Parareflexive Operators on Banach Spaces

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1. Introduction

Nilpotent operators on finite-dimensional spaces are all direct sums of Jordan cells. In infinite-dimensional Hilbert spaces this is not true in general; however, every nilpotent operator there is still quasi-similar to a direct sum of Jordan cells [1]. A slightly different approach yields quasi-similarity of an arbitrary nilpotent operator on a Hilbert space to a direct sum of Jordan models, that is, Jordan block-cells [1]. The latter approach seems more appropriate for transplanting these results into a Banach space. This is accomplished in Section 2, where we apply the notion of quasi-similarity that was extended to Banach spaces in [9].

The results of Section 2 are applied in Section 3 to extend to Banach spaces a Hilbert space result on parareflexive operators [1]. Unfortunately, some of our results are proven only in Banach spaces that satisfy a technical condition (see condition (A) introduced in §2). This condition is satisfied in particular by all Hilbert spaces and by all separable Banach spaces. Our main result is a characterization of parareflexive operators on Banach spaces that satisfy condition (A).

2. Nilpotent Operators

The most simple nilpotent operators on a Banach space are *Jordan operators,* that is, operators $J_m(X)$ acting on a direct sum $X^m = X \oplus X \oplus \cdots \oplus X$ of m copies of a Banach space X, where m is a positive integer supplied with (say) the l_1 norm and defined by

$$J_m(X)(x_1, x_2, \ldots, x_m) = (x_2, x_3, \ldots, x_m, 0).$$

Observe that this operator can be represented by an operator matrix

$$J_m(X) = \begin{bmatrix} 0 & I & 0 & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 0 & I \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

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We do not exclude here the possibility $m=1$, which gives merely the zero operator on the space X. Every nilpotent operator on a Hilbert space is quasi-similar to a direct sum of Jordan operators [1]. Here we give a Banach space extension of this result, using the notion of inner and outer representations [9]. An operator V between the Banach spaces X and Y is called a *quasi-affinity* if it has trivial kernel and dense range. An operator T_0 on a Banach space X_0 is said to be an *inner representation* of an operator T, acting on a given Banach space X, if there exists a quasi-affinity V_0 from X_0 into X such that $V_0 T_0 = T V_0$. An operator T^0 on a Banach space X^0 is said to be an *outer representation* of T if there exists a quasi-affinity V^0 from X into X^0 such that $T^0 V^0 = V^0 T$.

The main result of this section is the following theorem, which holds in every Banach space satisfying the technical condition defined here. (Of course, this condition is introduced merely for the purposes of this note.) We say that a Banach space X has property (A) if:

(A1) for any closed subspace U of X there exists a closed subspace V of X such that $V \cap U = \{0\}$ and $V \oplus U$ is dense in X; and
(A2) every subspace and every factor space of X has property (A1).

Observe that every Hilbert space has this property, since we may take $V = U^{\perp}$ in (A1) while the subspaces and the factor spaces of a Hilbert space are all Hilbert spaces again. By a result of Murray and Mackey (see [8], [7]), every separable Banach space has this property.

2.1. THEOREM. *Let T be a nilpotent operator of order n on a nontrivial Banach space X satisfying property* (A). *Then there exist Banach spaces Y_i and Z_i, $i = 1, 2, \ldots, n$, such that the operator*

$$R = \sum_{i=1}^{n} \oplus J_i(Y_i)$$

on the space

$$Y = \sum_{i=1}^{n} \oplus Y_i^i$$

is an inner representation of T, and the operator

$$S = \sum_{i=1}^{n} \oplus J_i(Z_i)$$

on the space

$$Z = \sum_{i=1}^{n} \oplus Z_i^i$$

is an outer representation of T.

REMARK. We do not exclude the possibility that some of the subspaces Y_i and Z_i are trivial. However, since the order of the nilpotent operator T is n, the subspaces Y_n and Z_n are necessarily nontrivial.

Proof. Let us first give the inner part of the proof. Start with a subspace Y_n such that

$$X = \operatorname{Ker} T^n = \operatorname{Cl}(Y_n \oplus \operatorname{Ker} T^{n-1})$$

and suppose inductively that we have already determined subspaces Y_n, $Y_{n-1}, \ldots, Y_{n-k+1}$ for a fixed index $k<n$, where $Y_{n-i} \subset \operatorname{Ker} T^{n-i}$ for $i=0,1, \ldots, k-1$. Denote by U_{n-k} the span of $\operatorname{Ker} T^{n-k-1}$ and the sets $T^i Y_{n-k+i}$ for $i=1,2,\ldots,k$, and observe that $U_{n-k} \subset \operatorname{Ker} T^{n-k}$. Using property (A) we can find a subspace $Y_{n-k} \subset \operatorname{Ker} T^{n-k}$ such that

$$\operatorname{Ker} T^{n-k} = \operatorname{Cl}(Y_{n-k} \oplus U_{n-k}). \tag{1}$$

Finally, define the mapping

$$V\colon Y = \sum_{i=1}^{n} \oplus Y_i^i \to X, \quad \text{and}$$

$$V\colon \sum_{i=1}^{n} \oplus (y_i^1, y_i^2, \ldots, y_i^i) \mapsto \sum_{i=1}^{n} \sum_{j=1}^{i} T^{i-j} y_i^j.$$

Observe that for the operator

$$R = \sum_{i=1}^{n} \oplus J_i(Y_i),$$

it holds that $VR = TV$. It remains to show that V is a quasi-affinity.

In order to prove that the kernel of V is trivial, choose any vector

$$\sum_{i=1}^{n} \oplus (y_i^1, \ldots, y_i^i) \in Y$$

such that

$$\sum_{i=1}^{n} \sum_{j=1}^{i} T^{i-j} y_i^j = 0.$$

Note that y_n^n is an element of $Y_n \cap \operatorname{Ker} T^{n-1}$ and is therefore equal to zero. Suppose, inductively, that $y_{n-j+i}^{n-j} = 0$ for all $0 \le i \le j < k$, where k is a fixed index, $0 \le k < n$. Observe at first that $y_{n-k}^{n-k} \in Y_{n-k}$ and is equal to

$$-\sum_{i=1}^{k} T^i y_{n-k+i}^{n-k} - \sum_{j=k+1}^{n-1} \sum_{i=0}^{j} T^i y_{n-j+i}^{n-j},$$

where the double sum is supposed to be trivial in the case $k+1 < n-1$.

The terms of the first sum are elements of $T^i Y_{n-k+i}$, while the terms of the second sum all belong to $\operatorname{Ker} T^{n-k-1}$; therefore, the whole sum lies in the space U_{n-k}, which yields $y_{n-k}^{n-k} = 0$ by (1). Using an induction argument on the index i, we shall actually show that all y_{n-k+i}^{n-k} are equal to zero. Suppose that we have already shown this for all indices i up to a fixed index $m \ge 0$. Then the vector $y_{n-k+m+1}^{n-k}$ is a sum of

$$-\sum_{i=m+2}^{k} T^{i-m-1} y_{n-k+1}^{n-k}$$

and of a vector contained in Ker T^{n-k+m}. Consequently, the vector $y^{n-k}_{n-k+m+1}$ belongs to the space $U_{n-k+m+1}$, and by (1) is therefore equal to zero. Induction on both indices gives us the desired result.

Next, show that V has dense range. Choose an arbitrary vector $x \in X$ and note that it can be approximated by a vector of the form $y^n_n + u_{n-1}$, where $y^n_n \in Y_n$ and $u_{n-1} \in \operatorname{Ker} T^{n-1}$. Suppose inductively that we have already approximated our vector x by a sum

$$\sum_{j=0}^{k-1} \sum_{i=0}^{j} T^i y^{n-j}_{n-j+i} + u_{n-k},$$

where $y^{n-j}_{n-j+i} \in Y_{n-j+i}$ and where $u_{n-k} \in \operatorname{Ker} T^{n-k}$ for a fixed index $k \geq 1$. Using (1) and the definition of the space U_{n-k}, we can approximate the vector u_{n-k} by a sum

$$\sum_{i=0}^{k} T^i y^{n-k}_{n-k+i} + u_{n-k-1},$$

with $y^{n-k}_{n-k+i} \in Y_{n-k+i}$ and $u_{n-k-1} \in \operatorname{Ker} Y^{n-k-1}$. Therefore, the sum

$$\sum_{j=0}^{k} \sum_{i=0}^{j} T^i y^{n-j}_{n-j+i} + u_{n-k-1}$$

approximates the original vector x. Since this induction ends after a finite number of steps, the final approximation may be made as close as desired, and the inner part of the proof is completed.

We turn now to the outer part of the proof. We begin by finding a closed subspace X_n of X such that

$$X = \operatorname{Cl}(X_n \oplus \operatorname{Cl}(\operatorname{Im} T^{n-1})),$$

and suppose inductively that subspaces $X_n, X_{n-1}, \ldots, X_{n-k+1}$ have already been found such that $X_{n-i} \supset \operatorname{Im} T^{n-i}$ for $i = 0, 1, \ldots, k-1$; here k is a fixed index, $1 < k < n$. Set

$$U_{n-k} = \operatorname{Cl}(\operatorname{Im} T^{n-k-1}) \cap \left(\bigcap_{i=1}^{k} T^{-i} X_{n-k+i} \right)$$

and note that $U_{n-k} \supset \operatorname{Im} T^{n-k}$. By condition (A) find a subspace X_{n-k} of X containing $\operatorname{Im} T^{n-k}$ such that

$$X/V_{n-k} = \operatorname{Cl}(X_{n-k}/V_{n-k} \oplus U_{n-k}/V_{n-k}), \tag{2}$$

where V_{n-k} denotes $\operatorname{Cl}(\operatorname{Im} T^{n-k})$. At the end of induction introduce the mapping

$$W: X \to Z = \sum_{i=1}^{n} \oplus Z^i_i,$$

where $Z_i = X/X_i$, by

$$W: x \mapsto \sum_{i=1}^{n} \oplus \sum_{j=1}^{i} \oplus (T^{j-1}x + X_i).$$

Since $T^i x \in X_i$, it follows easily from this definition that $WT = SW$, where

$$S = \sum_{i=1}^{n} \oplus J_i(Z_i)$$

is defined as an operator on Z. It remains to show that W is a quasi-affinity. But, for each $x \in \operatorname{Ker} W$, it follows that $T^i x \in X_r$ for all indices $i = 0, 1, \ldots, r-1$ and $r = 1, 2, \ldots, n$. This forces x to belong to U_1 and therefore to $V_1 = \operatorname{Cl}(\operatorname{Im} T)$, by equation (2) for $k = n-1$. Use this argument inductively to see that x belongs to all subspaces $\operatorname{Cl}(\operatorname{Im} T^k)$ for $k = 1, 2, \ldots, n$. Consequently, x must be equal to zero.

In order to see that the range of W is dense in the space Z, choose an arbitrary continuous functional on Z which annihilates $\operatorname{Im} W$. Recall the definition of Z to see that such a functional is determined by a double sequence of functionals $f_i^j \in X'$, for $j = 1, 2, \ldots, i$ and $i = 1, 2, \ldots, n$. With no loss of generality we may suppose that all functionals f_i^j annihilate X_i. Now, pick an arbitrary vector $x \in X$ and note that

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{i} f_i^j(T^{j-1}x) = 0. \tag{3}$$

If $x \in \operatorname{Im} T^{n-1}$, this equation reduces to

$$f_n^1(x) = 0$$

due to the fact that $T^{j-1}x = 0$ for $j > 1$ and $x \in \operatorname{Im} T^{n-1} \subset \operatorname{Im} T^{n-r} \subset X_{n-r}$ for $r \geq 1$. Since $x \in \operatorname{Im} T^{n-1}$ is arbitrary, and X_n and $\operatorname{Cl}(\operatorname{Im} T^{n-1})$ are disjoint, this forces f_n^1 to be trivial. Assume now inductively that we have already shown the functionals f_i^j to be trivial for all indices under consideration such that $i - j > n - k$, where k is a fixed integer, $1 < k < n$. For each vector $x \in \operatorname{Im} T^{n-k}$, equation (3) then reduces to

$$\sum_{i=n-k+1}^{n} f_i^{i+k-n}(T^{i+k-n-1}x) = 0$$

due to the following three facts: First, for $j > k-1$ we have $T^{j-1}x = 0$; second, for $j \leq k-1$ (but $j > i+k-n$) we obtain $T^{j-1}x \in \operatorname{Cl}(\operatorname{Im} T^r x) \subset X_r$ for each $r < n-k+j$ that includes $r = i$; third, for $j < i+k-n$, the functionals f_i^j are trivial by the induction hypothesis. Now choose first an arbitrary vector x from U_{n-k+1} to get from equation (2) (with index k decreased by 1) that f_{n-k+1}^1 is trivial. Continue by choosing x in such a way that Tx belongs to U_{n-k+2} and proceed inductively on i. After both inductions are completed we see that the functional f is trivial, and the theorem is proved. □

In Section 3 we shall need the following simple implication of this theorem.

2.2. PROPOSITION. *Under the assumptions of the theorem there exists a quasi-affinity from Y_n into Z_n. Therefore, Y_n is finite-dimensional if and only if the same holds for Z_n. In this case, their dimensions are equal and there also exists a quasi-affinity from Y_{n-1} into Z_{n-1}.*

Proof. Let the operators T on X, S on Y, and R on Z be defined as in the theorem. Denote by V the quasi-affinity from Y into X such that $TV = VS$ and by W the quasi-affinity from X into Z such that $WT = RW$. It follows that $R^k WV = WVS^k$ for all k, and therefore the restriction of WV to $\operatorname{Im} S^k$ (a closed subspace) is a quasi-affinity into $\operatorname{Im} R^k$ (which is also closed). The first assertion now follows from the fact that $\operatorname{Im} S^{n-1}$ is equal to the first copy of the space Y_n in the direct sum that defines Y, and $\operatorname{Im} R^{n-1}$ is equal to the first copy of Z_n in the defining sum for Z.

Similarly, $\operatorname{Im} S^{n-2}$ is the direct sum of the first two copies of Y_n and the first copy of Y_{n-1}, while $\operatorname{Im} T^{n-2}$ is the direct sum of the first two copies of Z_{n-1} and the first copy of Z_{n-1}; the proposition follows. □

3. Parareflexive Operators

A linear subspace M of a Banach space X will be called *paraclosed* if it is the range of some bounded linear mapping A from a Banach space Y into X. Note that we can always assume A to be one-to-one. Otherwise, take $Y/\operatorname{Ker} A$ instead of Y, and instead of A take the operator from $Y/\operatorname{Ker} A$ into X induced by A. Obviously, every closed subspace is paraclosed; however, in infinite-dimensional Banach spaces there exist paraclosed subspaces which are not closed. Also, in every Banach space of infinite dimension there exists a linear subspace which is not paraclosed. Indeed, in such a space we can find a sequence of vectors $\{x_i\}$ such that the distance of x_i to the linear span of the others is strictly positive for all i; then the linear span of these vectors is not paraclosed, as can easily be seen.

Observe that the intersection and the sum of two paraclosed subspaces is paraclosed. If $U_i = A_i Y_i$, $i = 1, 2$, then $U_1 + U_2 = AY$ where $Y = Y_1 \oplus Y_2$ and $A(y_1 \oplus y_2) = A_1 y_1 + A_2 y_2$ for $y_i \in Y_i$ $(i = 1, 2)$, while $U_1 \cap U_2 = AZ$ where $Z = \{y_1 \oplus y_2 \in Y;\ A_1 y_1 = A_2 y_2\}$.

Note that, when X is a Hilbert space, every subspace of X which is paraclosed in the sense of Foiaş [5] is also paraclosed in our sense; however, the converse is not true.

We begin this section with Banach space extensions of some results of Douglas and Foiaş [3]. Most of the proofs follow similar lines and will not be given in full detail. However, we shall introduce some new tricks to avoid the Nagy–Foiaş theory of contractions in Hilbert spaces.

In the following theorem, a vector $x \in X$ will be called *algebraic* for the operator T if there exists a polynomial p such that $p(T)x = 0$.

3.1. THEOREM. *Let S and T be operators on X, with S nonalgebraic and $\|S\| < 1$. Then $T = u(S)$ for a function $u \in H^\infty$ if and only if T leaves invariant the range of every bounded linear mapping A that intertwines S with a completely nonunitary contraction on a Hilbert space.*

Proof. If every vector $x \in X$ is algebraic for S, then S is algebraic by a well-known result of Kaplansky (see [6, Thm. 15]). Therefore, there must exist a nonalgebraic vector $x \in X$. Denote by U_x the smallest closed subspace that contains the vector x and is invariant under S.

Note that any function $\phi \in H^2$ is analytic on the open unit disc in the complex plane containing the spectrum of S. Consequently, we can define $A_x \phi = \phi(S)x$ to obtain a bounded linear mapping from H^2 into X. Assume that $A_x \phi = 0$ for a nonzero $\phi \in H^2$; then $\phi(S)x = 0$, which implies $\phi(S|_{U_x}) = 0$. It is well known (and easy to see, using the spectral mapping theorem) that this yields the existence of a polynomial p such that $p(S|_{U_x}) = 0$; this forces $p(S)x = 0$, contradicting the fact that the vector x is nonalgebraic. We have thus shown that the mapping A_x is one-to-one. Now observe that $SA_x = A_x M_+$, where M_+ is the unilateral shift on H^2 defined by $(M_+\phi)(\lambda) = \lambda\phi(\lambda)$, for $\phi \in H^2$ and λ from the unit disc. By the assumptions of the theorem the linear subspace $A_x H^2$ is invariant under T, and by the closed graph theorem $N = A_x^{-1} T A_x$ is a bounded linear operator on H^2. Moreover, it can easily be shown that N leaves invariant each closed subspace of H^2 that is invariant under M_+; it then follows by a result of Sarason (see [10, p. 514]) that there exists a function $u_x \in H^\infty$ such that $N = u_x(M_+)$ and finally $T|_{U_x} = u_x(S)|_{U_x}$. Using similar arguments as in the second part of the proof of Theorem 1 in [3], it can be shown that the function $u = u_x$ is actually independent of the choice of the nonalgebraic vector x. Also, the same result holds for possible algebraic vectors of S and therefore $T = u(S)$. The proof of the theorem's other implication is straightforward. □

3.2. THEOREM. *Let S and T be two operators on X, and let S be nonalgebraic. Then $T = u(S)$ for an entire function u if and only if T leaves invariant every paraclosed subspace invariant under S.*

Proof. Let us first settle the easier implication. Suppose that the paraclosed subspace $\operatorname{Im} A$, where A is a one-to-one bounded linear mapping from a Banach space Y into X, is invariant under S. Then the operator $R = A^{-1}SA$ is bounded (by the closed graph theorem), and

$$u(S)\operatorname{Im} A = u(S)AY = Au(R)Y \subset AY = \operatorname{Im} A$$

for every entire function u.

To prove the converse, define $S_r = rS$ for each real number r satisfying $0 < r < \|S\|^{-1}$. Clearly, we can apply Theorem 3.1 to the operator S_r to get a function $u_r \in H^\infty$ with $T = u_r(S_r)$. It can easily be shown that $u(\lambda) = u_r(r\lambda)$ is independent of r for all λ in the common definition set. It follows that u is an entire function, and since $T = u(S)$ the proof is complete. □

3.3. COROLLARY. *If S and T are two commuting operators on X, then T leaves invariant every paraclosed subspace of X invariant under S if and only if $T = u(S)$ for an entire function u.*

Proof. For nonalgebraic S the assertion follows immediately from Theorem 3.2. The algebraic case is covered by a result of Fillmore (see [4, Thm. 2]). □

An operator T on X is called *parareflexive* if each operator U on X leaving invariant all the paraclosed invariant subspaces of T is an entire function of T. This definition extends to operators on a Banach space the well-known Hilbert space notion of parareflexivity (see [1]). Here we give a result which may be of some independent interest.

3.4. THEOREM. *Every inner and outer representation of a parareflexive operator is parareflexive.*

Proof. Let T on Y be an outer representation of S on X, with the intertwining quasi-affinity denoted by V, and suppose that S is parareflexive. If T is not algebraic, then by Theorem 3.2 it is parareflexive and the proposition follows. Suppose now that T is algebraic; then the same holds for S. Choose an operator U leaving invariant every paraclosed subspace of Y invariant under T. In particular, U leaves invariant Im V, and by the closed graph theorem $W=V^{-1}UV$ is a bounded operator on X. Now, for every paraclosed subspace M of X that is invariant under S, the space VM is a paraclosed subspace of Y that is invariant under T and consequently also under U. Therefore, M is invariant under W and (since M was arbitrary) W is an entire function of S. It follows that U is (the same) entire function of T.

To obtain the inner part of the proof suppose that T, S, and V are the same as above with one exception: this time assume that T is parareflexive. Again, we may confine ourselves to the case where T and S are both algebraic, since the nonalgebraic case is covered by Theorem 3.2. Choose an operator U on X leaving invariant every paraclosed subspace of X that is invariant under S.

For each functional $f \in X'$, the dual of X, denote by M^f the smallest linear subspace of X' containing f that is invariant under S', the adjoint of S. Observe that M^f is finite-dimensional and therefore closed even in the X topololgy of X'. Denote by $M_\perp^f$ the annihilator of M^f in X, that is, the set of all $x \in X$ such that $g(x)=0$ for all $g \in M^f$. Note that $M_\perp^f$ is invariant under S, and since $M_\perp^f$ is closed, it must also be invariant under U. It follows that the annihilator $(M_\perp^f)^\perp$—that is, the set of all $g \in X'$ such that $g(x)=0$ for all $x \in M_\perp^f$—is invariant under U'. But, since M^f is closed in the X topology of X', we have necessarily that $M^f=(M_\perp^f)^\perp$. Consequently, there exists a polynomial p_f such that $U'f=p_f(S')f$.

Thus if $f \in \operatorname{Im} V'$ and $g \in X'$ with $f=V'g$, we have

$$U'f=p_f(S')V'g=V'p_f(T')g.$$

This implies that U' leaves Im V' invariant. It follows by the closed graph theorem that $Z=V'^{-1}U'V'$ is a bounded operator on Y'. On the other hand, the mapping $W=VUV^{-1}$ is densely defined and linear on Im $V \subset Y$ with values in Y. Moreover, for each $y \in \operatorname{Im} V$ and $x \in X$ such that $y=Vx$, we have

$$f(Wy)=f(WVx)=f(VUx)=(U'V'f)(x)=(V'Zf)(x)=(Zf)(y)$$

for all $f \in Y'$, which proves that $Z=W'$. The boundedness of Z now yields the same for W. Denote the unique everywhere defined bounded extension of W again by W. For each $f \in Y'$ we have

$$V'W'f = U'V'f = p_{V'f}(S')V'f = V'p_{V'f}(T')f.$$

This proves $W'f = p_{V'f}(T')f$. Fix now a vector $x \in X$ and choose $f \in Y'$ such that $f(T^k x) = 0$ for all indices k. Because $W'f = p_{V'f}(T')f$, it follows that $f(Wx) = 0$, and by the choice of $f \in Y'$ we get a polynomial p_x such that $Wx = p_x(T)x$. This implies that W leaves invariant every paraclosed subspace of Y that is invariant under T; since T is parareflexive by assumption, W must be an entire function of T. Consequently, U is (the same) entire function of S and our proof is finished. □

Before giving our main result we must introduce a Banach space version of the Deddens–Fillmore condition [2]. Here we use our results of Section 2. Naturally, we shall also impose condition (A) on the Banach space X.

Fix a nilpotent operator T, and choose an inner representation

$$R = \sum_{i=1}^{n} \oplus J_i(Y_i)$$

on the space

$$Y = \sum_{i=1}^{n} \oplus Y_i^i$$

of the operator T with the intertwining quasi-affinity denoted by V. Also choose an outer representation

$$S = \sum_{i=1}^{n} \oplus J_i(Z_i)$$

on the space

$$Z = \sum_{i=1}^{n} \oplus Z_i^i$$

of this operator T with intertwining quasi-affinity denoted by W. By Proposition 2.2, the Banach space Y_n is finite-dimensional if and only if the same holds for Z_n, and in this case there exists a quasi-affinity from Y_{n-1} into Z_{n-1}; in particular, Y_{n-1} is trivial if and only if the same holds for Z_{n-1}. We shall say that T satisfies the Deddens–Fillmore condition [2] if either $n = 1$ or the following holds: If any one of the spaces Z_n or Y_n (and hence if both of them) has dimension 1, then at least one of the spaces Y_{n-1} or Z_{n-1} (and hence then both of them) must be nontrivial. Note that, by the preceding considerations, this definition depends only on T and not on the concrete choice of inner or outer representations satisfying the implications of Theorem 2.1.

3.5. THEOREM. *A nilpotent operator on a Banach space satisfying* (A) *is parareflexive if and only if it satisfies the Deddens–Fillmore condition.*

Proof. In view of Theorem 2.1 and Theorem 3.4, we need only consider the case where T is a direct sum of Jordan operators. In order to simplify notations, suppose that $T = R$ on $X = Y$, where R and Y are as above. Further, suppose that T does not satisfy the Deddens–Fillmore condition. Then $n>1$, Y_n is one-dimensional, and Y_{n-1} is trivial. Identify Y_n with the scalar field $\mathbf{C}$; then every $x\in X$ can be written as

$$x=(\lambda_1,\lambda_2,\dots,\lambda_n)\oplus y,$$

where

$$y\in\sum_{i=1}^{n-2}\oplus Y_i^i.$$

Define a bounded operator P on X by

$$Px=(\lambda_{n-1},0,\dots,0)\oplus 0$$

for $x\in X$ as above. Since $TP=0$ and $PT\neq 0$, P cannot be an entire function of T. On the other hand, denote by M_x the smallest subspace of X invariant under T which contains the vector $x\in X$. Then the vectors $T^{n-1}x=(\lambda_n,0,\dots,0)\oplus 0$ and $T^{n-2}x=(\lambda_{n-1},\lambda_n,0,\dots,0)\oplus 0$ are elements of M_x, and so Px is also in M_x. It follows that P leaves invariant every subspace that is invariant under T, and T is not a parareflexive operator.

To get the converse implication of the theorem, suppose that T satisfies the Deddens–Fillmore condition. Let P be any operator on X which leaves invariant all the closed subspaces invariant under T. Fix a nonzero vector x from the last copy of Y_n. If Y_n is not one-dimensional, then fix another vector y from the last copy of Y_n that is linearly independent of x; elsewhere, fix a nonzero vector y from the last copy of Y_{n-1}. Now choose a vector $z\in X$, linearly independent of the vectors x and y, and denote by M_z the smallest subspace invariant under T that contains the vectors x, y, and z. It follows that M_z is a finite-dimensional subspace of X and $T|_{M_z}$ still satisfies "our" condition. By the Deddens–Fillmore theorem [2] there must exist a polynomial p_z such that $P|_{M_z}=p_z(T|_{M_z})$. With no loss of generality we may suppose that the degree of p_z is not greater than n. But this implies that $Px=p_z(T)x$ is independent of z, and therefore the polynomial $p=p_z$ is independent of z. It follows that $P=p(T)$ and T is parareflexive. □

3.6. THEOREM. *An operator T on a Banach space satisfying* (A) *is parareflexive if and only if either it is nonalgebraic or the nilpotents corresponding to the points of the spectrum of T satisfy the Deddens–Fillmore condition.*

Proof. Observe that in the algebraic case the spectral subspaces of T still satisfy condition (A), and use the above results. □

References

1. C. Apostol, R. G. Douglas, and C. Foiaş, *Quasi-similar models for nilpotent operators,* Trans. Amer. Math. Soc. 224 (1976), 407–415.
2. J. A. Deddens and P. A. Fillmore, *Reflexive linear transformations,* Linear Algebra Appl. 10 (1975), 89–93.

3. R. G. Douglas and C. Foiaş, *Infinite dimensional versions of a theorem of Brickmann-Fillmore,* Indiana Univ. Math. J. 25 (1976), 315–320.
4. P. A. Fillmore, *On invariant linear manifolds,* Proc. Amer. Math. Soc. 41 (1973), 501–505.
5. C. Foiaş, *Invariant para-closed subspaces,* Indiana Univ. Math. J. 21 (1972), 887–906.
6. I. Kaplansky, *Infinite Abelian groups,* rev. ed., Univ. Michigan Press, Ann Arbor, 1969.
7. G. W. Mackey, *Note on a theorem of Murray,* Bull. Amer. Math. Soc. 52 (1946), 322–325.
8. F. J. Murray, *Quasi-complements and closed projections in reflexive Banach spaces,* Trans. Amer. Math. Soc. 58 (1945), 77–95.
9. M. Omladič, *Some spectral properties of an operator,* Oper. Theory: Adv. Appl., 17, pp. 239–248, Birkhäuser, Basel, 1986.
10. D. Sarason, *Invariant subspaces and unstarred operator algebras,* Pacific J. Math. 17 (1966), 511–517.

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