}$. Note that the hypotheses imply that $\alpha(x)=O((\log x)^c)$ and $\beta(x)=O((\log x)^c)$ for some $c>0$. Using the hypothesis, from Theorem 2.7 we get that I is complemented in $A_{\mathbf{P}}$ if and only if there exists $m\in\mathbf{N}$ such that for each $k\in\mathbf{N}$ there are $n\in\mathbf{N}$ and $j_0\in\mathbf{N}$ with

$$\left(\frac{f_k(x_j)}{f_m(x_j)}\right)^\rho \le \frac{f_n(x_j)}{f_k(x_j)}, \quad j\in\mathbf{N},\ j\ge j_0,$$

which is equivalent to

$$q(x_j)(k-m)\rho+\left(\frac{1}{m}-\frac{1}{k}\right)\le q(x_j)(n-k)+\left(\frac{1}{k}-\frac{1}{n}\right), \quad j\in\mathbf{N},\ j\ge j_0.$$

Obviously, this is true if $\liminf_{j\to\infty} q(x_j)>0$ and false in the other case. □

2.10. EXAMPLES. The assertion of Example 2.9(1) is also true for the weight systems 1.5(a) and (c). The assertion of Example 2.9(2) is also true for the weight systems 1.5(b).

2.11. APPLICATION. We will sketch how, via duality theory, Theorem 2.7 applies to (FN) spaces of analytic functions on the disc being smooth up to the boundary.

Let

$$M=(M_{p,k})_{p\in\mathbf{N}_0,k\in\mathbf{N}}$$

be a matrix of positive numbers such that $(M_{p,k}/(pM_{p-1,k}))_{p\in\mathbf{N}}$ is increasing and unbounded for each $k\in\mathbf{N}$. Assume that for each $k\in\mathbf{N}$ there are $n\in\mathbf{N}$ and $p_0\ge 0$ such that, for all $p\ge p_0$,

$$M_{p,k+1}\le M_{p,k}, \quad M_{p+1,n}\le M_{p,k}, \quad M_{2p,n}\le M_{p,k}^2, \quad M_{p,n}2^p\le M_{p,k}$$

(e.g., $M_{p,k}=p!^s/k^p$, where $1<s$). Consider the Fréchet space

$$\mathcal{E}_M=\{f\in C(\bar{\mathbf{D}})\mid f\in A(\mathbf{D}),\ \tilde f\in C^\infty(\mathbf{R}),\ \tilde f(t):=f(e^{it}),\ t\in\mathbf{R},$$
$$\|f\|_k=\sup_{t\in\mathbf{R}}\ \sup_{p\in\mathbf{N}_0}|\tilde f^{(p)}(t)|/M_{p,k}<\infty \text{ for all } k\in\mathbf{N}\}.$$

By the Cauchy–Fantappié transform $\hat{}: \mathcal{E}'_{M_b}\to A_{\mathbf{P}}$,

$$\hat\mu(z)=\sum_{j=0}^{\infty}\mu_w(w^j)z^j=\mu_w\left(\frac{1}{1-zw}\right), \quad z\in\mathbf{D},$$

we can identify the strong dual $\mathcal{E}'_{M_b}$ with $A_{\mathbf{P}}$, where $\Psi=(\psi_k)_{k\in\mathbf{N}}$ and

$$\psi_k(x):=\sup_{p\in\mathbf{N}_0}\log\left(\frac{x^p p!}{M_{p,k}}\right), \quad x\ge 1,$$

has all the properties of a weight system, with the exception of 1.1(δ) (see Momm [12, §6] and Körner [5]; in our example, the weight functions ψ_k are equivalent to $\tilde\psi_k(x)=kx^{1/(s-1)}$).

By this identification, a closed linear subspace $W \subset \mathcal{E}_M$ is S-invariant,

$$S: \mathcal{E}_M \to \mathcal{E}_M, \quad Sf(z) = \frac{f(z)-f(0)}{z},$$

if and only if $W^\perp \subset A_{\mathbf{P}}$ is a closed ideal. In this case, $z \in \mathbf{D}$ is a zero of $W^\perp$ if and only if the function $e_z: w \mapsto 1/(1-zw)$ belongs to W.

Now, in addition assuming 1.1(δ) (i.e., $s<2$ in our example), from Theorem 2.7 we obtain: *A proper S-invariant subspace W of $\mathcal{E}_M$ is complemented if and only if there exists $m \in \mathbf{N}$ such that, for each $k \in \mathbf{N}$, there is $n \in \mathbf{N}$ with*

$$\psi_k\left(\frac{1}{1-|a|}\right)\psi_n^{-1}\left(\psi_k\left(\frac{1}{1-|a|}\right)\right) \le \frac{1}{1-|a|}\psi_m\left(\frac{1}{1-|a|}\right)$$

for almost all $a \in \mathbf{D}$ with $e_a \in W$.

If we apply this result to particular S-invariant subspaces, that is, to kernels of Toeplitz operators T_μ, $\mu \in \mathcal{E}'_M$, where

$$T_\mu: \mathcal{E}_M \to \mathcal{E}_M, \quad T_\mu f = \sum_{n=0}^{\infty} \mu_w(w^n) S^n f,$$

since $T_\mu = M_{\hat{\mu}}^t$, $M_{\hat{\mu}}: A_{\mathbf{P}} \to A_{\mathbf{P}}$, and $M_{\hat{\mu}} f = \hat{\mu} f$, we have: *A nonzero Toeplitz operator T_μ, $\mu \in \mathcal{E}'_M$, admits a continuous linear right inverse if and only if the zeros of the characteristic function $\hat{\mu}$ satisfy the condition of Theorem 2.7.*

3. (FN) Algebras

The results of the previous sections concerning (DFN) algebras have an extension to another type of weighted algebras. Hereafter, a weight system as in Definition 1.1 will be called an *inductive* weight system, in contrast to those defined next.

3.1. DEFINITION. Let $\Psi = (\psi_k)_{k \in \mathbf{N}}$ be a sequence of continuous, nonnegative, strictly increasing, unbounded functions on $x \ge 1$. Ψ will be called a *projective weight system* if for each $n \in \mathbf{N}$ there are $k \in \mathbf{N}$ and $x_0 \ge 1$ such that, for all $x \ge x_0$,

(α) $\psi_{k+1}(x) \le \psi_k(x)$, ($\beta$) $2\psi_k(x) \le \psi_n(x)$,

(γ) $\psi_k(2x) \le \psi_n(x)$, (δ) $x(\int_1^x \sqrt{\psi_k(t)/t^3}\, dt)^2 \le \psi_n(x)$.

3.2. DEFINITION. Let Ψ be a projective weight system as in Definition 3.1. We put

$$\mathbf{P} = (p_k)_{k \in \mathbf{N}}, \quad p_k(z) := \psi_k\left(\frac{1}{1-|z|}\right), \quad z \in \mathbf{D}, \ k \in \mathbf{N},$$

and define

$$A_{\mathbf{P}}^0 = \left\{ f \in A(\mathbf{D}) \,\middle|\, \|f\|_k = \sup_{z \in \mathbf{D}} |f(z)| e^{-p_k(z)} < \infty \text{ for each } k \in \mathbf{N} \right\}.$$

Endowed with the topology induced by the norms $(\| \|_k)_{k\in\mathbf{N}}$, $A^0_{\mathbf{P}}$ is an (FN) algebra; that is, $A^0_{\mathbf{P}}$ is a nuclear Fréchet space.

CONVENTION. In the sequel let Ψ and $\mathbf{P}$ be as in Definition 3.2.

3.3. REMARK. Most of the results of the first and second section are valid also for the (FN) algebra $A^0_{\mathbf{P}}$, only making minor changes in the assertions. Thus 1.2, 1.7, 1.9, and 1.11 are valid without further changes. 1.3 and 1.8 are valid with $(k\psi)_{k\in\mathbf{N}}$ replaced by $((1/k)\psi)_{k\in\mathbf{N}}$. In 1.6, 1.10, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, and 2.7 we have to make a systematic change of the quantifiers which are combined with positive integers k, m, n: So, replace "$\exists k$" by "$\forall k$", "$\forall k\,\exists n$" by "$\forall n\,\exists k$", "$\exists m\,\forall k\,\exists n$" by "$\forall n\,\exists k\,\forall m$", and replace "(DFN)" by "(FN)".

Let us state the main result for (FN) algebras:

3.4. THEOREM. *Let Ψ be a projective weight system and let $\mathbf{P}$ be defined as in Definition 3.2. A nonzero ideal I is complemented in $A^0_{\mathbf{P}}$ if and only if its zeros satisfy the following condition: For each $n\in\mathbf{N}$ there exists $k\in\mathbf{N}$ such that for each $m\in\mathbf{N}$*

$$\psi_k\left(\frac{1}{1-|a|}\right)\psi_n^{-1}\left(\psi_k\left(\frac{1}{1-|a|}\right)\right)\leq\frac{1}{1-|a|}\psi_m\left(\frac{1}{1-|a|}\right)$$

for almost all zeros a of I.

Proof. Let I be a nonzero closed ideal in $A^0_{\mathbf{P}}$ with infinitely many zeros. First, we must prove the analogue of Proposition 1.9.

To prove the localization result of 1.9, we use a modified version of the proof of Kelleher and Taylor [4, 4.6]: If $F\in I\setminus\{0\}$, then we define the following associated inductive weight system $\Phi=(\phi_k)_{k\in\mathbf{N}}$: Put

$$\phi\left(\frac{1}{1-r}\right)=\max_{|z|=r}\log|F(z)|,\quad 0\leq r<1,$$

and define inductively

$$\phi_0(x):=\max\{\phi(x),1\},\quad \phi_k(x):=4x\left(\int_1^{4x}\sqrt{\frac{\phi_{k-1}(t)}{t_3}}\,dt\right)^2,\quad x\geq 1,\ k\in\mathbf{N}.$$

With the arguments of the proof of Lemma 1.2, we get that Φ is an inductive weight system such that the corresponding (DFN) algebra A_Q contains the function F. Furthermore,

$$\phi_k(x)=o(\psi_n(x)),\quad k,n\in\mathbf{N}.$$

Now we repeat the proof of Kelleher and Taylor [4, 4.6], with A_Q instead of A_p and with I^* replaced by $\{f\in A_Q \mid fI_{\text{loc}}\subset I\}$, to obtain that this ideal (in A_Q) equals A_Q. Hence it contains the identity and $I=I_{\text{loc}}$. By the "jiggling of zeros" argument (see 1.9), we conclude that $I=I_{\text{loc}}(F_1, F_2)$ with appropriate $F_1, F_2\in A^0_{\mathbf{P}}\setminus\{0\}$.

To prove the analogue of 2.1, proceed as in Meise and Taylor [9, 2.7], but use the inductive weight system Φ constructed above, with F replaced by $(|F_1|^2+|F_2|^2)^{1/2}$.

The remaining parts of the proof of Theorem 3.4 are analogous to those of Theorem 2.7. For more details, we refer to Momm [12]. □

3.5. COROLLARY. *Let* $\Psi=((1/k)\psi)_{k\in\mathbf{N}}$ *with* ψ *as in Lemma 1.3* (see Remark 3.3). *Then only those closed ideals are complemented in* $A^0_\mathbf{P}$ *which are generated by a polynomial.*

Proof. By Lemma 1.2, we may assume that $x\mapsto\psi(x)/x$ is increasing. Hence we have $(1/k)\psi(x)\geq\psi((1/k)x)$ for all $k\in\mathbf{N}$ and $x\geq 1$. For this reason there exists $n\in\mathbf{N}$ (take $n=1$) such that for each $k\in\mathbf{N}$ there is $m\in\mathbf{N}$ (choose $m>k^2$) with

$$\frac{1}{k}\psi(x)\psi^{-1}\left(\frac{n}{k}\psi(x)\right)>x\frac{1}{m}\psi(x),\quad x\geq 1.$$

Thus, from Theorem 3.4, we get the assertion. □

3.6. EXAMPLE. Let $\rho>1$ and define $\Psi=(\psi_k)_{k\in\mathbf{N}}$ as follows:

$$\psi_k(x)=x^\rho f_k(x),\quad x\geq 1,$$

where

$$f_k(x)=\alpha(x)^{-k}\beta(x)^{1+1/k},\quad x\geq 1,$$

with α and β as in 2.9. In addition, let the function $x\mapsto x^\epsilon/\alpha(x)$ be increasing for each $\epsilon>0$ and for sufficiently large $x\geq 1$. Then Ψ is a projective weight system (after a suitable change of the functions ψ_k for small $x\geq 1$). With q as in 2.9, we again consider three cases.

1. $\liminf_{x\to\infty}q(x)>0$. Then only those closed ideals are complemented in $A^0_\mathbf{P}$ which are generated by a polynomial.
2. $\lim_{x\to\infty}q(x)=0$. Then each closed ideal is complemented in $A^0_\mathbf{P}$.
3. $0=\liminf_{x\to\infty}q(x)<\limsup_{x\to\infty}q(x)$. Then there are noncomplemented closed ideals as well as complemented nonzero ideals with infinitely many zeros.

Proof. If $(a_j)_{j\in\mathbf{N}}$ are the zeros of a closed ideal I of $A^0_\mathbf{P}$, put $x_j:=1/(1-|a_j|)$, $j\in\mathbf{N}$. As in 2.9, we get that I is complemented if and only if for each $n\in\mathbf{N}$ there is $k\in\mathbf{N}$ such that, for each $m\in\mathbf{N}$, there exists $j_0\in\mathbf{N}$ with

$$q(x_j)(m-k)\rho+\left(\frac{1}{k}-\frac{1}{m}\right)\leq q(x_j)(k-n)+\left(\frac{1}{n}-\frac{1}{k}\right),\quad j\in\mathbf{N},\ j\geq j_0.$$

Obviously, this is true if $\lim_{j\to\infty}q(x_j)=0$ and false in the other case. □

3.7. EXAMPLES. The assertion of Example 3.6(1) is also true for $\psi_k(x)=(1/k)x^\rho$, $\rho>1$. The assertion of Example 3.6(2) is also true for $\psi_k(x)=x^{\rho+1/k}$, $\rho\geq 1$.

3.8. APPLICATION. Changing the quantifiers, analogous to 2.11, we get an application of Theorem 3.4 to shift invariant subspaces of (DFN) spaces of analytic functions on the disc being smooth up to the boundary (see Momm [12]).

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