Analytic Functions That Have Convex Successive Derivatives

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1. Introduction

In a series of papers [4]–[8], Shah and Trimble studied the family of univalent functions such that an infinite sequence of the successive derivatives (possibly each successive derivative) is univalent. Many of their results involved functions that map the unit disk onto domains that have certain geometric properties such as close-to-convexity or convexity. For example, Theorems 1 and 2 in [8] give necessary and sufficient conditions on β and $\{z_k\}_{k=1}^N$ (with each z_k on a given ray) so that the function f given by

$$f(z)=ce^{\beta z}\prod_{k=1}^{N}\left(1-\frac{z}{z_k}\right)$$

is close-to-convex (or convex) and has each successive derivative close-to-convex (or convex). They conjecture that the function e^z-1 has many extremal properties within the family of functions f that are analytic in $D\equiv\{|z|<1\}$, are normalized by $f(0)=0$ and $f'(0)=1$, and have the property that $f(D), f'(D), f''(D), \ldots$ are all convex. For example, they conjecture that for such an f with $f(z)=z+a_2z^2+\cdots$,

$$1-e^{-|z|}\le|f(z)|\le e^{|z|}-1 \quad\text{and}\quad |a_k|\le\frac{1}{k!}.$$

In [9] we showed that if f as above satisfies the conditions described and if, in addition, each coefficient a_k is positive, then $a_k\le 1/k!$ and then clearly $|f(z)|\le e^{|z|}-1$.

In [1], Barnard and Suffridge showed that if f is analytic in D, $f(0)=0$, $f'(0)=1$, and $f(D)$ and $f'(D)$ are convex, then the coefficients a_k, $k\ge 2$, satisfy $|a_k|\le 4/(3k)$ with equality if and only if $f(z)=-\frac{z}{3}-\frac{4}{3}\log(1-z)$ (so that $f'(z)=1+\frac{4}{3}(z/(1-z))$ or a rotation $e^{-i\alpha}f(ze^{i\alpha})$ of this function). In this paper, we prove some general theorems concerning convexity of f when $f'(z)=1+2ag(z)$ and g is convex. We then study some extremal problems in K_n, where $K=K_0=\{f\colon f$ is analytic in D, $f(0)=0$, $f'(0)=1$, and $f(D)$ is convex$\}$, $K_{n+1}=\{f\in K_n\colon f^{(n+1)}(D)$ is convex or $f^{(n+1)}$ is constant$\}$, and $K_\infty=\bigcap_{n=0}^{\infty}K_n$. We find the sharp coefficient bounds in the family K_2, and

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we have a conjecture for the coefficient bounds and the extremal function in K_n, $n=1,2,3,\dots$.

2. The Family K_1

If $f\in K_n$, $n\geq 1$, then

$$f'(z)=1+2ag(z),\quad f(0)=0, \tag{1}$$

where $g\in K_{n-1}$ and $a=a_2=f''(0)/2$. We wish to begin with (1) where $g\in K_0$ and find conditions on a (related to the function g) so that $f\in K_1$. Of course, such conditions will apply to the situation $g\in K_{n-1}$, $f\in K_n$, $n\geq 1$.

When convenient, we will assume that a in (1) is positive. For our purpose, this is no restriction since for $g\in K_0$, $f'(0)=1+2ae^{i\alpha}g(z)$ is the derivative of $f\in K_1$ if and only if

$$f'(ze^{-i\alpha})=1+2ae^{i\alpha}g(ze^{-i\alpha})=1+2a(e^{i\alpha}g(ze^{-i\alpha}))$$

is the derivative of $e^{i\alpha}f(ze^{-i\alpha})\in K_1$. We begin with the following straightforward result.

THEOREM 1. *Let $g\in K_0$ $(a\neq 0)$. Then f given by* (1) *is in the family K_1 if and only if $|1/a+2g(z)+zg'(z)|\geq|g'(z)|$ for all $z\in D$.*

Proof. The necessary and sufficient condition for $f\in K_1$ (given $g\in K_0$) is that $\operatorname{Re}[1+zf''(z)/f'(z)]>0$. That is, we require $\operatorname{Re}[1+2azg'(z)/(1+2ag(z))]>0$. This is equivalent to $|2+2azg'(z)/(1+2ag(z))|\geq|2azg'(z)/(1+2ag(z))|$ (i.e., $\operatorname{Re} w>0$ if and only if $|w+1|\geq|w-1|$). After dividing by $2a$ and multiplying by $|1+2ag(z)|$, the result is $|1/a+2g(z)+zg'(z)|\geq|zg'(z)|$. Since the right side of this inequality is zero only when $z=0$, the left side is never 0. The theorem now follows from Schwarz' lemma applied to

$$\frac{zg'(z)}{1/a+2g(z)+zg'(z)}. \qquad \square$$

It is convenient to set $\rho_f=\sup_{r<1}\min_{|z|=r}|f(z)|$ for functions analytic in D with $f(0)=0$. The following theorem is very useful.

THEOREM 2. *If $g\in K_0$ $(a\neq 0)$, and if f given by* (1) *is in the family K_1, then $|a|\leq 1/2(\rho_g+\rho_{zg'})$.*

Proof. Using Theorem 1, we require $|1/a+2g(z)+zg'(z)|\geq|zg'(z)|$. The function $1/a+2g(z)$ maps the unit disk D onto a convex domain that does not contain the origin. It is convex because $g\in K_0$, and it does not contain 0 because $f'(z)=1+2ag(z)$ is the derivative of a univalent function. For fixed r, $0<r<1$, there is a z_0 with $|z_0|=r$ such that $\min_{|z|=r}|1/a+2g(z)|=|1/a+2g(z_0)|$. Since $zg'(z)$ is an outer normal to the curve $1/a+2g(z)$, $|z|=r$, we conclude that

$$\left|\frac{1}{a}+2g(z_0)+z_0 g'(z_0)\right|=\left|\frac{1}{a}+2g(z_0)\right|-|z_0 g'(z_0)|.$$

Therefore, $|1/a+2g(z_0)|\geq 2|z_0 g'(z_0)|$. Now choose z_1 so that $ag(z_1)<0$, $|z_1|=r$. Then

$$\begin{aligned}\frac{1}{|a|}-2\min_{|z|=r}|g(z)|&\geq\frac{1}{|a|}-2|g(z_1)|=\left|\frac{1}{a}+2g(z_1)\right|\\&\geq\min_{|z|=r}\left|\frac{1}{a}+2g(z)\right|=\left|\frac{1}{a}+2g(z_0)\right|\\&\geq 2|z_0 g'(z_0)|\geq 2\min_{|z|=r}|zg'(z)|.\end{aligned}$$

Thus, for each r, $0<r<1$,

$$\frac{1}{|a|}\geq 2\min_{|z|=r}|g(z)|+2\min_{|z|=r}|zg'(z)|.$$

Since both terms on the right increase with r, we have $1/|a|\geq 2\rho_g+2\rho_{zg'}$ and the theorem follows. □

Note that for $g\in K_0$, $\rho_g\geq\frac{1}{2}$ and $\rho_{zg'}\geq\frac{1}{4}$ so $|a|\leq\frac{2}{3}$, which is the result obtained in [1]. Further, it is clear in this case that $|a|<\frac{2}{3}$ unless g is a rotation of $z/(1-z)$.

In light of Theorem 2, the value of $\rho_{zf'}$ for $f\in K_1$ should be useful in the study of K_2.

THEOREM 3. *If $f\in K_1$ then $\rho_{zf'}\geq\frac{1}{3}$ (hence $|f'(z)|\geq\frac{1}{3}$), with equality if and only if f is a rotation of $-\frac{z}{3}-\frac{4}{3}\log(1-z)$.*

Proof. Write $f'(z)=1+2ag(z)$, $g\in K_0$. We know that $\operatorname{Re}[zg'(z)/g(z)]\geq\frac{1}{2}+\epsilon$, where $\epsilon\geq 0$ with $\epsilon>0$ unless $g(z)=z/(1-z)$ (or a rotation). Therefore $|zg'(z)/g(z)|\geq\frac{1}{2}+\epsilon$ and $|zg'(z)|\geq(\frac{1}{2}+\epsilon)|g(z)|$. Thus,

$$\begin{aligned}\min_{|z|=r}|f'(z)|&=\min_{|z|=r}|1+2ag(z)|=|1+2ag(z_0)|\\&\geq 2|a||z_0 g'(z_0)|\geq|a|(1+2\epsilon)|g(z_0)|\end{aligned}$$

for some z_0, $|z_0|=r$. If $2|a||z_0 g'(z_0)|\geq(1+\epsilon)/3$ then clearly we have that $\min_{|z|=r}|f'(z)|\geq(1+\epsilon)/3$. If $2|a||z_0 g'(z_0)|<(1+\epsilon)/3$ then

$$|a||g(z_0)|(1+2\epsilon)\leq 2|a||z_0 g'(z_0)|<\frac{1+\epsilon}{3}$$

and we have

$$\begin{aligned}\min_{|z|=r}|f'(z)|&=\min_{|z|=r}|1+2ag(z)|=|1+2ag(z_0)|\geq 1-2|a||g(z_0)|\geq 1-\frac{2}{3}\frac{1+\epsilon}{1+2\epsilon}\\&=\frac{1}{3}+\frac{2}{3}\frac{\epsilon}{1+2\epsilon}\geq\frac{1+\epsilon}{3}\end{aligned}$$

because $\epsilon<\frac{1}{2}$.

In any case, $|f'(z)| \geq \frac{1}{3} + \epsilon/3$, where $\epsilon \geq 0$ with equality only if $g(z) = z/(1-z)$ or a rotation. The proof of theorem is now complete. □

Since $z/(1-z)$ is the extremal function for many extremal problems in K_0 (in fact, the set of extreme points in K_0 is precisely $\{z/(1-ze^{i\alpha})\colon \alpha \text{ is real}\}$), it is instructive to find the values of a such that f, given by (1), when $g(z) = z/(1-z)$, is convex. The answer is somewhat surprising.

THEOREM 4. *Set* $f'(z) = 1 + 2az/(1-z)$, $a \neq 0$, $f(0) = 0$. *Then* $f \in K_1$ *if and only if* $|a - \frac{1}{2}| \leq \frac{1}{6}$.

Proof. Using Theorem 1, we obtain the inequality

$$\left|1 + \frac{2az}{1-z} + \frac{az}{(1-z)^2}\right| \geq \left|\frac{az}{(1-z)^2}\right|.$$

On multiplying by $|(1-z)^2|$ we obtain

$$|(1-2a)z^2 + (3a-2)z + 1| \geq |az|, \quad |z| \leq 1, \tag{2}$$

as the necessary and sufficient condition for $f \in K_1$. Note that $|1-2a| < 1$ is necessary because otherwise the left side of (2) has a zero in $0 < |z| \leq 1$, which is not possible. We rewrite (2) as

$$\left|(1-2a)\left(z^2 - \frac{3}{2}z\right) + 1 - \frac{z}{2}\right|^2 \geq \frac{|(1-2a)z - z|^2}{4}, \quad |z| \leq 1,$$

and

$$|1-2a|^2|z|^2\left(\left|z - \frac{3}{2}\right|^2 - \frac{1}{4}\right) - 2\,\mathrm{Re}\left[(1-2a)z\left(\left(\frac{3}{2} - z\right)\left(1 - \frac{\bar{z}}{2}\right) - \frac{\bar{z}}{4}\right)\right] + \left|1 - \frac{z}{2}\right|^2 - \frac{|z|^2}{4} \geq 0.$$

Thus,

$$|1-2a|^2(|z|^2)(|z|^2 - 3|z|\cos\theta + 2) - 2\,\mathrm{Re}\left[(1-2a)z\left(\frac{3}{2} - 2|z|\cos\theta + \frac{|z|^2}{2}\right)\right] + 1 - |z|\cos\theta \geq 0,$$

where $z = |z|e^{i\theta}$. This can be rewritten as

$$\begin{aligned} &3|1-2a|^2|z|^3(1-\cos\theta) - 4\,\mathrm{Re}[(1-2a)z(|z|)(1-\cos\theta) + |z|(1-\cos\theta)] \\ &\quad + |1-2a|^2|z|^2(2 - 3|z| + |z|^2) \\ &\quad - 2\,\mathrm{Re}\left[(1-2a)z\left(\frac{3}{2} - 2|z| + \frac{|z|^2}{2}\right)\right] + 1 - |z| \geq 0. \end{aligned} \tag{3}$$

Now let $|z| \to 1$, divide by $1 - \cos\theta$, and let $(1-2a)z \to |1-2a|$ to see that

$$3|1-2a|^2 - 4|1-2a| + 1 \geq 0, \qquad (1 - |1-2a|)(1 - 3|1-2a|) \geq 0.$$

Since $1 > |1-2a|$ (as observed previously), we see that $\frac{1}{6} \geq |a - \frac{1}{2}|$ is a necessary condition.

Returning to (3), assume $1-3|1-2a|\geq 0$. We see that

$$\begin{aligned}
&|z|(1-\cos\theta)(3|1-2a|^2|z|^2-4\operatorname{Re}[(1-2a)z]+1)\\
&\quad+(1-|z|)(|1-2a|^2|z|^2(2-|z|)-\operatorname{Re}[(1-2a)z](3-|z|)+1)\\
&\geq |z|(1-\cos\theta)(1-4|1-2a||z|+3|1-2a|^2|z|^2)\\
&\quad+(1-|z|)(|1-2a|^2|z|^2-2|1-2a||z|+1)\\
&\quad+(1-|z|)^2(|1-2a|^2|z|^2-|1-2a||z|)\\
&\geq |z|(1-\cos\theta)(1-|1-2a||z|)(1-3|1-2a||z|)\\
&\quad+(1-|z|)(1-|1-2a||z|)^2\geq 0
\end{aligned}$$

when $|z|\leq 1$, and the theorem is proved. □

REMARK. Assume f is given by (1) with $g\in K_0$. Since g is convex and we require $f'(z)\neq 0$, $|z|<1$, f' maps the disk onto a convex set that does not contain the origin. Consider $f'(|z|=r)$ (see Figure 1).

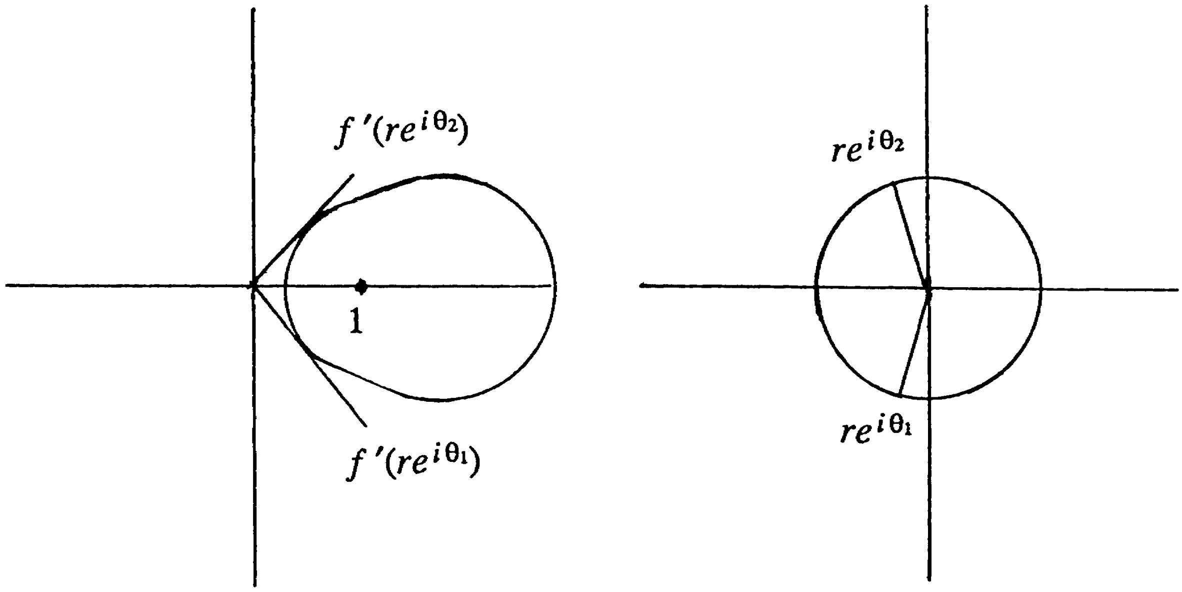

Figure 1

We want $zf'(z)$ to be starlike so that $\partial[\arg re^{i\theta}f'(re^{i\theta})]/\partial\theta\geq 0$; that is, $[\partial\arg f'(re^{i\theta})]/\partial\theta\geq -1$. From Figure 1 it is clear that $[\partial\arg f'(re^{i\theta})]/\partial\theta\geq 0$ when $\theta_1\leq\theta\leq\theta_2$, so we need only consider $\theta_2<\theta<\theta_1+2\pi$. Consider $g(z)=z/(1-z)$, $a>0$. When $a=\frac{2}{3}$,

$$\left.\frac{\partial\arg f'(e^{i\theta})}{\partial\theta}\right|_{\theta=\pi}=-1.$$

Now consider what happens to $(\partial\arg f'(e^{i\theta}))/\partial\theta$ near $\theta=0$ as a decreases. Since $[\partial\arg g(e^{i\theta})]/\partial\theta=\frac{1}{2}$, $\theta\neq 2k\pi$, we know that if $0<\theta_0<\pi$ then the arc $0<\theta<\theta_0$ maps to the half-line indicated in Figure 2. Clearly the angle ϕ increases as a decreases. Finally, if $0<a<\frac{1}{3}$ then θ_0 can be chosen so that $\phi>\theta_0$. This implies that $[\partial\arg f'(e^{i\theta})]/\partial\theta<-1$ for some θ_0.

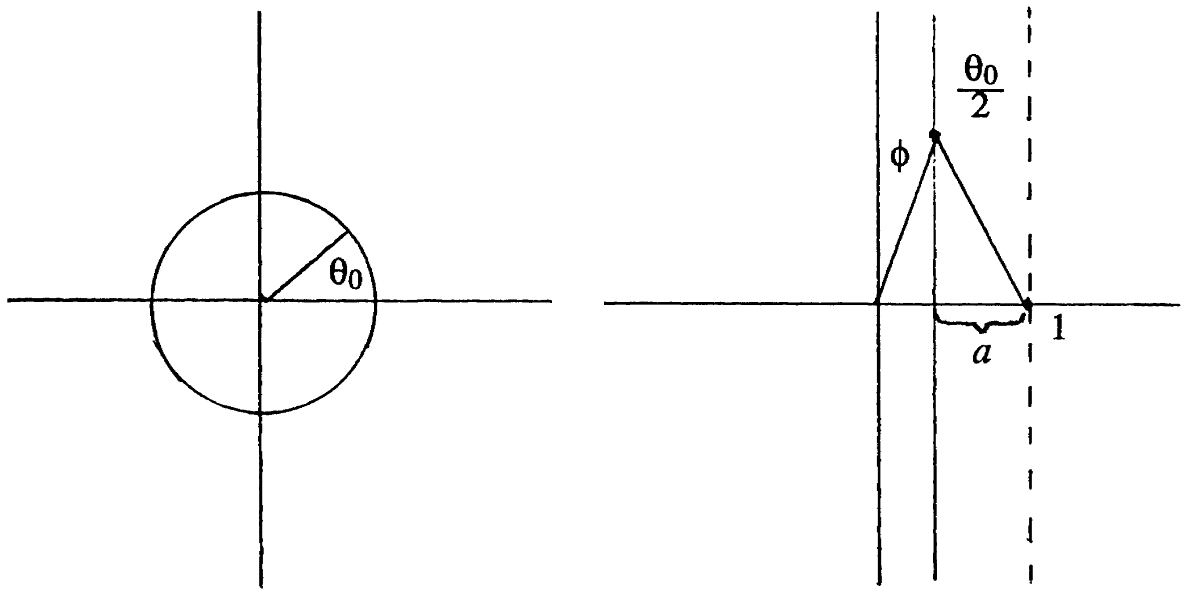

Figure 2

Similar considerations apply if a is not real. Assume f' maps onto a half-plane that makes an angle $\pi/2+\phi$ with the positive real axis, $0<\phi<\pi/2$, so that the point on $\partial f(D)$ that is at minimum distance from the origin is $\rho e^{i\phi}$. Using (1) with $g(z)=z/(1-z)$, we see that

$$\frac{1-(1-2a)z}{1-z}=f'(z)=\frac{1-(1-2(\cos\phi-\rho)e^{i\theta})z}{1-z}.$$

If $a=\frac{1}{2}-\frac{1}{6}e^{-i\alpha}$ (so that $|a-\frac{1}{2}|=\frac{1}{6}$), we have $\tan\phi=\sin\alpha/(3-\cos\alpha)$ and $\tan\phi\leq\sqrt{2}/4$ with equality when $\cos\alpha=\frac{1}{3}$ and $\sin\alpha=2\sqrt{2}/3$. Further, $\rho=4/3\sqrt{10-6\cos\alpha}$.

As demonstrated in the proof of Theorem 4, equality holds in (3) when $(1-2a)e^{i\theta}=|1-2a|$; that is, with $a=\frac{1}{2}-\frac{1}{6}e^{-i\alpha}$ when $\theta=\alpha$. Thus, if we choose $a=\frac{1}{2}-\frac{1}{6}e^{-i\alpha}$ and let α vary from $-\pi$ to π with $f'(z)=2az/(1-z)$, the regions $f'(|z|<1)$ will be the half-planes

$$\operatorname{Re}(we^{-i\theta})>\rho,\qquad \phi=\sin^{-1}\left(\frac{\sin\alpha}{\sqrt{10-6\cos\alpha}}\right),\qquad \rho=\frac{4}{3\sqrt{10-6\cos\alpha}}.$$

Given a function $g\in K_0$, there is no reason to suppose there is an $a\neq 0$ so that f given by (1) will be in the family K_1. For example, consider the case where g maps the disk onto an infinite strip, not a half-plane. If a is chosen so that $f'(z)=0$ for some z, $|z|\leq 1$, then clearly $zf'(z)$ is not starlike. For other values of a, we use the fact that $\operatorname{Re}[zg'(z)/g(z)]\to\infty$ as z tends to either of the discontinuities on $|z|=1$ to see that there will always be a value of θ such that $[\partial\arg(f'(e^{i\theta}))]/\partial\theta<-1$.

Actually, any f given by (1) has f' subordinate to one of the functions $1+2az/(1-z)$, where $a=\frac{1}{2}+\frac{1}{6}e^{i\alpha}$.

THEOREM 5. *If $f\in K_1$ then there exists a, $|a-\frac{1}{2}|=\frac{1}{6}$, such that $f'(z)\prec 1+2az/(1-z)$. Furthermore, we may assume that $a=\frac{1}{2}+\frac{1}{6}e^{i\alpha}$ where $|\alpha|\leq\cos^{-1}(2\sqrt{2}/3)$.*

Proof. The last part of the theorem follows readily from the first, because (from the remark above) the half-plane that is the image of $1+2az/(1-z)$ for some a ($a-\frac{1}{2}=\frac{1}{6}e^{i\beta}$, $|\beta|>\cos^{-1}(2\sqrt{2}/3)$) is contained in the half-plane that is the image of $1-2az/(1-z)$ ($a-\frac{1}{2}=\frac{1}{6}e^{i\alpha}$ for some α, where $|\alpha|<\cos^{-1}(2\sqrt{2}/3)$). We prove the theorem by contradiction. Assume there is an $f\in K_1$ that is not subordinate to any of the functions

$$h_\alpha(z)=1+\left(1+\frac{1}{3}e^{i\alpha}\right)\frac{z}{1-z},\quad -\pi\le\alpha\le\pi.$$

Then it is easy to see that there is an $r<1$ such that $f_r(z)=f(rz)$ is not subordinate to any h_α. That is, we may assume f is analytic on $|z|\le 1$. Taking $\alpha=0$, we see that $\operatorname{Re} f'(z_0)<\frac{1}{3}$ for some z_0, $|z_0|\le 1$. The proof will proceed as follows. We want to find ρ and α so that $f'(\rho z)\prec h_\alpha(z)$, that is, $f'(\rho z)=h_\alpha(w(z))$ where $|w(z)|\le|z|$ and $w(e^{i\theta})=e^{i\phi}$ for some θ and ϕ. We may assume $\operatorname{Im} f'(z_0)>0$ (otherwise replace $f(z)$ by $\overline{f(\bar{z})}$) because $\operatorname{Im} f'(z_0)\ne 0$ since $|f'(z)|\ge\frac{1}{3}$. Since $f'(D)$ is convex, $\operatorname{Re} f'(z)>\frac{1}{3}$ whenever $\operatorname{Im} f'(z)\le 0$. Thus, $\alpha>0$ and we want $\phi=\pi-\alpha$, so that

$$\operatorname{Re}\left[\frac{e^{i\phi}h'(e^{i\phi})}{h(e^{i\phi})}\right]=\left[\frac{\partial\arg h(e^{i\theta})}{\partial\theta}\right]_{\theta=\phi}=-1$$

(see the proof of Theorem 4). Then we will have that $e^{i\theta}w'(e^{i\theta})/w(e^{i\theta})$ is real and greater than 1 by a result known as Jack's lemma (see [2, p. 28] and [10, p. 777]). Then

$$\left[\frac{\rho e^{i\theta}f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\right]=\frac{\partial\arg f'(\rho e^{i\theta})}{\partial\theta}=\frac{e^{i\theta}w'(e^{i\theta})}{w(e^{i\theta})}\cdot\operatorname{Re}\frac{e^{i\theta}h'_\alpha(e^{i\theta})}{h_\alpha(e^{i\theta})}<-1,$$

contradicting the fact that $f\in K_1$.

Observe that

$$\begin{aligned} h_\alpha(e^{i(\pi-\alpha)})&=1+\frac{(1+\frac{1}{3}e^{i\alpha})(-e^{-i\alpha})}{1+e^{-i\alpha}}\\ &=\frac{2}{3(1+e^{-i\alpha})}=\frac{1}{3\cos\alpha/2}e^{i\alpha/2}=\frac{1}{3}+\frac{i}{3}\tan\frac{\alpha}{2}\end{aligned}$$

are the images of the values $z=e^{i(\pi-\alpha)}$ where $\operatorname{Re}[zh'_\alpha(z)/h_\alpha(z)]=-1$. The line $\operatorname{Re} w=\frac{1}{3}$ intersects the curve $f(e^{i\theta})$, $0\le\theta\le 2\pi$, in exactly two points in the upper half-plane. Let the points be $\frac{1}{3}+\frac{i}{3}\tan\alpha_1$ and $\frac{1}{3}+\frac{i}{3}\tan\alpha_2$, $\alpha_2>\alpha_1$. Figure 3 makes it clear that a ρ, θ and ϕ as described above exist.

Let l_1 and l_2 be the boundary lines for $h_{\alpha_1}(D)$ and $h_{\alpha_2}(D)$, respectively. The normal n_1 to the boundary curve $f'(e^{i\theta})$ at $\frac{1}{3}+\frac{i}{3}\tan\alpha_1$ is below the normal to l_1, and the normal n_2 at $\frac{1}{3}+\frac{i}{3}\tan\alpha_2$ is above the normal to l_2. The vector $zf''(z)$ assumes every direction between the direction of n_1 and n_2 along the pre-image of the line segment. Therefore, there is a $\rho e^{i\theta}=z$ such that $f'(\rho e^{i\theta})=\frac{1}{3}+\frac{i}{3}\tan\alpha$ ($\alpha_1<\alpha<\alpha_2$) and such that the normal to the curve $f'(|z|=\rho)$ at $\rho e^{i\theta}$ is in the same direction as the normal to $h_\alpha(|z|=1)$. This yields the required ρ,θ,α and the proof is complete. □

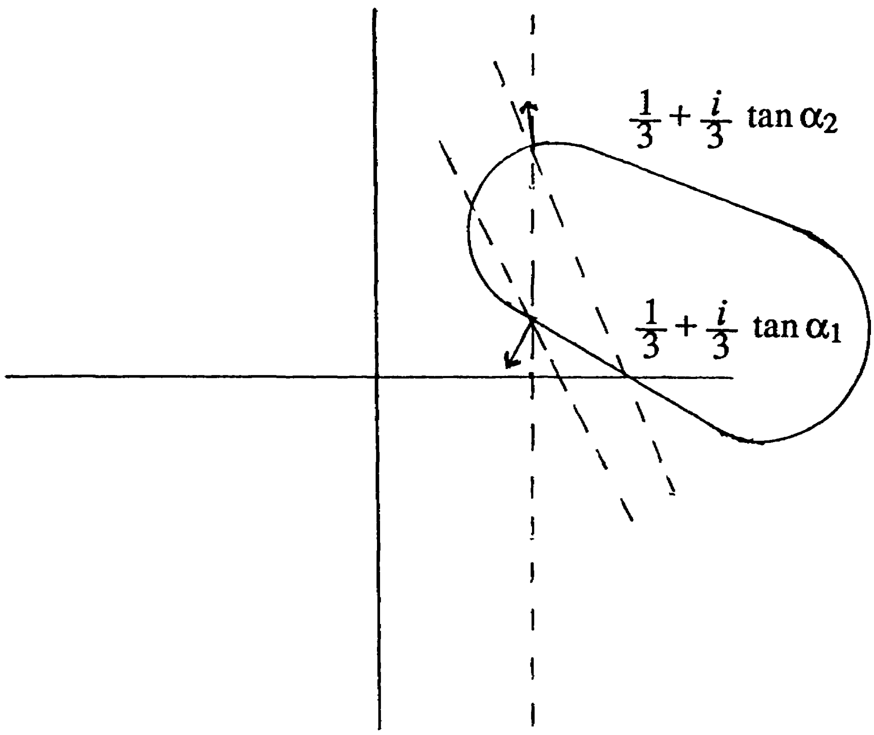

Figure 3

THEOREM 6. *If $f \in K_1$ then there exists a probability measure μ on $[0, 2\pi]$ and an $a = \frac{1}{2}+\frac{1}{6}e^{i\alpha}$, $|\alpha| \le \cos^{-1}(2\sqrt{2}/3)$, such that*

$$f(z) = \int_0^{2\pi} [z(1-2a) - 2ae^{-it}\log(1-ze^{it})]\, d\mu(t).$$

Proof. By Theorem 5 and Herglotz' formula, there is an a as described above and a probability measure μ on $[0, 2\pi]$ such that

$$\begin{aligned} f'(z) &= 1+2a\int_0^{2\pi} \frac{ze^{it}}{1-ze^{it}}\, d\mu(t) \\ &= \int_0^{2\pi}\left(1+2a\frac{ze^{it}}{1-ze^{it}}\right) d\mu(t). \end{aligned}$$

The theorem now follows by integration. □

THEOREM 7. *The functions $f \in K_1$ given by*

$$f'(z) = 1+2a\frac{z}{1-z}, \qquad a = \frac{1}{2}+\frac{1}{6}e^{i\alpha}, \qquad |\alpha| \le \cos^{-1}\frac{2\sqrt{2}}{3}$$

are extreme points of K_1.

Proof. Assume not, so that

$$\begin{aligned} 1+2a\frac{z}{1-z} &= tg_1'(z)+(1-t)g_2'(z) \\ &= 1+t(g_1'(z)-1)+(1-t)(g_2'(z)-1) \end{aligned}$$

for some a as in the theorem and $g_1, g_2 \in K_1$, $0<t<1$. Let $b=\frac{1}{2}g_1''(0)$, $c=\frac{1}{2}g_2''(0)$,

$$G_1(z)=\frac{g_1'(z)-1}{2b}, \qquad G_2(z)=\frac{g_2'(z)-1}{2c}.$$

Then $z/(1-z)=(b/a)tG_1(z)+(c/a)(1-t)G_2(z)$, where $G_1, G_2 \in K_0$ (or G_1 or G_2 is constant). We may assume G_1 has the property $(1-z)G_1(z) \not\to 0$ as $z \to 1$. Since $G_1 \in K_0$, this implies $G_1(z)=z/(1-z)$ because $G_1 \notin H^1$ [3, Lemma 8.8]. Thus,

$$\left(1-\frac{b}{a}t\right)\frac{z}{1-z}=\frac{c}{a}(1-t)G_2(z)$$

and there are two possibilities. First, if $G_2(z) = z/(1-z)$ then $a = bt + c(1-t)$ and

$$\frac{1}{6}=\left|a-\frac{1}{2}\right|\le t\left|b-\frac{1}{2}\right|+(1-t)\left|c-\frac{1}{2}\right|$$
$$\le \frac{1}{6}t+\frac{1}{6}(1-t)=\frac{1}{6};$$

it follows that $a=b=c$, contradicting the assumption that f was not an extreme point. The second possibility is $c=0$ so $G_2(z)=0$ and $1-(b/a)t=0$. Then $b/a=1/t>1$ which is not possible under the assumption on a. This concludes the proof. □

THEOREM 8. *If $f \in K_1$ then $|f'(z)| \ge (3-|z|)/3(1+|z|)$ and $|f(z)| \ge \frac{4}{3}\log(1+|z|)-|z|/3$, with equality if and only if*

$$f'(z)=1+\frac{4}{3}\frac{z}{1-z}=\frac{3+z}{3(1-z)}$$

or a rotation of this function so that $f(z)=-z/3-\frac{4}{3}\log(1-z)$ or a rotation.

Proof. We know $f'(z) \prec 1+(2az/(1-z))$ for some $a=\frac{1}{2}+\frac{1}{6}e^{i\alpha}$, so

$$|f'(z_0)| \ge \min_{|z|=|z_0|}\left|\frac{1+\frac{1}{3}e^{i\alpha}z}{1-z}\right| \ge \frac{1-\frac{1}{3}|z_0|}{1+|z_0|}=\frac{3-|z_0|}{3(1+|z_0|)}.$$

Equality is clearly only possible when $\alpha=0$. Further,

$$|f(z)|=\left|\int_0^z f'(u)\,du\right|=\left|\int_\gamma f'(u)\,du\right|=\int_\gamma |f'(u)||du|$$
$$\ge \int_0^{|z|}\left(\frac{4}{3}\frac{1}{1+\rho}-\frac{1}{3}\right)d\rho=\frac{4}{3}\log(1+|z|)-\frac{|z|}{3},$$

where

$$\gamma=f^{-1}(tf(z)), \quad 0\le t\le 1,$$

so that

$$f'(u)du=\frac{f(z)}{|f(z)|}|f'(u)||du|.$$

□

3. The Family K_n, $n \geq 1$

We believe that in K_n there is an extremal function f_n with positive coefficients that maximizes every coefficient. Further, it appears that the sequence $\{f_n\}$ satisfies $f'_{n+1}(z)=1+2af_n(z)$, where $2a=1/(\rho_{f_n}+\rho_{zf'_n})$. Furthermore, it seems to be true that

$$-f_n(-|z|)\leq|f(z)|\leq f_n(|z|) \quad\text{and}\quad f'_n(-|z|)\leq|f'(z)|\leq f'_n(|z|)$$

for all f in K_n. Here,

$$f_0(z)=\frac{z}{1-z},\qquad f_1(z)=-\frac{z}{3}-\frac{4}{3}\log(1-z),$$

and

$$f_2(z)=z\left(1+\frac{1}{\log 2}\right)-\frac{z^2}{8\log 2}+\frac{1-z}{\log 2}\log(1-z).$$

We will show $f_2\in K_2$ and will find the functions f_n generated using $f'_{n+1}=1+[1/(\rho_{f_n}+\rho_{zf'_n})]f_n(z)$.

THEOREM 9. *If $f\in K_2$ and $f'(z)=1+2ag(z)$ with $g\in K_1$ and $a>0$, then $a\leq 3/(8\log 2)$. This result is sharp, with equality if and only if*

$$f(z)=z\left(1+\frac{1}{\log 2}\right)-\frac{z^2}{8\log 2}+\frac{1-z}{\log 2}\log(1-z).$$

Proof. Using Theorems 2, 3, and 8, we see that $g\in K_1$ implies

$$\rho_g\geq\frac{4}{3}\log 2-\frac{1}{3},\qquad \rho_{zg'}\geq\frac{1}{3},$$

$$2\rho_g+2\rho_{zg'}\geq\frac{8\log 2}{3},\quad\text{and}\quad a\leq\frac{3}{8\log 2}.$$

It remains to show that f given in the theorem is in the family K_2. Clearly $f'(D)$ and $f''(D)$ are convex. It remains to show $f(D)$ is convex. We require (by Theorem 1) that

$$\left|1+\frac{3}{4\log 2}\left(-\frac{z}{3}-\frac{4}{3}\log(1-z)\right)+\frac{3}{8\log 2}\left(z+\frac{4}{3}\frac{z^2}{1-z}\right)\right| \geq\frac{3}{8\log 2}\left|z+\frac{4}{3}\frac{z^2}{1-z}\right|.$$

After multiplying by $|1-z|$, both sides are continuous in $\{|z|\leq 1\}$ so it is sufficient to prove the inequality on $|z|=1$. By symmetry, we may take $z=e^{i\theta}$, $0<\theta\leq\pi$. Thus, we require

$$\left|1-\frac{z}{4\log 2}-\frac{\log(1-z)}{\log 2}\right|^2 +\frac{3}{4\log 2}\operatorname{Re}\left[\left(z+\frac{4}{3}\frac{z^2}{1-z}\right)\left(1-\frac{\bar{z}}{3}-\frac{\log(\overline{1-z})}{\log 2}\right)\right]\geq 0,$$

which becomes

$$\left|1-\frac{e^{i\theta}}{4\log 2}-\frac{\log(2\sin\theta)}{\log 2}+i\frac{\pi-\theta}{2\log 2}\right|^2$$
$$+\frac{3}{4\log 2}\operatorname{Re}\left[e^{i\theta}\left(\frac{1}{3}+\frac{2i}{3}\frac{1+\cos\theta}{\sin\theta}\right)\right.$$
$$\left.\times\left(1-\frac{e^{-i\theta}}{4\log 2}-\frac{\log(2\sin\theta)}{\log 2}-i\frac{\pi-\theta}{2\log 2}\right)\right]\geq 0.$$

This is equivalent to

$$4\log^2\frac{2}{1-\cos\theta}-2\log\frac{2}{1-\cos\theta}(2+3\cos\theta)$$
$$+2(\pi-\theta)\frac{-1+2\cos\theta+3\cos^2\theta}{\sin\theta}+4(\pi-\theta)^2\geq 0.$$

or

$$\text{(4)}\quad \begin{aligned}&2\log\frac{2}{1-\cos\theta}\left(2\log\frac{2}{1-\cos\theta}-2-3\cos\theta\right)\\&\qquad+\left(2(\pi-\theta)(2(\pi-\theta)-\frac{\sin\theta(1-3\cos\theta)}{1-\cos\theta}\right)\geq 0.\end{aligned}$$

Set

$$h(\theta)=2(\pi-\theta)-\frac{\sin\theta(1-3\cos\theta)}{1-\cos\theta}\quad\text{so}\quad h'(\theta)=\frac{-4-\cos\theta+3\cos^2\theta}{1-\cos\theta}\leq 0.$$

Thus, $h(\theta)$ is a decreasing function and $(\pi-\theta)h(\theta)$ is decreasing and is 0 when $\theta=\pi$. Thus, $(\pi-\theta)h(\theta)\geq 0$.

Set $K(\theta)=2\log[2/(1-\cos\theta)]-2-3\cos\theta$, so that

$$K'(\theta)=\frac{\sin\theta(1-3\cos\theta)}{1-\cos\theta}\begin{array}{ll}>0 & \text{if } \cos\theta<\frac{1}{3},\\ <0 & \text{if } \cos\theta>\frac{1}{3}.\end{array}$$

Since $K(\pi/6)>0$ it follows that (4) holds when $0<\theta\leq\pi/6$. Also, $K(2\pi/3)>0$ so (4) holds when $2\pi/3\leq\theta\leq\pi$.

When $\pi/2\leq\theta\leq 2\pi/3$,

$$2(\pi-\theta)h(\theta)\geq\frac{2\pi}{3}h\left(\frac{2\pi}{3}\right)>1.36,$$

while

$$2\log\frac{2}{1-\cos\theta}K(\theta)\geq 2\log 2(2\log 2-2)\geq -0.9$$

so (4) holds for these values. Finally, if $\pi/6<\theta<\pi/2$,

$$2(\pi-\theta)h(\theta)\geq\pi h\left(\frac{\pi}{2}\right)=\pi(\pi-1)>6.7$$

while

$$2\log\frac{2}{1-\cos\theta}K(\theta)\geq 2\log\frac{2}{1-\cos(\pi/6)}K\left(\cos^{-1}\frac{1}{3}\right)$$
$$=2\log\frac{4}{2-\sqrt{3}}(2\log 3-3)>-4.4.$$

Thus, the proof is complete. □

COROLLARY 1. *If* $f\in K_2$ *and* $f(z)=z+a_2z^2+\cdots$ *then* $|a_2|\leq 3/(8\log 2)$ *and* $|a_k|\leq 1/[k(k-1)\log 2]$, $k\geq 3$, *with equality if and only if* $f(z)$ *is a rotation of*

$$z\left(1+\frac{1}{\log 2}\right)-\frac{z^2}{8\log 2}+\frac{1-z}{\log 2}\log(1-z).$$

Proof. We know that $g\in K_1$ and $g(z)=z+b_2z^2+\cdots$ implies $|b_k|\leq 4/3k$ $(k\geq 2)$, with equality if and only if $g(z)=-\frac{z}{3}-\frac{4}{3}\log(1-z)$. Further, by Theorem 9, if $f\in K_2$ and $f'(z)=1+2ag(z)$ with $a>0$ and $g\in K_1$, then $a\leq 3/(8\log 2)$ with equality if and only if g is as described above. The corollary now follows. □

REMARK. Since $e^z-1\in K_\infty\subset K_n$ for each n, we know that

$$\frac{1}{2}\leq\max_{g\in K_n}\{a: f'(z)=1+2ag(z),\ f\in K_n,\ g\in K_{n-1}\}<\frac{3}{8\log 2}<0.542$$

when $n\geq 3$.

4. An Interesting Lemma and the Conjectured Extremal for K_n

In order to find the conjectured extremal for K_n, $n\geq 3$, we require the following rather curious lemma.

LEMMA 1. *If* $n\geq 3$ *then*

$$\sum_{p=1}^{n-1}\frac{1}{(p-1)!\,(n-p-1)!}\left[(n-p)\sum_{l=p}^{n-1}\frac{1}{l}+p\sum_{l=1}^{p-1}\frac{(-1)^l}{l}\right]=\frac{2n-1}{(n-1)!}2^{n-3}-\frac{1}{(n-2)!}. \tag{5}$$

Proof. Set

$$\phi(x)=\sum_{p=1}^{n-1}\frac{n-p}{(p-1)!\,(n-p-1)!}\sum_{l=p}^{n-1}\frac{x^l}{l}$$

and

$$\psi(x)=\sum_{p=1}^{n-1}\frac{p}{(p-1)!\,(n-p-1)!}\sum_{l=1}^{p-1}\frac{(-x)^l}{l}.$$

Then $\phi(0)=0=\psi(0)$ and $\phi(1)+\psi(1)$ is the left side of (5). We have

$$\phi'(x)=\sum_{p=1}^{n-1}\frac{n-p}{(p-1)!\,(n-p-1)!}\sum_{l=p}^{n-1}x^{l-1}$$
$$=\sum_{p=1}^{n-1}\frac{n-p}{(p-1)!\,(n-p-1)!}\,\frac{x^{p-1}-x^{n-1}}{1-x}.$$

This series can be obtained in closed form by replacing p by $p+1$ and using the fact that

$$\sum_{p=0}^{n-2}\frac{(n-2)!}{p!\,(n-p-2)!}x^p=(1+x)^{n-2}$$

and

$$\sum_{p=0}^{n-2}\frac{p(n-2)!}{p!\,(n-p-2)!}x^{p-1}=(n-2)(1+x)^{n-3}.$$

The result is

$$\phi'(x)=\frac{1}{(n-2)!}\times\left[\frac{n((1+x)^{n-3}-2^{n-3}x^{n-3})}{1-x}-(1+x)^{n-3}+n2^{n-3}(x^{n-3}+x^{n-2})\right].$$

Similarly,

$$\psi'(x)=\frac{1}{(n-2)!}\left(-n\frac{(2^{n-3}-(1-x)^{n-3})}{1+x}-(n-1)(1-x)^{n-3}\right).$$

We want to find $\phi(1)+\psi(1)=\int_0^1[\phi'(x)+\psi'(x)]\,dx$. Consider the first terms

$$s_n=\int_0^1\frac{(1+x)^{n-3}-2^{n-3}x^{n-3}}{1-x}\,dx\quad\text{and}\quad t_n=\int_0^1\frac{2^{n-3}-(1-x)^{n-3}}{1+x}\,dx.$$

Note that $s_3=0=t_3$ while $s_{n+1}-2s_n=1/(n-2)=t_{n+1}-2t_n$. This implies that $s_n=t_n$ when $n\geq 3$. Therefore

$$\phi(1)+\psi(1)=\int_0^1\left[\frac{n2^{n-3}}{(n-2)!}(x^{n-3}+x^{n-2})-\frac{(1+x)^{n-3}}{(n-2)!}-\frac{(n-1)(1-x)^{n-3}}{(n-2)!}\right]dx$$
$$=\frac{2^{n-3}(2n-1)}{(n-1)!}-\frac{1}{(n-2)!},$$

and the proof is complete. □

Now take

$$f_0(z)=\frac{z}{1-z},\qquad f_1(z)=-\frac{z}{3}-\frac{4}{3}\log(1-z),$$

and

$$f_2(z)=z\left(1+\frac{1}{\log 2}\right)-\frac{z^2}{8\log 2}+\frac{1-z}{\log 2}\log(1-z).$$

We want $f'_{n+1}(z)=1+[1/(f'_n(-1)-f_n(-1))]f_n(z)$. Clearly, $f_{n+1}(z)$ has the form

$$\sum_{k=1}^{n+1} \alpha_k^{(n+1)} z^k - (1-\alpha_1^{(n+1)})(1-z)^n \log(1-z). \tag{6}$$

Write

$$\begin{aligned} 1+\gamma_n f_n(z) &= f'_{n+1}(z) \\ &= \sum_{k=1}^{n+1} k\alpha_k^{(n+1)} z^{k-1} + (1-\alpha_1^{(n+1)})[(1-z)^{n-1}(1+n\log(1-z))]. \end{aligned}$$

Equating coefficients of the term $(1-z)^{n-1}\log(1-z)$, we see that

$$\gamma_n = -\frac{n(1-\alpha_1^{(n+1)})}{1-\alpha_1^{(n)}}.$$

On dividing by $(1-\alpha_1^{(n+1)})$ and equating coefficients of z^k in

$$\sum_{k=1}^{n} \frac{(k+1)\alpha_{k+1}^{(n+1)}}{1-\alpha_1^{(n+1)}} z^k + \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k} z^k = \frac{-n}{1-\alpha_1^{(n)}} \sum_{k=1}^{n} \alpha_k^{(n)} z^k,$$

we obtain

$$\frac{(n+1)\alpha_{n+1}^{(n+1)}}{1-\alpha_1^{(n+1)}} + \frac{n\alpha_n^{(n)}}{1-\alpha_1^{(n)}} = 0 \tag{7}$$

and

$$\frac{(n+1-p)\alpha_{n+1-p}^{(n+1)}}{1-\alpha_1^{(n+1)}} + \frac{n\alpha_{n-p}^{(n)}}{1-\alpha_1^{(n)}} = (-1)^{n-p-1}\frac{(n-1)!}{(n-p)!\,(p-1)!}, \tag{8}$$

where the substitution $k=n-p$, $1 \le p \le n-1$, has been made in (8).

In (7), let $u_n = n\alpha_n^{(n)}/(1-\alpha_1^{(n)})$ to see that $u_{n+1}+u_n=0$. The solution is $n\alpha_n^{(n)}/(1-\alpha_1^{(n)}) = c(-1)^n$, and using $\alpha_1^{(1)} = -\frac{1}{3}$ yields

$$\alpha_n^{(n)} = \frac{(-1)^n}{4n}(1-\alpha_1^{(n)}), \quad n \ge 1. \tag{9}$$

In (8), let $u_n = \alpha_{n-p}^{(n)}/(1-\alpha_1^{(n)})$ to get

$$(n+1-p)u_{n+1} + nu_n = (-1)^{n-p-1}\frac{(n-1)!}{(n-p)!\,(p-1)!}.$$

The homogeneous solution of this equation is

$$u_n^{(c)} = \frac{c(n-1)!}{(n-p)!}(-1)^n,$$

while a particular solution is

$$u_n = \frac{(-1)^{n-p}(n-1)!}{(n-p)!\,(p-1)!}\sum_{l=1}^{n-1}\frac{1}{l}.$$

Therefore

$$\alpha_{n-p}^{(n)} = \frac{(-1)^{n-p}(n-1)!}{(n-p)!\,(p-1)!}\left[K_p + \sum_{l=1}^{n-1}\frac{1}{l}\right](1-\alpha_1^{(n)}), \quad 1 \le p \le n-1.$$

Setting $p=n-1$ yields

$$K_p = -\sum_{l=1}^{p} \frac{1}{l} - \frac{\alpha_1^{(p+1)}}{p(1-\alpha_1^{(p+1)})},$$

so that

$$\alpha_{n-p}^{(n)} = \frac{(-1)^{n-p}(n-1)!}{(n-p)!\,(p-1)!} \times \left[\sum_{l=p}^{n-1} \frac{1}{l} - \frac{1}{p(1-\alpha_1^{(p+1)})} \right] (1-\alpha_1^{(n)}), \quad 1 \le p \le n-1. \tag{10}$$

It remains to find the $\alpha_1^{(k)}$. The $\alpha_1^{(k)}$ are chosen successively to make

$$f'_{n+1}(z) = 1 - \frac{n(1-\alpha_1^{(n+1)})}{1-\alpha_1^{(n)}} f_n(z) = 1 + \frac{1}{f'_n(-1) - f_n(-1)} f_n(z).$$

That is, we require $[zf''_n(z) + f'_n(z)]_{z=-1} = 0$. Using (6), with n replacing $n+1$, we conclude

$$\sum_{k=1}^{n} k^2 \alpha_k^{(n)} (-1)^{k-1} + (1-\alpha_1^{(n)})(2^{n-3})(2n-1+n(n-1)\log 2) = 0.$$

Thus,

$$\begin{aligned} &2^{n-3}(2n-1+n(n-1)\log 2)(1-\alpha_1^{(n)}) \\ &\quad = \sum_{p=1}^{n-1} (n-p)^2 \alpha_{n-p}^{(n)} (-1)^{n-p} - (-1)^n n^2 \alpha_n^{(n)} \\ &\quad = \left[\sum_{p=1}^{n-1} \frac{(n-p)(n-1)!}{(n-p-1)!\,(p-1)!} \sum_{l=p}^{n-1} \frac{1}{l} \right. \\ &\qquad \left. - \sum_{p=1}^{n-1} \frac{(n-p)(n-1)!}{(n-p-1)!\,p!} \frac{1}{1-\alpha_1^{(p+1)}} + \frac{n}{4} \right] (1-\alpha_1^{(n)}). \end{aligned}$$

We will show that

$$\frac{1}{1-\alpha_1^{(n)}} = (-1)^{n-1} \left[\frac{1}{4} \sum_{p=0}^{n-1} \frac{(n-1)!}{p!} + \frac{1}{2} \right] + (n-1)\left(\sum_{p=1}^{n-1} (-1)^{p-1} \frac{1}{p} - \log 2 \right). \tag{11}$$

It is easy to check that (11) holds for $n = 1, 2$.

In order to prove (11) for $n \ge 3$, set

$$\frac{1}{1-\alpha_1^{(p+1)}} = p!\, u_{p+1} - p \log 2 + \frac{(-1)^p}{2}.$$

Then

$$\begin{aligned} &2^{n-3}(2n-1+n(n-1)\log 2) \\ &\quad = \sum_{p=1}^{n-1} \frac{(n-p)(n-1)!}{(n-p-1)!\,(p-1)!} \sum_{l=p}^{n-1} \frac{1}{l} - \sum_{p=1}^{n-1} \frac{(n-p)(n-1)!}{(n-p-1)!} u_{p+1} \\ &\qquad + \left(\sum_{p=1}^{n-1} \frac{(n-p)(n-1)!}{(n-p-1)!\,(p-1)!} \right) \log 2 - \frac{1}{2} \sum_{p=1}^{n-1} \frac{(n-p)(n-1)!}{(n-p-1)!\,p!} (-1)^p + \frac{n}{4}. \end{aligned}$$

The last series on the right has the sum $n/2$ when $n \geq 3$ because it is

$$-\left[\left(\frac{n}{2}(1+x)^{n-2}-1\right)+(n-1)(1+x)^{n-3}\right]_{x=-1}.$$

The next-to-last series is

$$\log 2[(n-1)^2(1+x)^{n-2}-(n-1)(n-2)\cdot(1+x)^{n-3}]_{x=1}=n(n-1)2^{n-3}\log 2.$$

Therefore we need

$$2^{n-3}\frac{(2n-1)}{(n-1)!}=\sum_{p=1}^{n-1}\frac{n-p}{(n-p-1)!\,(p-1)!}\sum_{l=p}^{n-1}\frac{1}{l}$$
$$-\sum_{p=1}^{n-1}\frac{n-p}{(n-p+1)!}u_{p+1}+\frac{3n}{4(n-1)!}.$$

That is,

$$u_n=-\sum_{p=0}^{n-2}\frac{n-p}{(n-p-1)!}u_{p+1}+\frac{1}{(n-1)!}-\sum_{l=p}^{n-1}\frac{p}{(p-1)!\,(n-p-1)!}\sum_{l=1}^{p-1}\frac{(-1)^l}{l},$$

where we have used the lemma and the fact that $u_1=\frac{1}{4}$. Suppose that

$$u_p=(-1)^{p-1}\frac{1}{4}\sum_{l=0}^{p-1}\frac{1}{l!}+\frac{1}{(p-2)!}\sum_{l=1}^{p-1}\frac{(-1)^l}{l}$$

when $1\leq p\leq l$ (this is true for $p=1,2$). Then

$$u_n=\frac{1}{4}\sum_{p=0}^{n-2}\frac{n-p}{(n-p-1)!}(-1)^{p-1}\sum_{l=0}^{p}\frac{1}{l!}+\frac{1}{(n-1)!}$$
$$+\sum_{p=1}^{n-2}\frac{n-p}{(n-p-1)!\,(p-1)!}\sum_{l=1}^{p}\frac{(-1)^l}{l}-\sum_{p=2}^{n-1}\frac{p}{(p-1)!\,(n-p-1)!}\sum_{l=1}^{p-1}\frac{(-1)^l}{l}$$
$$=\frac{1}{4}s_1+\frac{1}{(n-1)!}+s_2-s_3.$$

Here

$$s_1=\sum_{p=1}^{n-1}\frac{n-p+1}{(n-p)!}(-1)^p\sum_{l=0}^{p-1}\frac{1}{l!}=\sum_{p=1}^{n-1}(-1)^p\left[\frac{1}{(n-p-1)!}+\frac{1}{(n-p)!}\right]\sum_{l=0}^{p-1}\frac{1}{l!}$$
$$=\sum_{p=1}^{n-1}(-1)^p\frac{1}{(n-p-1)!}\sum_{l=0}^{p-1}\frac{1}{l!}-\sum_{p=0}^{n-2}(-1)^p\frac{1}{(n-p-1)!}\sum_{l=0}^{p}\frac{1}{l!}$$
$$=(-1)^{n-1}\sum_{l=0}^{n-2}\frac{1}{l!}-\frac{1}{(n-1)!}-\sum_{p=1}^{n-2}\frac{(-1)^p}{(n-p-1)!\,p!}$$
$$=(-1)^{n-1}\sum_{l=0}^{n-1}\frac{1}{l!}-\sum_{p=0}^{n-1}\frac{(-1)^p}{(n-p-1)!\,p!}$$
$$=(-1)^n\sum_{l=0}^{n-1}\frac{1}{l!}.$$

Also,

$$s_2=\sum_{p=2}^{n-1}\frac{n-p+1}{(n-p)!\,(p-2)!}\sum_{l=1}^{p-1}\frac{(-1)^{l-1}}{l}$$
$$=\sum_{l=1}^{n-2}\frac{(-1)^{l-1}}{l}\sum_{p=l+1}^{n-1}\left[\left(\frac{1}{(n-p-1)!}+\frac{1}{(n-p)!}\right)\frac{1}{(p-2)!}\right.$$
$$=\sum_{l=1}^{n-2}\frac{(-1)^{l-1}}{l}\sum_{p=l+1}^{n-1}\frac{1}{(p-2)!\,(n-p-1)!}$$
$$\left.+\sum_{p=l}^{n-2}\frac{1}{(n-p-1)!\,(p-1)!}\right]$$
$$=\sum_{l=1}^{n-2}\frac{(-1)^{l}}{l}\left[\sum_{p=l+1}^{n-1}\frac{p}{(p-1)!\,(n-p-1)!}-\frac{1}{(n-2)!}+\frac{1}{(n-l-1)!\,(l-1)!}\right]$$
$$=s_3+\sum_{l=1}^{n-2}\frac{(-1)^{l-1}}{(n-2)!\,l}+\sum_{l=1}^{n-2}\frac{(-1)^{l}}{l!\,(n-l-1)!}$$
$$=s_3+\sum_{l=1}^{n-2}\frac{(-1)^{l-1}}{(n-2)!\,l}+\frac{1}{(n-1)!}[(1+x)^{n-1}-1-(x)^{n-1}]_{x=-1}$$
$$=s_3+\frac{1}{(n-2)!}\sum_{l=1}^{n-1}\frac{(-1)^{l-1}}{l}-\frac{1}{(n-1)!}.$$

This shows that

$$u_n=\frac{(-1)^{n-1}}{4}\sum_{l=0}^{n-1}\frac{1}{l!}+\frac{1}{(n-2)!}\sum_{l=1}^{n-1}\frac{(-1)^{p-1}}{l},$$

which was to be proved. □

CONJECTURE. The function f_n defined by $f_0(z)=z/(1-z)$ and

$$f_n(z)=\sum_{k=1}^{n}\alpha_k^{(n)}z^k-(1-\alpha_1^{(n)})(1-z)^{n-1}\log(1-z),$$

with the $\alpha_k^{(n)}$ given by (9), (10), and (11), is extremal in K_n in the sense that

(i) $f_n\in K_n$; and

(ii) if $f_n(z)=\sum_{k=1}^{\infty}A_kz^k$ and $g(z)=\sum_{k=1}^{\infty}a_kz^k\in K_n$, then: $|a_k|\le A_k$ for each k, $-f_n(-|z|)\le|g(z)|\le f_n(|z|)$, and

$$\frac{-|z|f_n'(-|z|)}{f_n(-|z|)}\le\operatorname{Re}\frac{zg'(z)}{g(z)}.$$

Note that for the second coefficient A_2 of f_n, we have

$$A_2=\frac{1}{2}\left[\frac{\displaystyle\sum_{p=0}^{n-2}\frac{1}{p!}+\frac{2}{(n-2)!}-\frac{4(-1)^n}{(n-2)!}\left((n-2)\left(\log 2-\sum_{p=1}^{n-2}\frac{(-1)^{p-1}}{p}\right)\right)}{\displaystyle\sum_{p=0}^{n-1}\frac{1}{p!}+\frac{2}{(n-1)!}+\frac{(-1)^n\cdot 4}{(n-2)!}\left(\log 2-\sum_{p=1}^{n-1}\frac{(-1)^{p-1}}{p}\right)}\right],$$

which is of course always greater than $\frac{1}{2}$ but approaches $\frac{1}{2}$ quite rapidly. For example, with $n=12$, $A_2-\frac{1}{2}<10^{-9}$.

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