Solution Operators for Partial Differential Equations in Weighted Gevrey Spaces

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The existence of continuous linear right inverses for concrete linear operators has recently been studied by many authors ([10], [8], [6], and the literature cited there). These results are mainly based on the general splitting theorem of Vogt for exact sequences of power series spaces of infinite type [15, Thm. 7.1], while the nonexistence of a continuous linear right inverse is often proved by the noncompatibility of certain linear topological invariants (such as (DN) and $(\bar{\Omega})$), which were developed in the recent study of the structure of nuclear (F)-spaces ([14], [15]).

In this paper, partial differential equations in (weighted) spaces of ultradistributions of Roumieu type are considered. These spaces locally are isomorphic to a power series space of finite type. So one has to consider exact sequences of power series spaces of finite type, which need not be split, contrary to the infinite type case [14].

To obtain splitting theorems for these sequences, one has to fix a special norm system (defining the topology) such that any linear mapping of the exact sequence (or used in the proofs) shifts the counting of this norm system at most by a multiplicative constant. In other words, one has to work with graded (F)-spaces and tame linear maps. The necessary tools and a basis free version of the tame splitting theorem of Vogt ([13], [11]) for tame exact sequences of power series spaces of finite type are contained in the first section.

The weighted spaces and the assumptions used in this paper are as follows:

$$W(x) := \sum_{i \le N} W_i(|x_i|), \tag{1}$$

where $W_i \in C^1[0,\infty)$ and $w_i := W_i'$ is increasing from 0 to ∞ on $[0,\infty)$;

$$w_i(t) = o(W_i(t)^{\delta_i}); \tag{2}$$

$$W_i \circ w_i^{-1}(2t) = o(W_i \circ w_i^{-1}(t)), \tag{3}$$

where $\delta = (\delta_i)_{i \le N} > 1$. Let

$$\Gamma^\delta(\pm W) := \left\{ f \in C^\infty(\mathbf{R}^N) \mid p_n^\pm(f) := \sup_{x,k} \frac{|D^k f(x) e^{(\pm 1 + 1/n)W(x)}|}{n^{\delta \cdot k} k^{\delta k}} < \infty \quad \text{for some } n \ge 1 \right\}.$$

Received February 2, 1988.
Michigan Math. J. 37 (1990).

Let $P(D)$ be a $r\times s$ system of partial differential operators with constant coefficients (on $\mathbf{R}^N$) and let $Q(D)$ be the matrix of relations implied by $P(D)$. Then the main result of this paper is the following (Theorem 4.2).

THEOREM. *Let W satisfy* (1)–(3). *Then $P(D)$ has tame linear right inverses*

$$R_-: \operatorname{Ker} Q(D)\cap\Gamma^\delta(-W)^r\to\Gamma^\delta(-W)^s$$

and

$$R_+: \operatorname{Ker} Q(D)\cap(\Gamma^\delta(W)_b')^r\to(\Gamma^\delta(W)_b')^s.$$

This implies that hypoelliptic systems have continuous linear right inverses in the weighted space of C^∞-functions

$$C_1^\infty(W):=\left\{f\in C^\infty(\mathbf{R}^N)\mid \sup_{\substack{k\le n\\ x}}|D^kf(x)e^{-(1+1/n)W(x)}|<\infty \text{ for any } n\in\mathbf{N}\right\}.$$

This improves on results obtained in [5]. Notice that $C_1^\infty(W)$ is not isomorphic to a power series space.

The steps in the proof of the main theorem are similar to those in [6], with the technical complication that any linear isomorphism used in the proof must be a tame isomorphism. A tame isomorphic sequence space representation and a tame Paley–Wiener-type theorem is proved for the weighted Gevrey spaces in Sections 2 and 3. The final step in the proof is a tame version of the Ehrenpreis principle, which is obtained for these spaces in Section 4.

1. Tameness

The notion of a graded (F)-space was used (e.g. in [1]) in connection with the Nash–Moser inverse function theorem. It means that the counting of a system of 0-neighbourhoods (defining the topology) is fixed. To simplify later notation, we will always use $[1,\infty)$ as an index set for the system of 0-neighbourhoods.

1.1. DEFINITION. (a) A grading of a (FS)-space is a fixed decreasing system $\{U_n\mid n\ge 1\}$ of absolutely convex closed 0-neighbourhoods defining the topology of E. (E, U_n) is called a *graded* (FS)-*space* (g-(FS)-space).

(b) A grading of a (DFS)-space E is a fixed increasing system $\{B_n\mid n\ge 1\}$ of absolutely convex closed bounded sets absorbing each bounded set. (E, B_n) is called a *graded* (DFS)-*space* (g-(DFS)-space).

Subspaces and quotients of a graded space E are always considered with the canonical grading induced by the grading of E. (The quotient grading of a g-(DFS)-space absorbs each bounded set, by Theorem 8 in [4].) The strong dual E_b' of a g-(FS)-space (E, U_n) will always be considered as a g-(DFS)-space with the dual grading

$$B_n:=U_n^0:=\{x'\in E'\mid |\langle x',x\rangle|\le 1 \text{ for any } x\in U_n\}.$$

Let $\alpha=(\alpha_n)$ be an increasing unbounded sequence of positive numbers. The power series space of finite type

$$\Lambda_0(\alpha):=\left\{(c_j)\in \mathbf{C}^{\mathbf{N}} \mid \|(c_j)\|_n:=\sum_{j=1}^{\infty}|c_j|e^{-\alpha_j/n}<\infty \quad \text{for any } n\geq 1\right\}$$

will always be considered with the grading defined by the norms $\| \ \|_n$.

1.2. DEFINITION. (a) Let (E, U_n^E) and (F, U_n^F) be g-(FS)-spaces and let $T: E\to F$ be linear. T is called *linearly tame* (or tame, for short) if and only if

$$\exists C\ \forall n\geq C\ \exists C_1: T(U_{Cn}^E)\subset C_1 U_n^F.$$

(b) Let (E, B_n^E) and (F, B_n^F) be g-(DFS)-spaces and let $T: E\to F$ be linear. T is called *tame* if and only if

$$\exists C\ \forall n\geq C\ \exists C_1: T(B_n^E)\subset C_1 B_{Cn}^F.$$

Tame maps (between g-(FS)- or g-(DFS)-spaces) are continuous. With the conventions from above, a linear map $T: (E, U_n^E)\to(F, U_n^F)$ is tame between two g-(FS)-spaces (E, U_n^E) and (F, U_n^F) if and only if the transpose $T': (F_b', B_n^{F'})\to(E_b', B_n^{E'})$ is tame for $B_n^{F'}=(U_n^F)^0$ and $B_n^{E'}=(U_n^E)^0$.

Two graded spaces E and F are tamely isomorphic (t-isomorphic) if and only if there is a linear isomorphism $T: E\to F$ such that T and T^{-1} are tame. A linear mapping $T: E\to F$ is tamely open (t-open) if and only if $\tilde{T}^{-1}: T(F)\to E/\operatorname{Ker} T$ is tame. A sequence

$$0\to E\xrightarrow{i}F\xrightarrow{q}G\to 0$$

of graded spaces E, F, and G is called tamely exact (t-exact) if and only if the sequence is exact and i is a t-isomorphism onto the subspace $i(E)\subset F$ and G is t-isomorphic to the quotient $F/i(E)$ of F.

1.3. DEFINITION. Let (E, U_n) be a g-(FS)-space with corresponding seminorms $\| \ \|_n$.

(a) $E\in(\underline{DN})_t$ if and only if

$$\exists \delta\geq 1, C\geq 1\ \forall n\geq C\ \exists C_1, m: \| \ \|_n\leq C_1 \| \ \|_\delta^{1/(Cn)} \| \ \|_m^{1-1/(Cn)}.$$

(b) $E\in(\bar{\Omega})_t$ if and only if

$$\forall p\ \exists D\geq p\ \forall n\geq D, k\geq 1\ \exists C_1\ \forall t>0: U_{nD}\subset t^{1/n}U_k+\frac{C_1}{t^{1-1/n}}U_p.$$

The classes $(\underline{DN})_t$ and $(\bar{\Omega})_t$ are the appropriate t-isomorphic invariant specializations of the linear topological invariants $(\underline{DN})$ and $(\bar{\Omega})$ introduced by Vogt [14]. $(\underline{DN})_t$ is inherited to subspaces and $(\bar{\Omega})_t$ to quotients (with their canonical gradings!). The class $(\underline{DN})_t$ is used in [8] to show the nonexistence of extension operators for certain classes of ultradifferentiable functions of Roumieu type.

1.4. LEMMA. (a) *A g-(FS)-space* $(E, U_n) \in (\bar{\Omega})_t$, *if the g-(DFS)-space* (E', B_n) *satisfies (with* $B_n := U_n^0$*):*

$$\forall p\, \exists D \geq p\, \forall n \geq D, k \geq 1\, \exists C_1\, \forall t > 0: B_k/t^{1/n} \cap B_p t^{1-1/n}/C_1 \subset B_{nD}.$$

(b) $\Lambda_0(\alpha) \in (\underline{\mathrm{DN}})_t \cap (\bar{\Omega})_t$.

Proof. Part (a) follows from the bipolar theorem.

(b) $\Lambda_0(\alpha) \in (\underline{\mathrm{DN}})_t$ by Hölder's inequality ((2.2) in [8]). (a) implies that $\Lambda_0(\alpha) \in (\bar{\Omega})_t$ if the following holds:

$$(*) \qquad \forall p\, \forall n, k \geq 1\, \forall t > 0: \sup |c_j| e^{\alpha_j/(np)} \leq 1,$$

if $\sup_j |c_j| \max\{e^{\alpha_j/k} t^{1/n}, e^{\alpha_j/p}/t^{1-1/n}\} \leq 1$.

Fix n, p, k, and t. Then $(*)$ is trivial if $t \leq e^{\alpha_j/p}$. For $t > e^{\alpha_j/p}$ we have

$$|c_j| e^{\alpha_j/(np)} \leq e^{\alpha_j(1/(np)-1/k)}/t^{1/n} \leq e^{-\alpha_j/k} \leq 1. \qquad \square$$

The significance of the classes $(\bar{\Omega})_t$ and $(\underline{\mathrm{DN}})_t$ comes from the following tame version of the basic Theorem 1.6 in [14].

1.5. THEOREM. *The following are equivalent for a g-(FN)-space E:*

(i) *E is t-isomorphic to* $\Lambda_0(\alpha)$ *for some sequence* α;

(ii) *E is t-isomorphic to a complemented subspace of* $\Lambda_0(\beta)$ *for some sequence* β;

(iii) $E \in (\underline{\mathrm{DN}})_t \cap (\bar{\Omega})_t$.

Proof. "(i) ⇒ (ii)" is evident.

"(ii) ⇒ (iii)" We may assume that E is a complemented subspace of $(\Lambda_0(\beta), U_r)$ via a projection Π. $E \in (\underline{\mathrm{DN}})_t$ by Lemma 1.4(b). Lemma 1.3(b) and the continuity of Π imply, for $V_j := U_j \cap E$,

$$\forall p\, \exists p', C_2, D \geq p'\, \forall n \geq D, k\, \exists k', C_1, C_3\, \forall t > 0:$$

$$V_{nD} = \Pi(V_{nD}) \subset t^{1/n}\Pi(U_{k'}) + C_1 \Pi(U_{p'})/t^{1-1/n} \subset C_3 t^{1/n} V_k + C_1 C_2 V_p/t^{1-1/n}$$

"(iii) ⇒ (i)" This follows by an improved version of the Mitiagin-Henkin procedure (see [14, §1]).

(a) $(\bar{\Omega})_t$ implies the existence of a bounded set $B \subset E$ such that the following holds for the dual "norms"

$$\|x'\|_n^* := \sup\{|\langle x, x'\rangle| \; \|x\|_n \leq 1\}$$

(see [14, Lemma 1.4]):

$$(1.1) \qquad \forall p\, \exists D\, \forall n \geq D\, \exists C_1: \|x'\|_{2Dn}^* \leq C_1 (\|x'\|_B^*)^{1-1/n} (\|x'\|_p^*)^{1/n}.$$

The proof of Satz 1.6 in [14] shows the existence of a Hilbert space $(E_\infty, \langle\ ,\ \rangle_\infty)$ such that

$$(1.2) \qquad E_\infty \text{ is continuously embedded in } E,$$

$$(1.2') \qquad E_B \subset E_\infty \quad \text{and} \quad \|\ \|_\infty \leq \|\ \|_B.$$

$(\underline{\mathrm{DN}})_t$ and (1.1)–(1.2′) imply the existence of δ, C, and $D(\delta)$ such that

$$(\alpha)_t \qquad \forall n \geq C\ \exists C_1 : \|\ \|_n \leq C_1 \|\ \|_\delta^{1/(Cn)} \|\ \|_m^{1-1/(Cn)} \leq C_2 \|\ \|_\delta^{1/(nC)} \|\ \|_\infty^{1-1/(nC)};$$

$$(\omega)_t \qquad \forall n \geq D\ \exists C_3 : \|\ \|_{2Dn}^* \leq C_3 (\|\ \|_\delta^*)^{1/n} (\|\ \|_\infty^*)^{1-1/n}.$$

We may thus apply the proof of Satz 1.1 in [14] to obtain sequences

$$\{e_k \mid k \in \mathbf{N}\} \subset E \quad \text{and} \quad \{\phi_k \mid k \in \mathbf{N}\} \subset E'$$

such that

$$(1.3) \qquad x = \sum \phi_k(x) e_k \quad (\text{in } E)$$

$$(1.4) \qquad \|\phi_k\|_\delta^* = 1/a_k, \qquad \|\phi_k\|_\infty^* = 1,$$

and

$$(1.5) \qquad \|e_k\|_\delta = a_k, \qquad \|e_k\|_\infty = 1,$$

where $(a_n) \in \bigcap_{p>0} l^p$ is decreasing.

(b) Let $T: E \to \omega$ be defined by $T(x) := (\phi_n(x))_{n \in \mathbf{N}}$. Then T is a t-isomorphism from E onto $\Lambda_0(\alpha)$, where $\alpha_n := \max(0, \ln(1/a_n))$.

Proof. (1.3), $(\alpha)_t$, and (1.5) show the tameness of T^{-1}.

$$\|x\|_n \leq \sum_k |\phi_k(x)| \, \|e_k\|_n \leq C_2 \sum_k |\phi_k(x)| \, \|e_k\|_\delta^{1/(nC)} \|e_k\|^{1-1/(nC)}$$
$$= C_2 \sum_k |\phi_k(x)| \, a_n^{1/(nC)} \leq C_4 \|T(x)\|_{nC}^\Lambda,$$

where $\|\ \|_n^\Lambda$ is the canonical nth norm in $\Lambda_0(\alpha)$.

$(\omega)_t$ and (1.4) imply:

$$\|\phi_k\|_{4Dn}^* \leq C_3 (\|\phi_k\|_\delta^*)^{1/(2n)} (\|\phi_k\|_\infty^*)^{1-1/(2n)} = C_3 (1/a_n)^{1/(2n)}$$

and therefore

$$\|T(x)\|_n^\Lambda \leq \sum_k |\phi_k(x)| a_k^{1/n} \leq C_3 \left(\sum_k a_k^{1/(2n)} \right) \|x\|_{4Dn}. \qquad \square$$

Combining Theorem 1.5 with the splitting theorem of Vogt for t-exact sequences of power series spaces of finite type ([13], see also [11]), we obtain the following basic theorem.

1.6. THEOREM. *Let E, F, G be g-(FN)-spaces and let*

$$0 \to E \xrightarrow{i} F \xrightarrow{q} G \to 0$$

be a t-exact sequence. Let $E, G \in (\bar{\Omega})_t \cap (\underline{\mathrm{DN}})_t$. Then the sequence is t-split; that is, q has a tame right inverse.

Proof. E and G are t-isomorphic to some $\Lambda_0(\alpha)$ (resp. $\Lambda_0(\beta)$) by Theorem 1.5. So the resulting sequence

$$0 \to \Lambda_0(\alpha) \xrightarrow{\tilde{i}} F \xrightarrow{\tilde{q}} \Lambda_0(\beta) \to 0$$

is t-exact, and $\tilde{q}$ has a tame right inverse $\tilde{R}$ (see [13]), which immediately gives a tame right inverse R for q. □

2. Weighted Gevrey Classes of Roumieu Type

For $i \leq N$ let

$$W_i(x) := \int_0^{|x|} w_i(t)\, dt \quad \text{with } 0 < \epsilon < w_i \in C([0,\infty)). \tag{2.1}$$

Let

$$x \cdot \xi := \sum_{i \leq N} x_i \xi_i \quad \text{for } x, \xi \in \mathbf{C}^N$$

and let

$$k^{\delta k} := \prod_{i \leq N} k_i^{\delta_i k_i} \quad \text{for } k, \delta \in \mathbf{R}_+^N.$$

2.1. DEFINITION. Let $W(x) := \sum_{i \leq N} W_i(x_i)$ and let $\delta = (\delta_i)_{i \leq N}$ with $\delta_i > 1$ for $i \leq N$.

(a) $$\Gamma^\delta(-W) := \left\{ f \in C^\infty(\mathbf{R}^N) \mid p_n^-(f) := \sup_{x,k} \frac{|D^k f(x) e^{(-1+1/n)W(x)}|}{n^{\delta \cdot k} k^{\delta k}} < \infty \text{ for some } n \geq 1 \right\}.$$

(b) $$\Gamma^\delta(W) := \left\{ f \in C^\infty(\mathbf{R}^N) \mid p_n(f) := \sup_{x,k} \frac{|D^k f(x) e^{(1+1/n)W(x)}|}{n^{\delta \cdot k} k^{\delta k}} < \infty \text{ for some } n \geq 1 \right\}.$$

The spaces $\Gamma^\delta(\pm W)$ will always be equipped with the above gradings.

We will consider partial differential equations in $\Gamma^\delta(-W)$ and $\Gamma^\delta(W)'_b$ in Section 4. It is shown in the present section that $\Gamma^\delta(\pm W)'_b \in (\underline{\mathrm{DN}})_t \cap (\bar{\Omega})_t$. To simplify notation, only functions of one variable will be considered in the remaining part of this section.

We first construct special cut-off functions (see also [7, Lemma 1.2.] and [9, p. 103ff]).

2.2. LEMMA. *Let $\delta > 1$. Then there is $C > 0$ such that for $j > 0$ there are $0 \leq \mu_j \in C^\infty(\mathbf{R})$ such that the following hold:*

$$\int \mu_j(t)\, dt = 1 \quad \textit{and} \quad \operatorname{supp} \mu_j \subset [-j/64, j/64];$$

$$\forall n \geq C \;\exists C_1 \;\forall j: \sup_{k,x} \frac{|\mu_j^{(k)}(x)|}{(nk)^{\delta k}} \leq \frac{C_1}{j} e^{C/(nj^{1/(\delta-1)})}.$$

Proof. This is evident for $j > 1$. Let $j \leq 1$. Let $p_0 := [1/j^{1/(\delta-1)}]$, $k_0 := 512/j$ and

$$k_0 := \begin{cases} 256 p_0 / j & \text{for } p \leq p_0, \\ 128 \cdot 2^{\delta-1} p^\delta / (\delta - 1) & \text{for } p > p_0. \end{cases}$$

Let

$$\Phi_p(x) := \begin{cases} k_p/2 & \text{for } |x| \le 1/k_p, \\ 0 & \text{for } |x| > 1/k_p, \end{cases}$$

and let $\psi_0 := \Phi_0 * \Phi_0$ and $\psi_p := \psi_{p-1} * \Phi_p$ for $p \ge 1$. Φ_p and ψ_p are nonnegative and $\int \Phi_p(x)\,dx = \int \psi_p(x)\,dx = 1$ for $p \ge 0$.

$$\operatorname{supp} \psi_p \subset \left\{ x \,\middle|\, |x| \le 2/k_0 + \sum_{p=1}^{\infty} 1/k_p \right\} \subset [-j/64, j/64]$$

by the choice of k_p. $\psi_p \in C^p(\mathbf{R})$ and

$$\|\psi_p^{(a)}\|_\infty \le \|\psi_a^{(a)}\|_\infty \le K_a \|\psi_0\|_\infty = 256 K_a/j$$

for $0 \le a < p$ and $K_0 := 1$, $K_a := \prod_{p=1}^{a} k_p$.

So, a subsequence of ψ_p converges in $C^\infty(\mathbf{R})$ to a function $\mu_j \in C^\infty(\mathbf{R})$, and $\|\mu_j^{(a)}\|_\infty \le 256 K_a/j$ for $a > 0$.

Let $1 \le a \le p_0$. Then

$$\frac{K_a}{(na)^{\delta a}} \le \left[\frac{(C/(nj^{1/(\delta-1)}))^a}{a!} \right]^\delta \le e^{\delta C/(nj^{1/(\delta-1)})}.$$

Let $a > p_0$. Then

$$\frac{K_a}{(n_a)^{\delta a}} \le e^{\delta C/(nj^{1/(\delta-1)})} \frac{(K_a/K_{p_0})}{(na)^{\delta(a-p_0)}}$$
$$\le e^{\delta C/(nj^{1/(\delta-1)})}$$

for $n \ge (128 \cdot 2^{\delta-1}/(\delta-1))^{1/\delta}$. The lemma follows immediately. □

We will frequently use special cut-off functions defined by $\{\mu_j\}$ in the following way: We suppose that

(2.2) $$w(t) = o(W(t)^\delta),$$

so there is a strictly positive increasing and unbounded function $g \in C(\mathbf{R})$ such that

(2.3) $$g(t) = o(t^\delta) \quad \text{and} \quad w(t) \le g(W(t)).$$

Let $h(t) := g(t)/t^{\delta-1}$. Then

(2.4) $$h(t) = o(t).$$

For $y > 0$ let $\tilde{y} > 0$ be defined by

(2.5) $$W(\tilde{y}) = W(y) + h(2W(y)).$$

$\tilde{y}$ is well defined and $\tilde{y} > y$, as $w = W'$ and h are positive. Let $\epsilon(y) := \tilde{y} - y$.

Let Φ_y denote the convolution of a characteristic function with $\mu_{\epsilon(t)}$, where $t = y$ or $t = \tilde{y}$ or let Φ_y be the sum of two such functions. We also suppose that

$$\operatorname{supp} \Phi_y \subset I_y := [y, \tilde{y}]. \tag{2.6}$$

Let $\mathfrak{F}$ or $\hat{}$ be the Fourier transform defined by $\hat{f}(z) := \langle {}_x f, e^{-ix\cdot z}\rangle$.

2.3. LEMMA. *Let ξ, Φ_y be defined as above and let W satisfy* (2.1) *and* (2.2). *Then* $\exists A \geq 1\, \forall n \geq A\, \exists C_1\, \forall y \geq A\, \forall f \in \Gamma^\delta(\mathbf{R})$:

$$\sup_{z\in\mathbf{C}} |\widehat{f\Phi_y}(z) e^{|z|^{1/\delta}/n}| \leq C_1 \sup_{\substack{x\in I_y\\ k}} |f^{(k)}(x)| e^{AW(x)/n} \left(\frac{A}{nk}\right)^{\delta k} \int_{I_y} e^{x \operatorname{Im} z}\, dx.$$

Proof. (a)

$$\forall t \geq e^\delta: \sup_{k\in\mathbf{N}} \left(\frac{t}{k^\delta}\right)^k \leq e^{\delta t^{1/\delta}/e} = \sup_{k\geq 1} \left(\frac{t}{k^\delta}\right)^k \leq t \sup_{k\in\mathbf{N}} \left(\frac{t}{k^\delta}\right)^k. \tag{2.7}$$

Combining this with Lemma 2.2, we get for $|z| \geq C_1(n)$

$$\begin{aligned} \sup_z |\widehat{f\Phi_y}(z) e^{|z|^{1/\delta}/n}| &\leq \sup_{\substack{k\in\mathbf{N}\\ z}} \left| \widehat{(f\Phi)^{(k)}}(z) \left(\frac{2e}{kn\delta}\right)^{\delta k} \right| \\ &\leq C_2(n) \int_{I_y} e^{x \operatorname{Im} z}\, dx \sup_{\substack{x\in I_y\\ k\in\mathbf{N}}} \left| f^{(k)}(x) \left(\frac{4e}{kn\delta}\right)^{\delta k} \right| e^{\tilde{C}/(n\epsilon(t))^{1/(\delta-1)}}. \end{aligned}$$

(b) By (2.4),

$$W(\tilde{y}) \leq 2W(\tilde{y}) \leq 4W(y) \quad \text{for large } y. \tag{2.8}$$

$$\begin{aligned} \frac{1}{\epsilon(y)} = \frac{1}{\tilde{y}-y} &= \frac{h(2W(y))}{W^{-1}(W(y)+h(2W(y)) - W^{-1}(W(y))} \frac{1}{h(2W(y))} \\ &\leq \frac{w(\tau)}{h(2W(y))} \leq \frac{g(W(\tilde{y}))}{h(2W(y))} \leq (2W(y))^{\delta-1} \end{aligned} \tag{2.9}$$

for some $\tau < \tilde{y}$ by the mean value theorem, (2.3), and (2.8). Similarly,

$$1/\epsilon(\tilde{y}) \leq (2W(\tilde{y}))^{\delta-1} \leq (4W(y))^{\delta-1}.$$

The conclusion follows (at first for $|z| \geq C_1(n)$) by (2.8), since W is increasing. □

2.4. THEOREM. *Let W satisfy* (2.1) *and* (2.2). *Then*

$$\Gamma^\delta(\pm W)'_b \in (\underline{\mathrm{DN}})_t \cap (\bar{\Omega})_t$$

and $\Gamma^\delta(\pm W)'_b$ is nuclear.

Proof. (a) With $h(t) = g(t)/t^{\delta-1}$ as in (2.3) and with large C, let $x_0 := 0$, $x_1 := C$, and $W(x_{r+1}) = W(x_r) + h(2W(x_r))$ for $r \geq 1$ (i.e., $x_{r+1} = \tilde{x}_r$; see (2.5)). x_r is strictly increasing and unbounded (otherwise $x = \lim x_r$ exists, and $W(x) = \lim W(x_{r+1}) = W(x) + h(2W(x))$, hence $h(2W(x)) = 0$, a contradiction).

For negative r, x_r is defined by $x_r := -x_{-r}$. Let $\epsilon(r) := \epsilon(x_r) := x_{|r|+1} - x_{|r|}$.

(b) Let $I_r := (x_r, x_{r+2})$ and $I_{-r} := -I_{-r}$ for $r \in \mathbf{N}$ ($I_0 := (-x_2, x_2)$). Let χ_M be the characteristic function of $M \subset \mathbf{R}$. For $r \geq 1$, let $\mu_{\epsilon(r)}$ be defined as in Lemma 2.2. Let $\tilde{\Phi}_r := \chi_{B_r} * \mu_{\epsilon(r)}$, $\Phi_r := \tilde{\Phi}_{r+1} - \tilde{\Phi}_r$, and $\Phi_{-r}(t) := \Phi_r(-t) := \check{\Phi}_r(t)$, where $B_r := (-\infty, x_r + \epsilon(r)/2)$ ($\Phi_0(t) := \tilde{\Phi}_1(|t|)$). $\{\Phi_r \mid r \in \mathbf{Z}\}$ is a resolution of the identity subordinate to $\{I_r \mid r \in \mathbf{Z}\}$, which satisfies the assumptions of Lemma 2.3. These are also satisfied by

$$\check{\psi}_{-r} := \psi_r := \chi_{B_r''} * \mu_{\epsilon(r+1)} - \chi_{B_r'} * \mu_{\epsilon(r)},$$

where $B_r' := (-\infty, x_r + \epsilon(r)/16)$ and $B_r'' := (-\infty, x_{r+1} + 9\epsilon(r+1)/16)$ for $r \geq 1$. $\psi_0 \in \Gamma_0^\delta(-x_2, x_2)$ is chosen to be 1 on supp Φ_0. $\psi_r \equiv 1$ on a neighbourhood of supp Φ_r for $r \in \mathbf{Z}$.

(c) Let $S_x f := f(\cdot - x)$ for $f \in C^\infty(\mathbf{R})$. Let $\kappa_1^\pm : C^\infty(\mathbf{R}) \to s \hat{\otimes}_\pi \omega$ and $\kappa_2^\pm : s \hat{\otimes}_\pi \omega \to C^\infty(\mathbf{R})$ be defined by

$$\kappa_1^\pm(f) := \frac{1}{\gamma(r)} \left(S_{-x_r}(f\Phi_r))\hat{\ } \left(2\pi \frac{s}{\gamma(r)} \right) e^{\pm W(x_r)} \right)_{(s,r) \in \mathbf{Z} \times \mathbf{Z}}$$

and

$$\kappa_2^\pm((c_{sr}))(t) := \sum_r \psi_r S_{x_r} \sum_s e^{2\pi i s t/\gamma(r)} e^{\mp W(x_r)} c_{sr}$$

for $\gamma(r) = x_{|r|+2} - x_{|r|} = \epsilon(r+1) + \epsilon(r)$. It is easily seen that $\kappa_i^\pm$ is defined and $\kappa_2^\pm \circ \kappa_1^\pm$ is the identity on $C^\infty(\mathbf{R})$.

(d) Let

$$\Lambda := \{(c_{sr}) \mid \sup_{(s,r) \in \mathbf{Z} \times \mathbf{Z}} |c_{sr}| e^{(|s/\gamma(r)|^{1/\delta} + W(x_r))/n} =: q_n((c_{sr})) < \infty \quad \text{for some } n \geq 1\}.$$

To show that $\kappa_1^\pm : \Gamma^\delta(\pm W) \to \Lambda$ and $\kappa_2^\pm : \Lambda \to \Gamma^\delta(\pm W)$ are tame, we use the following consequences of (2.5) and (2.4):

$$\text{(2.10)} \qquad \forall n \, \exists C_1 \, \forall r : \begin{cases} W(x_{r+2}) \leq (1 + 1/n) W(x_r) + C_1, \\ -W(x_r) \quad \leq (-1 + 1/n) W(x_{r+2}) + C_1. \end{cases}$$

Lemma 2.3 implies

$$\begin{aligned} q_n(\kappa_1^\pm(f)) &\leq \sup_{s,r} \frac{1}{\gamma(r)} \left| (f\Phi_r)\hat{\ } \left(\frac{2\pi s}{\gamma(r)} \right) \right| e^{\pm W(x_r) + [|s/\gamma(r)|^{1/\delta} + W(x_r)]/n} \\ &\leq C_2 \sup_r \sup_{x \in I_r} |f^{(k)}(x)| \left(\frac{A}{nk} \right)^{\delta k} e^{(\pm 1 + (A+2)/n) W(x)} \\ &\leq C_2 p_{n/(A+2)}^\pm(f). \end{aligned}$$

Lemma 2.2 and (2.7)-(2.10) imply

$$\begin{aligned} p_n^\pm(\kappa_2^\pm(c_{sr})) &\leq \sum_{s,r} p_n^\pm(\psi_r e^{2\pi i(\cdot - x_r)/\gamma(r)}) |c_{sr}| e^{\mp W(x_r)} \\ &\leq \sum_{s,r} \sup_k \|\psi_r^{(k)}\|_\infty \left(\frac{2}{nk} \right)^{\delta k} \sup_k \left(\frac{2\pi}{\gamma(r)} \right)^k \left(\frac{2}{nk} \right)^{\delta k} |c_{sr}| e^{4W(x_r)/n} \leq \end{aligned}$$

$$\leq C_3 \sum_{s,r} e^{(8CW(x_r)+\tilde{C}|s/\delta(r)|^{1/\delta})/n}|c_{sr}|$$

$$\leq C_4 q_{n/C'}((c_{sr})).$$

The final estimate follows, as

$$\exp\left(\frac{-(W(x_r)+|s/\gamma(r)|^{1/\delta})}{n}\right) \in l^1. \tag{2.11}$$

Proof.

$$\epsilon(r) = \tilde{x}_r - x_r \leq \frac{h(2W(x_r))}{w(\tau)} \leq \frac{h(2W(x_r))}{\epsilon} \leq \frac{W(x_r)}{6} \quad \text{for large } r$$

by (2.1) and (2.4). For large r, (2.8) implies that

$$\frac{1}{\gamma(r)} \geq \frac{2}{W(x_r)}$$

and that

$$W(x_r) + \left|\frac{s}{\gamma(r)}\right|^{1/\delta} \geq W(x_r) + \left|\frac{2s}{W(x_r)}\right|^{1/\delta} \geq |s|^{1/(2\delta)} + \frac{W(x_r)}{2}.$$

The conclusion follows, as

$$\sum_r e^{-W(x_r)/n} \leq \sum_r \frac{1}{\epsilon(r)} e^{-W(x_r)/(4n)} \int_{x_r}^{x_{r+1}} e^{-W(x)/(4n)}\, dx$$

$$\leq \int_0^\infty e^{-W(x)/(4n)}\, dx < \infty$$

by (2.8) and (2.9).

(e) $\kappa_1^\pm \circ \kappa_2^\pm$ is a (tame) projection in Λ onto a subspace E, which is t-isomorphic to $\Gamma^\delta(\pm W)$ via $(\kappa_1^\pm)^{-1}|_E = \kappa_2|_E$. As Λ is t-isomorphic to $\Lambda_0(\alpha)$ for some α, we get $\Gamma^\delta(\pm W) \in (\underline{\mathrm{DN}})_t \cap (\overline{\Omega})_t$. $\Gamma^\delta(\pm W)'_b$ is nuclear, as Λ is nuclear by the Grothendieck–Pietsch criterion (see (2.11)). □

In fact, $\Gamma^\delta(\pm W)$ is t-isomorphic to

$$\tilde{\Lambda} := \left\{(c_{sr}) \in \mathbf{C}^{\mathbf{Z}\times\mathbf{Z}} \mid \sum_{s,r} |c_{sr}| e^{(|s|^{1/\delta}+W(r))/n} := q_n(c_{sr}) < \infty \text{ for some } n \geq 1\right\},$$

but we will not need this in the subsequent sections. (For an indication of the proof see Proposition 1.5 and Theorem 1.6 in [6].)

In the literature it is usually assumed that $\Gamma^\delta(\pm W)$ is stable for shifts, that is,

$$W(t+1) = o(W(t)). \tag{2.12}$$

This implies that $W(t) \leq e^t$ for large t.

Condition (2.2) is much weaker than (2.12). It does not imply *a priori* bounds for the growth of W (see §4, where we give simple examples of weight functions that satisfy any of the conditions needed in this paper).

3. Fourier Transformation

It is shown in this section that the Fourier transformation is a tame isomorphism of $\Gamma^\delta(-W)'_b$ (and $\Gamma^\delta(W)$) onto certain spaces of entire functions defined below. Moreover, special plurisubharmonic (psh.) functions (needed to apply the fundamental principle of Ehrenpreis) may be constructed in a tame way.

Let U^* be the Young conjugate of a convex function U defined by

$$U^*(y) := \sup_{x \in \mathbf{R}^k} y \cdot x - U(x).$$

Let

$$W_i(t) := \int_0^{|t|} w_i(\tau)\, d\tau, \tag{3.1}$$

where $w_i \in C[0,\infty)$ is increasing and bijective on $[0,\infty)$.

3.1. DEFINITION. Let $W(t) = \sum_{i=1}^N W_i(t_i)$ satisfy (3.1).

(a)

$$\mathcal{H}_- := \{f \in \mathcal{H}(\mathbf{C}^N) \mid q_n^-(f) := \sup_z |f(z) e^{-((1-1/n)W)^*(\operatorname{Im} z) - |z|^{1/\delta}/n}| < \infty \quad \text{for any } n \geq 1\}.$$

(b)

$$\mathcal{H}_+ := \{f \in \mathcal{H}(\mathbf{C}^N) \mid q_n(f) := \sup_z |f(z) e^{-((1+1/n)W)^*(\operatorname{Im} z) + |z|^{1/\delta}/n}| < \infty \quad \text{for some } n \geq 1\}.$$

Again, for the sake of simplicity, only functions of one variable are considered in this section.

3.2. PROPOSITION. *The Fourier transformation is a t-isomorphism between*

(a) $\Gamma^\delta(W)$ *and* $\mathcal{H}_+$;

(b) $\Gamma^\delta(-W)'_b$ *and* $\mathcal{H}_-$, *if* W *also satisfies* (2.2).

Proof. (a) With $A := 2e/\delta$, by (2.7) we have

$$q_n(\hat{f}) \leq C_1(n) \sup_{k,z} |\hat{f}(z)| e^{-((1+1/n)W)^*(\operatorname{Im} z)} |z|^k (A/(nk))^{\delta k}$$
$$\leq C_1 p_{n/A_1}(f) \int e^{-W(x)/n}\, dx \leq C_2 p_{n/A_1}(f)$$

for $A_1 \geq \max\{A, 2\}$. The Fourier inversion formula implies

$$f^{(k)}(x)=\frac{1}{2\pi}\int \hat{f}(\xi+iy_x)(\xi+iy_x)^k e^{ix(\xi+iy_x)}\,d\xi$$

for any $y_x\in\mathbf{R}$. Choose y_x such that $x\cdot y_x-(1+1/n)W(x)=((1+1/n)W)^*(y_x)$. Then

$$\begin{aligned}p_n(f)&\le C_3(n)\sup_x\int|\hat{f}(\xi+iy_x)|e^{\delta|x+iy_x|^{1/\delta}/(ne)-((1+1/n)W)^*(y_x)}\,d\xi\\&\le C_4 q_{n/A_2}(f)\end{aligned}$$

for $A_2\ge\max(2\delta/e,1)$, by (2.7).

(b.i) The canonical grading of $\Gamma^\delta(-W)'_b$ is defined by

$$\|T\|_n^-:=\sup\{|\langle T,f\rangle|\mid p_n^-(f)\le 1\}\quad\text{for } T\in\Gamma^\delta(-W)'_b.$$

With $A\ge\max(1,\delta/e)$ and

$$f_z(x):=\exp(-ixz-|z|^{1/\delta}/n-((1-1/n)W)^*(\operatorname{Im} z))$$

we get, from (2.7),

$$p_{nA}^-(f_z)\le 1\quad\text{and}\quad q_n^-(\hat{T})=\sup_{z\in\mathbf{C}}|\langle T,f_z\rangle|\le\|T\|_{nA}^-. \tag{3.2}$$

Hence the Fourier transform is tame from $\Gamma^\delta(-W)'_b$ into $\mathcal{H}_-$.

(b.ii) For $g\in\mathcal{H}$ and $f\in\Gamma_0^\delta(\mathbf{R})$ we define

$$\langle H(g),\check{f}\rangle:=\frac{1}{2\pi}\int_{\mathbf{R}} g(x)\hat{f}(x)\,dx.$$

$H(g)$ is defined (e.g., by part (a)) since $\Gamma_0^\delta(\mathbf{R})$ is contained in $\Gamma^\delta(W)$ and $H(g)$ is linear on $\Gamma_0^\delta(\mathbf{R})$. Choose a partition of unity $\{\Phi_r\mid r\in\mathbf{Z}\}$ as in part (b) of the proof of Theorem 2.4. This implies, for any $y_r\in\mathbf{R}$, that

$$\begin{aligned}&|\langle H(g),\check{f}\rangle|\\&\le\sum_r|\langle H(g),\check{f}\check{\Phi}_r\rangle|\le\frac{1}{2\pi}\sum_r\left|\int g(x+iy_r)\widehat{f\Phi_r}(x+iy_r)\,dx\right|\\&\le\frac{1}{2\pi}q_n^-(g)\int e^{-|x|^{1/\delta}/n}\,dx\sum_r\sup_x\left|\widehat{f\Phi_r}(x+iy_r)e^{((1-2/n)W)^*(y_r)}e^{2|x+iy_r|^{1/\delta}/n}\right|\\&\le C_1(n)q_n^-(g)p_{n/(4A)}^-(f)\sum_r\int_{I_r}e^{xy_r}\,dx\,e^{((1-2/n)W)^*(y_r)}e^{(1-A/n)W(x_r)},\end{aligned} \tag{3.3}$$

by Lemma 2.3 and (2.10). The sum is finite if $A\ge 3$ and if we choose y_r for $|r|\ge 1$ such that $\sup_{x\in I_r}xy_r=-|x_ry_r|$ and

$$-|x_ry_r|+((1-2/n)W)^*(y_r)=-(1-2/n)W(x_r).$$

Let

$$E_n:=\{f\in\Gamma^\delta(-W)\mid \lim_{|x|\to\infty}\sup_k|f^{(k)}(x)e^{(-1+1/n)W(x)}/(nk)^{\delta k}|=0\quad\text{for some } n\ge 1\}$$

with the norm p_n^-. Then $H(g)$ defines continuous linear forms $H_n(g)$ on $E_n \cap \Gamma_0^\delta(\mathbf{R})$ for any n, which may be uniquely extended to $\tilde{H}_n(g) \in E_n'$, as $\Gamma_0^\delta(\mathbf{R}) \subset E_n$ is dense in E_n. More precisely:

$$\forall n \geq 1\, \exists C: \tilde{\mathfrak{B}}_n \subset \overline{C\tilde{\mathfrak{B}}_n \cap \Gamma_0^\delta(\mathbf{R})}^{E_n}, \tag{3.4}$$

where $\tilde{\mathfrak{B}}_n$ is the unit ball in E_n.

$\{\tilde{H}_n(g) \mid n \geq 1\}$ define a continuous linear mapping $\tilde{H}(g)$ on $\operatorname{ind\,lim} E_n$, which is t-isomorphic to $\Gamma^\delta(-W)$. (3.3) and (3.4) then show that $\tilde{H}$ is tame from $\mathcal{H}_-$ into $\Gamma^\delta(-W)_b'$.

(b.iii) $\tilde{H}$ is bijective and $\tilde{H}^{-1}$ is the Fourier transform.

Proof. Let

$$0 = \langle \tilde{H}(g), \check{f} \rangle = \frac{1}{2\pi} \int g(x) \hat{f}(x)\, dx \tag{3.5}$$

for any $f \in \Gamma_0^\delta(\mathbf{R})$. Then (3.5) also holds for any $f \in \Gamma^\delta(x^2/2)$, as both sides of (3.5) are continuous on $\Gamma^\delta(x^2/2)$, which contains $\Gamma_0^\delta(\mathbf{R})$ as a dense subspace. As $e^{-x^2} f \in \mathcal{H}_+^\delta(x^2/2)$ $(:= \mathfrak{F}(\Gamma^\delta(x^2/2))$ for any $f \in \Gamma_0^\delta(\mathbf{R}))$, we have

$$0 = \int (g(x) e^{-x^2}) \hat{f}(x)\, dx \quad \text{for any } f \in \Gamma_0^\delta(\mathbf{R}).$$

As $ge^{-x^2} \in L^2(\mathbf{R})$ and as $\mathfrak{F}(\Gamma_0^\delta(\mathbf{R}))$ is dense in $L^2(\mathbf{R})$, this shows that $ge^{-x^2} = 0$ a.e. and $g(z) \equiv 0$, as g is entire. So, $\tilde{H}$ is injective.

To show the surjectivity of $\tilde{H}$, we only have to show that

$$\tilde{H} \circ \mathfrak{F} = \mathrm{Id}_{\Gamma^\delta(-W)_b'}$$

on the dense subspace $\mathfrak{D}(\mathbf{R})$ of $\Gamma^\delta(-W)_b'$. But this is just Parseval's formula:

$$\langle \tilde{H} \circ \mathfrak{F}(\varphi), \check{f} \rangle = \frac{1}{2\pi} \int \hat{\varphi}(x) \hat{f}(x)\, dx = \int \varphi(x) \check{f}(x)\, dx$$

for $\varphi \in \mathfrak{D}(\mathbf{R})$ and $f \in \Gamma_0^\delta(\mathbf{R})$.

Thus the proposition is completely proved. ❑

In particular, the inductive (resp. projective) limits defining $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) are compact.

Let $L_n^\pm(z) := ((1 \pm 1/n) W)^*(\operatorname{Im} z) \mp |z|^{1/\delta}/n$ (these are the weights on $\mathcal{H}_\pm$). The proposition below contains the technical tools needed to apply the fundamental principle (or, at least, the division-and-extension theorem [2]).

3.3. PROPOSITION. (a) $\forall C, n\, \exists C_1$:

(i) $\sup_{|\eta| \leq C} L_n(z+\eta) + C \ln(1+|z|^2) \leq L_{2n}(z) + C_1$;

(ii) $\sup_{|\eta| \leq C} L_{2n}^-(z+\eta) + C \ln(1+|z|^2) \leq L_n^-(z) + C_1$.

(b.i) *L_n^- is subharmonic (sh.) for any $n \geq 1$.*

(b.ii) *Let (2.2) be valid. There is $\mathfrak{B} > 0$ such that for any $n \geq 1$ there are $C_i > 0$ and sh. functions h_n such that $L_{n/2}(z) \leq C_1 + h_n(z) \leq C_2 L_{\mathfrak{B}n}(z)$.*

Proof. (a.i) The only nontrivial estimate is seen as follows:

$$((1+2/n)W)^*(t+\tau) \le ((1+1/n)W(x)-C|x|)^*(t) \\ \le C_1+((1+1/(2n))W)^*(t)$$

for $|\tau| \le C$ by (3.1). (a.ii) follows similarly.

(b.i) $((1-1/n)W)^*(\operatorname{Im} z)$ is continuous and is the supremum of the sh. functions $x \cdot \operatorname{Im} z-(1-1/n)W(x)$. Hence it is sh. [3, Thm. 3.6.2]. $|z|^{1/\delta}/n$ is also sh. [3, §1.6]. Thus L_n^- is sh.

(b.ii) For fixed $n \ge 1$ and $r \in \mathbf{N}$ let x_r and y_r be defined by $y_1=1$,

$$((1+1/n)W)^*(y_r)=((1+2/n)W)^*(y_{r+1}),$$

and

$$x_r=w^{-1}(y_r/(1+1/n)).$$

Let $\gamma(r):=\tilde{\tilde{x}}_r-x_r$, where $\tilde{x}_r$ is defined by (2.5). Then

$$W(\tilde{x}_r)=W(x_r)+h(2W(x_r)).$$

y_r is strictly increasing to ∞, as $y=0$ is the only solution of

$$((1+1/(2n)W)^*(y)=((1+1/n)W)^*(y).$$

With $\mu_{\gamma(r)}$ as in Lemma 2.2, we take

$$\psi_r:=\frac{1}{\gamma(r)}\mu_{\gamma(r)}*\chi_{\mathfrak{B}_r}e^{-(1+1/n)W(x_r)};$$

$$\mathfrak{B}_r:=\left[x_r+\frac{\gamma(r)}{4}, x_r+\frac{3\gamma(r)}{4}\right].$$

Lemma 2.3, (2.10), and (2.11′) imply ($f\equiv 1$):

$$|\hat{\psi}_r(z)|e^{|z|^{1/\delta}/(16An)} \le C_1 e^{W(\tilde{x}_r)/(16n)+|\tilde{x}_r \operatorname{Im} z|-(1+1/n)W(x_r)} \\ \le C_1 e^{-W(x_r)/(2n)+((1+1/(16n))W)^*(\operatorname{Im} z)}.$$

That is,

$$\ln|\hat{\psi}_r(z)| \le C_2+L_{16An}(z)-W(x_r)/(2n). \tag{3.6}$$

Let $|\operatorname{Re} z| \le 1/\gamma(r)$ and $y_{r+1} \ge \operatorname{Im} z \ge y_r$ for $r \in \mathbf{N}$. Then

$$|\hat{\psi}_r(z)|=e^{-(1+1/n)W(x_r)}\left|\mu_{\gamma(r)}\left(\cdot-\frac{\gamma(r)}{64}\right)^{\wedge}(z)\right||\hat{\chi}_{\mathfrak{B}_r}(z)|;$$

$$|\chi_{\mathfrak{B}_r}(z)| \ge e^{x_r y_r}\left|\int_0^{\gamma(r)/2} e^{-iyz}\,dy\right| \ge \frac{\gamma(r)}{2}\cos(1/2)\,e^{x_r y_r};$$

$$\left|\mu_{\gamma(r)}\left(\cdot-\frac{\gamma(r)}{64}\right)^{\wedge}(z)\right| \ge \int_0^{\gamma(r)/32}\mu_{\gamma(r)}\left(x-\frac{\gamma(r)}{64}\right)\cos(x\operatorname{Re} z)\,dx \ge \cos(1/2).$$

The choice of x_r and y_r implies, for $|\operatorname{Re} z| \le 1/\gamma(r)$ ad $y_{r+1} \ge \operatorname{Im} z \ge y_r$:

$$(3.7)\qquad \begin{aligned} |\hat{\psi}_r(z)| &\geq C_2 e^{x_r y_r - (1+1/n)W(x_r)} = C_2 e^{((1+1/n)W)^*(y_r)} \\ &= C_2 e^{((1+2/n)W)^*(y_{r+1})} \geq C_2 e^{((1+2/n)W)^*(\operatorname{Im} z)}. \end{aligned}$$

By §1.6 of [3],

$$\psi_{rs}(z) := \ln|\hat{\psi}_r(z - s/\gamma(r))| - |s/\gamma(r)|^{1/\delta}/n$$

is sh. for $r, s \in \mathbf{Z}$. Let $\tilde{h}_n(z) := \sup_{r,s}\{\psi_{rs}(z), \psi_{rs}(-z)\}$. Then

$$(3.8)\qquad \begin{aligned} \psi_{rs}(z) &\leq C_2 + L_{16An}(\pm z - s/\gamma(r)) - |s/\gamma(r)|^{1/\delta}/n - W(x_r)/(2n) \\ &\leq C_2 + L_{16An}(z) - (W(x_r) + |s/\gamma(r)|^{1/\delta})/(2n). \end{aligned}$$

For $z \in \mathbf{C}$ (with $\operatorname{Im} z \geq 1$) there is $r \in \mathbf{N}$ (and then $s \in \mathbf{Z}$) such that

$$y_{r+1} \geq \operatorname{Im} z \geq y_r \quad \text{and} \quad |\operatorname{Re} z - s/\gamma(r)| \leq 1/\gamma(r).$$

(3.7) implies:

$$(3.9)\qquad \begin{aligned} \tilde{h}_n(z) &\geq \ln|\hat{\psi}_r(z - s/\gamma(r)| - |s/\gamma(r)|^{1/\delta}/n \\ &\geq C_3 + ((1+2/n)W)^*(\operatorname{Im} z) - |s/\gamma(r)|^{1/\delta}/n \\ &\geq C_3 + L_{n/2}(z). \end{aligned}$$

The supremum defining $\tilde{h}_n$ is locally finite by (3.8). Hence $\tilde{h}_n(z)$ is continuous and sh. by Theorem 1.6.2 in [3]. $\tilde{h}_n$ satisfies the desired estimates for $|\operatorname{Im} z| \geq 1$. We may now take h_n as the supremum of $\tilde{h}_n$ and

$$\{\ln|\hat{\phi}(z-s)| - |s|^{1/\delta}/n \mid s \in \mathbf{Z}\},$$

where $\phi \in \gamma_0^\delta(\mathbf{R})$ is fixed such that $|\hat{\phi}(z)| \geq 1$ for $|z| \leq 1$. □

Notice that the difference of $L_{n/2}$ and $L_{\circledR n}$ may be very small (for large W). It is, however, sufficient to correct the nonsubharmonicity of $-|z|^{1/\delta}/n$.

4. Tame Right Inverses for Systems of Partial Differential Operators

We may now apply the tame splitting theorem of Section 1 and the structural results of Sections 2 and 3 to obtain tame right inverses for systems of partial differential operators with constant coefficients in the graded spaces $(\Gamma^\delta(W)'_b)^s$ and $\Gamma^\delta(-W)^s$. The main tool is a tame version of the fundamental principle (see Proposition 4.1 below), whose proof was prepared in Section 3.

Let $R(D)$ be an $r \times s$ system of partial differential operators with constant coefficients, and let $\{(\partial_j, V_j) \mid j = 1, \ldots, J\}$ be a Noetherian operator for ${}^tR(-z)$ (see [2]), with linear differential operators $\partial_j(z, D_z)$ (of size $1 \times s$) with polynomial coefficients and algebraic varieties V_j contained in the characteristic variety of R:

$$V_R := \{z \in \mathbf{C}^N \mid \operatorname{rank} {}^tR(-z) < s\}.$$

Let ρ be defined by

$$\rho(f) := (\partial_j f \mid_{V_j})_{j \le J} \quad \text{for } f \in \mathcal{H}(\mathbf{C}^N)^s.$$

The range of ρ is denoted by

$$\mathcal{H}(V_R) := \left\{ (f_j) \in \prod_{j \le J} \mathcal{H}(V_j) \mid \rho(f) = (f_j) \quad \text{for some } f \in \mathcal{H}(\mathbf{C}^N)^s \right\},$$

$$\mathcal{H}_+(V_R) := \{(f_j) \in \mathcal{H}(V_R) \mid \|(f_j)\|_n := \max_{j \le J} \sup_{z \in V_j} |f_j(z) e^{-L_n(z)}| < \infty \quad \text{for some } n \ge 1\},$$

and

$$\mathcal{H}_-(V_R) := \{(f_j) \in \mathcal{H}(V_R) \mid \|(f_j)\|_n^- := \max_{j \le J} \sup_{z \in V_j} |f_j(z) e^{-L_n^-(z)}| < \infty \quad \text{for any } n \ge 1\},$$

where $L_n^{\pm}(z) := ((1 \pm 1/n) W)^*(\operatorname{Im} z) \mp |z|^{1/\delta}/n$ are the weights defining also $\mathcal{H}_\pm$ (see 3.1). $\mathcal{H}_\pm(V_R)$ carry their natural projective (resp. inductive) topology.

4.1. PROPOSITION. *Let W satisfy* (3.1) *and* (2.2).

(a) $\mathcal{H}_+(V_R)$ *(and $\mathcal{H}_-(V_R)$) are g-(DFS)-spaces (resp. g-(FS)-spaces) with the gradings defined by* $\| \ \|_n^{\pm}$.

(b) ρ *is a t-isomorphism* (I) *from* $(\mathcal{H}_+)^s/{}^tR(-z)(\mathcal{H}_+)^r$ *onto* $\mathcal{H}_+(V_R)$ *and* (II) *from* $(\mathcal{H}_-)^s/{}^tR(-z)(\mathcal{H}_-)^r$ *onto* $\mathcal{H}_-(V_R)$.

Proof. "(b) ⇒ (a)" ρ is continuous from $(\mathcal{H}_\pm)^s$ onto $\mathcal{H}_\pm(V_R)$ by (b). Hence ${}^tR(-z)(\mathcal{H}_\pm)^r = \operatorname{Ker}\rho$ is closed in $(\mathcal{H}_\pm)^s$, and $(\mathcal{H}_\pm)^s/{}^tR(-z)(\mathcal{H}_\pm)^r$ are (FS)-spaces (resp. (DFS)-spaces). Applying (b) (and Theorem 8 of [4]) again, this implies that the projective (resp. inductive) spectra defining $\mathcal{H}_\pm(V_R)$ are compact. This proves (a).

(b.I.i) We first show that $\rho: (\mathcal{H}_+)^s \to \mathcal{H}_+(V_R)$ is tame:

$$\begin{aligned} \|\rho(f)\|_{2n} &= \max_{j \le J} \sup_{z \in V_j} |\partial_j(z, D_z) f(z) e^{-L_{2n}(z)}| \\ &\le C_1 \sup_{\substack{z \in \mathbf{C}^N \\ |\eta| \le C_1}} |(1+|z|)^{C_1} f(z+\eta) e^{-L_{2n}^-(z)}| \le C_2 q_n(f) \end{aligned}$$

for some $C_i > 0$ by Proposition 3.3(a)(i), as ∂_j are differential operators with polynomial coefficients.

(b.I.ii) The division-and-extension theorem (D/E theorem, see [2, p. 240]) implies that ${}^tR(-z)(\mathcal{H}_+)^r \subset \operatorname{Ker}\rho \cap (\mathcal{H}_+)^s$. Conversely, let $g \in \operatorname{Ker}\rho$ and $q_n(g) < \infty$. Choose a psh. function h_n and $C_i > 0$ according to Proposition 3.3(b)(ii) such that

$$-C_2 - L_{2\mathfrak{B}n}(z) \le -C_1 - h_n(z) \le -L_n(z). \tag{4.1}$$

Then

$$\sup_{|\eta| \le C} h_n(z+\eta) + C \ln(2+|z|^2) \le C_j + L_{4\mathfrak{B}n}(z)$$

by Proposition 3.3(a)(i).

Now the D/E theorem implies the existence of $v \in \mathcal{H}(\mathbf{C}^N)^r$, with

$$g = {}^tR(-z)v \quad \text{and} \quad q_{4\mathcal{B}n}(v) \leq C_4 \sup_{z \in \mathbf{C}^N} |g(z)| e^{-h_n(z)} \leq C_5 q_n(g).$$

So Ker $\rho \cap (\mathcal{H}_+)^s = {}^tR(-z)(\mathcal{H}_+)^r$ and ρ induces a tame (injective) mapping

$$\rho_+ : (\mathcal{H}_+)^s / {}^tR(-z)(\mathcal{H}_+)^r \to \mathcal{H}_+(V_R).$$

(b.I.iii) For $(f_j) \in \mathcal{H}_+(V_R)$ with $\|(f_j)\|_n \leq 1$, we may choose $f \in \mathcal{H}(\mathbf{C}^N)$ such that $\rho(f) = (f_j)$. Using h_n as in (4.1) and the D/E theorem, one shows the existence of $g \in \mathcal{H}(\mathbf{C}^N)$ such that $\rho(g) = \rho(f) = (f_j)$ and

$$\begin{aligned} q_{4\mathcal{B}n}(g) &\leq C_6 \max_{j \leq J} \sup_{z \in V_j} |\partial_j(z, D_z) f(z)| e^{-h_n(z)} \\ &\leq C_7 \|(f_j)\|_n. \end{aligned}$$

Hence ρ_+ is surjective and $(\rho_+)^{-1}$ is tame. This proves (b.I).

(b.II.i) It is proved above that $\rho : (\mathcal{H}_-)^s \to \mathcal{H}_-(V_R)$ is tame. Similarly,

(4.2) $$ {}^tR(-z)(\mathcal{H}_-)^r \subset \text{Ker } \rho \cap (\mathcal{H}_-)^s$$

and

(4.3) $$q_n^-({}^tR(-z)f) \leq C_1 q_{2n}^-(f).$$

Using Proposition 3.3 and the D/E theorem, one gets the following. First, $\exists B \, \forall n \, \exists C_1 \, \forall g \in \text{Ker } \rho \, \exists h_n \in \mathcal{H}(\mathbf{C}^N)^r$,

(4.4) $$g = {}^tR(-z)h_n \quad \text{and} \quad q_n^-(h_n) \leq C_1 q_{\mathcal{B}n}^-(g).$$

Next, $\exists B \, \forall n \, \exists C_1 \, \forall (f_j) \in \mathcal{H}_-(V_R) \, \exists g_n \in \mathcal{H}(\mathbf{C}^N)$,

(4.5) $$\rho(g_n) = (f_j) \quad \text{and} \quad q_n^-(g_n) \leq C_1 \|(f_j)\|_{\mathcal{B}n}^-.$$

Let $\mathcal{H}_{-k} := \{ f \in \mathcal{H}(\mathbf{C}^N) \mid q_n^-(f) < \infty \}$. Then

(4.6) $$\exists B_1 : \mathcal{H}_- \text{ is dense in } \mathcal{H}_{-\mathcal{B}_1 k} \text{ for the norm } q_{2k}^-.$$

(Use convolution and multiplication with Fourier transforms of cut-off function $\varphi \in \gamma_0^\delta$.)

Combining (4.2)–(4.6), we have:

$$\begin{aligned} \mathcal{H}^s_{-\mathcal{B}_1 \mathcal{B} k} \cap \text{Ker } \rho \subset {}^tR(-z)\mathcal{H}^r_{-\mathcal{B}_1 k} &\subset {}^tR(-z)\overline{(\mathcal{H}^r_-)}^{q_{2k}^-} \\ &\subset \overline{({}^tR(-z)\mathcal{H}^r_-)}^{q_k^-} \subset \overline{(\mathcal{H}^s_- \cap \text{Ker } \rho)}^{q_k^-}. \end{aligned}$$

So, $\mathcal{H}^s_- \cap \text{Ker } \rho$ is dense in $(\mathcal{H}_{-\mathcal{B}_1 \mathcal{B} k})^s \cap \text{Ker } \rho$ in the topology of $(\mathcal{H}_{-k})^s$. The Mittag-Leffler procedure and (4.5) show that,

$$\exists \mathcal{B}_2 \, \forall n \, \exists C_1 \, \forall (f_j) \in \mathcal{H}_-(V_R) \, \exists g \in (\mathcal{H}_-)^s,$$

$$\rho(f) := (f_j) \quad \text{and} \quad q_n^-(g_n) \leq C_1 \|(f_j)\|_{\mathcal{B}_2 n}^-.$$

$\rho_- : \mathcal{H}^s_- / (\text{Ker } \rho \cap \mathcal{H}^s_-) \to \mathcal{H}_-(V_R)$ is a t-isomorphism.

(b.II.ii) It remains to show that ${}^tR(-z)(\mathcal{H}_-)^r$ contains $(\mathcal{H}_-)^s \cap \operatorname{Ker} \rho$.

Proof. We must show that

$$(*) \qquad {}^tR(-z)h = g$$

is solvable with $h \in (\mathcal{H}_-)^r$, if $\rho(g) = 0$ and $g \in (\mathcal{H}_-)^s$.

(α) If ${}^tR(-z)$ is injective on $\mathbf{C}[z]^r$, then ${}^tR(-z)$ is injective on $\mathcal{H}(\mathbf{C}^N)^r$ and (4.4) holds with $h \equiv h_n$ for any n, so $h \in (\mathcal{H}_-)^r$ and ${}^tR(-z)h = g$.

(β) If ${}^tR(-z)f = 0$ for some $0 \neq f \in \mathbf{C}[z]^r$, then there is a $r_1 \times r$ matrix Q of polynomials such that ${}^tR(-z)f = 0$ for $f \in \mathbf{C}[z]^r$ if and only if $f = {}^tQ(-z)g$ for some $g \in \mathbf{C}[z]^{r_1}$.

Then $\{({}^t\check{R}_j, \mathbf{C}^N) \mid j \le s\}$ is a Noetherian operator for ${}^tQ(-z)$, where R_j are the columns of R. The argument in (b.i) is now applied (with $\{({}^t\check{R}_j, \mathbf{C}^N)\}$ instead of ρ and ${}^tQ(-z)$ instead of ${}^tP(-z)$) to show that $(*)$ is solvable as desired. □

Let $P(D)$ be a $r \times s$ system of partial differential operators with constant coefficients (on $\mathbf{R}^N$), and let N_P be the kernel of $P(D)$ (e.g., in the hyperfunctions). Let $Q(D)$ be the matrix of relations implied by $P(D)$ (i.e., ${}^tP(-z)f = 0$ for $f \in \mathbf{C}[z]^r$ if and only if $f = {}^tQ(-z)g$ for some $g \in \mathbf{C}[z]^{r_1}$). Q may be 0.

The main result of this paper is now concerned with the following two sequences:

$$(4.7) \qquad 0 \to N_P \cap \Gamma^\delta(-W)^s \to \Gamma^\delta(-W)^s \xrightarrow{P(D)} N_Q \cap \Gamma^\delta(-W)^r \to 0;$$

$$(4.8) \qquad 0 \to N_P \cap (\Gamma^\delta(W)_b')^s \to (\Gamma^\delta(W)_b')^s \xrightarrow{P(D)} N_Q \cap (\Gamma^\delta(W)_b')^r \to 0.$$

Let $W(t) := \sum_{i \le N} W_i(t_i)$ and

$$(3.1) \qquad W_i(t) := \int_0^{|t|} w_i(\tau)\, d\tau,$$

where $w_i \in C[0,\infty)$ is increasing and bijective on $[0,\infty)$, where

$$(2.2) \qquad w_i(t) = o(W_i(t)^\delta),$$

and where $W_i \circ w_i^{-1}$ is stable; that is,

$$(4.9) \qquad \exists C\colon W_i \circ w_i^{-1}(2t) \le C W_i \circ w_i^{-1}(t) \quad \text{for large } t.$$

4.2. THEOREM. *Let W satisfy* (3.1), (2.2) *and* (4.9). *Then the sequences* (4.7) *and* (4.8) *are t-exact and split tamely; that is, $P(D)$ has tame right inverses*

$$R_-\colon N_Q \cap \Gamma^\delta(-W)^r \to \Gamma^\delta(-W)^s;$$
$$R_+\colon N_Q \cap (\Gamma^\delta(W)_b')^r \to (\Gamma^\delta(W)_b')^s.$$

Proof. (a) The gradings of $\Gamma^\delta(\pm W)$ are fixed in Definition 2.1. $\Gamma^\delta(\pm W)_b'$ carry the canonical dual grading defined by

$$p_n^{\pm *}(T) := \sup\left\{ |\langle T, f\rangle| \;\Big|\; p_n(f) = \sup_{x,k} \frac{|f^{(k)}(x) e^{(1 \pm 1/n) W(x)}|}{n^{\delta \cdot k} k^{\delta k}} \le 1 \right\}.$$

We consider the sequences of (FN)-spaces

$$0 \to (\Gamma^\delta(-W)'_b)^r / \overline{{}^tQ(-D)(\Gamma(-W)'_b)^{r_1}} \xrightarrow{{}^tP(-D)} (\Gamma^\delta(-W)'_b)^s \tag{4.10}$$
$$\xrightarrow{q} (\Gamma^\delta(-W)'_b)^s / {}^tP(-D)(\Gamma^\delta(-W)'_b)^r \to 0;$$

$$0 \to N_p \cap (\Gamma^\delta(W)'_b)^s \to (\Gamma^\delta(W)'_b)^s \xrightarrow{P(D)} (\Gamma^\delta(W)'_b)^r \cap N_Q \to 0. \tag{4.8}$$

These are t-exact (by Proposition 3.2) if and only if

$$ {}^tP(-z) : (\mathcal{H}_\pm)^r / \overline{{}^tQ(-z)(\mathcal{H}_\pm)^{r_1}} \to (\mathcal{H}_\pm)^s \tag{4.11}$$

is a t-isomorphism (into $(\mathcal{H}_\pm)^s$).

Let $Q \neq 0$. Then we already noticed that $\{({}^t\check{P}_j, \mathbf{C}^N) \mid j \leq s\}$ is a Noetherian operator for ${}^tQ(-z)$ and that (4.11) is valid by Proposition 4.1. If $Q \equiv 0$, then an $r \times r$ submatrix of ${}^tP(-z)$ is nonsingular, ${}^tP(-z)$ is injective on $(\mathcal{H}_\pm)^r$, and we may use Cramer's rule and the Malgrange–Ehrenpreis lemma to show that ${}^tP(-z)$ is t-open. ${}^tP(-z)$ is obviously tame (see 4.1). So t-exactness of (4.10) and (4.8) is proved and we may also skip the closure in (4.10).

(b) (4.9) implies the following ($W_i =: W$). (i)

$$(1/C)(C_2 - C_1) W \circ w^{-1}(t) - A \leq (C_1 W)^*(t) - (C_2 W)^*(t)$$
$$\leq C(C_2 - C_1) W \circ w^{-1}(t) + A$$

for any $1/2 \leq C_1 < C_2 \leq 2$ and some A. Also, (ii)

$$W^*(t) + W \circ w^{-1}(t)/(Cn) - A_n \leq ((1 - 1/n) W)^*(t) \tag{4.12}$$
$$\leq W^*(t) + C W \circ w^{-1}(t)/n + A_n;$$

$$W^*(t) - C W \circ w^{-1}(t)/n - A_n \leq ((1 + 1/n) W)^*(t) \tag{4.13}$$
$$\leq W^*(t) - W \circ w^{-1}(t)/(Cn) + A_n$$

for $n \geq 2$.

Proof. $W^*(t) = \int_0^t w^{-1}(\tau)\, d\tau$, as W satisfies (3.1). (4.9) implies:

$$(C_1 W)^*(t) - (C_2 W)^*(t) \geq \int_0^{w^{-1}(t/C_2)} (C_2 w(\tau) - C_1 w(\tau))\, d\tau$$
$$= (C_2 - C_1) W \circ w^{-1}(t/C_2) \geq \frac{C_2 - C_1}{C} W \circ w^{-1}(t) - A;$$

$$(C_1 W)^*(t) - (C_2 W)^*(t) \leq \int_0^{w^{-1}(t/C_1)} (C_2 w(\tau) - C_1 w(\tau))\, d\tau$$
$$= (C_2 - C_1) W \circ w^{-1}(t/C_1)$$
$$\leq C(C_2 - C_1) W \circ w^{-1}(t) + A.$$

This proves (i). (ii) follows by choosing $C_1 = 1 - 1/n$ and $C_2 = 1$ (resp. $C_1 = 1$ and $C_2 = 1 + 1/n$).

(c) We now consider (4.10). $\Gamma^\delta(-W)'_b \in (\underline{\mathrm{DN}})_t \cap (\bar{\Omega})_t$ by Theorem 2.4. Hence,

$$(\Gamma^\delta(-W)_b')^r/{}^tQ(-D)(\Gamma^\delta(-W)_b')^{r_1}\in(\underline{\mathrm{DN}})_t\cap(\bar{\Omega})_t,$$

as (4.10) is tame and as $(\bar{\Omega})_t$ is inherited to quotients. Similarly,

$$(\Gamma^\delta(-W)_b')^s/{}^tP(-D)(\Gamma^\delta(-W)_b')^r\in(\bar{\Omega})_t.$$

$(\mathcal{H}_-)^s/{}^tP(-z)(\mathcal{H}_-)^r$ is t-isomorphic (via ρ) to $\tilde{\mathcal{H}}_-(V_P)$, where $\tilde{\mathcal{H}}_-(V_P)$ is the space $\mathcal{H}_-(V_P)$ equipped with the grading defined by

$$|(f_j)|_n^- := \max_{j\le J}\sup_{z\in V_j}|f_j(z)e^{-\tilde{L}_n^-(z)}|,$$

where

$$\tilde{L}_n^-(z) := W^*(\operatorname{Im} z)+\frac{1}{n}\left(\sum_{i\le N}W_i\circ w_i^{-1}(\operatorname{Im} z_i)+|z|^{1/\delta}\right).$$

Indeed, $\tilde{\mathcal{H}}_-(V_P)$ and $\mathcal{H}_-(V_P)$ are t-isomorphic by (4.12). $(\underline{\mathrm{DN}})_t$ is now easily proved for $\tilde{\mathcal{H}}_-(V_P)$ (and hence for $(\Gamma^\delta(-W)_b')^s/{}^tP(-D)(\Gamma^\delta(-W)_b')^r$).

Theorem 1.6 now implies that (4.10) is (t-) split, so (4.7) is also split and $P(D)$ has a continuous linear right inverse, which is automatically tame ([14, 5.1] and [8, (1.1)]), as all spaces are t-isomorphic to duals of power series spaces of finite type.

(d) We now consider (4.8). $N_P\cap(\Gamma^\delta(W)_b')^s\in(\underline{\mathrm{DN}})_t$ (and $N_Q\cap(\Gamma^\delta(W)_b')^r\in(\underline{\mathrm{DN}})_t\cap(\bar{\Omega})_t$), being a subspace (and a quotient) of some product of $\Gamma^\delta(W)_b'$. $(N_P\cap(\Gamma^\delta(W)_b')^s)$ is t-isomorphic to $(\mathcal{H}_+)^s/{}^tP(-z)(\mathcal{H}_+)^r$, which is t-isomorphic (via ρ) to $\tilde{\mathcal{H}}_+(V_P)$, where the grading of $\tilde{\mathcal{H}}_+(V_P)$ is defined by

$$|(f_j)|_n := \max_{j\le J}\sup_{z\in V_j}|f_j(z)e^{-\tilde{L}_n(z)}|,$$

where

$$\tilde{L}_n(z) := W^*(\operatorname{Im} z)-\frac{1}{n}\left[\sum_{i\le N}W_i\circ w_i^{-1}(\operatorname{Im} z_i)+|z|^{1/\delta}\right].$$

$\tilde{\mathcal{H}}_+(V_P)$ is t-isomorphic to $\mathcal{H}_+(V_P)$ by (4.13). Using $\tilde{\mathcal{H}}_+(V_P)$, one proves as in Lemma 1.4. that $(N_P\cap(\Gamma^\delta(W)_b')^s)\in(\bar{\Omega})_t$. Theorem 1.6 now implies the existence of a tame right inverse for $P(D)$. □

Condition (4.9) is satisfied, for example, in the following cases ($W := W_i$):

(i) W satisfies (3.1) and $2w(t)\le w(Ct)\le C'w(t)$ for some $C>1$ and large t;

(ii) $W(t)=\exp(\tilde{W}(t))$, where $\tilde{W}$ satisfies (3.1).

It was shown in Remark 3.6 of [6] that, for any G satisfying (3.1), there is W satisfying (3.1) such that:

$$G(t)\le W(t)\le G(t+C)+1 \quad \text{for some } C>0 \text{ and large } t;$$

$$w(t)\le W(t)^\epsilon \quad \text{for any } \epsilon>1 \text{ and large } t.$$

So, any continuous function is bounded by some function satisfying the assumptions of Theorem 4.2. So no *a priori* bounds on the growth of W are implied by these assumptions (see also the final remarks of Section 2).

Notice that these assumptions are stable for taking compositions in the following sense: $W \circ V$ satisfies (3.1), (2.2), and (4.9) if W and V satisfy (3.1) and (4.9), if V satisfies (2.2), and if W satisfies $w(t) = O(W(t)^{\delta}/t^{\delta})$.

We now give some simple explicit examples.

4.3. EXAMPLES. The following functions satisfy the assumptions of Theorem 4.2 (for any $\delta > 1$):

(a) $W(t) = t^{\alpha}(\ln t)^{\beta}$, $\alpha > 1, \beta \geq 0$
(b) $W(t) = e^{(\ln t)^{\alpha}}$, $\alpha > 1$
(c) $W(t) = e^{t^{\alpha}}$, $\alpha > 0$
(d) $W(t) = k$-fold composition of $e^{t^{\alpha}}$, $\alpha > 0$, $k \geq 1$.

Let W satisfy the assumptions of Theorem 4.2. If P is hypoelliptic, then the solution spaces of $P(D)$ in $\Gamma^{\epsilon}(W)'_b$ and in

$$C_1^{\infty}(W) := \{f \in C^{\infty}(\mathbf{R}^N) \mid p_n(f) := \sup_{j \leq n} |f^{(j)}(x) e^{-(1+1/n)W(x)}| < \infty \quad \text{for any } n \geq 1\}$$

coincide algebraically and topologically if

$$\epsilon > \delta \cdot \rho, \tag{4.14}$$

where ρ is the index of hypoellipticity. In fact, $C_1^{\infty}(W)$ is a localizable analytically uniform space [5, Lemma 2.2b] and, on the characteristic variety of P, the weight systems determining $\mathcal{K}_+$ and the Fourier transform of $C_1^{\infty}(W)'_b$ are equivalent by (2.2), (4.13), and (4.14).

Notice that $C_1^{\infty}(W)$ is isomorphic to the π tensor product of a power series space of finite type with (s), and hence $C_1^{\infty}(W)$ is not a power series space. The kernel $N_p \cap C_1^{\infty}(W)^s$ is however isomorphic to a power series space of finite type if (3.1), (4.9), and a condition weaker than (2.2) are satisfied [5]. Theorem 4.2 now implies the following improvement of this result.

4.4. COROLLARY. *Let the assumption of Theorem 4.2 be satisfied* (*for some* δ) *and let* P *be hypoelliptic.*

(a) $N_p \cap (C_1^{\infty}(W))^s$ *is t-isomorphic to some* $\Lambda_0(\alpha_n)$ (*for the grading defined by* p_n).
(b) P *has a continuous linear right inverse* $R: C_1^{\infty}(W)^r \cap N_Q \to C_1^{\infty}(W)^s$.

Proof. (a) For $f \in C_1^{\infty}(W)$ and p_n as in 2.1 we get:

$$p_n^*(f) := \sup\{|\langle f, g\rangle| \mid p_n(g) \leq 1\} \leq \int e^{-(1+1/n)W(x)} |f(x)|\, dx$$

$$\leq \tilde{p}_{2n}(f) \int e^{-W(x)/(2n)}\, dx.$$

So $\mathrm{Id}: N_P \cap C_1^{\infty}(W)^s \to N_P \cap (\Gamma^{\epsilon}(W)'_b)^s$ is tame. As $N_P \cap (C_1^{\infty}(\gamma W))^s$ and $N_P \cap (\Gamma^{\epsilon}(\gamma W)'_b)^s$ coincide topologically for $\gamma \geq 1$, we obtain, for $\tilde{p}_{n,\gamma} := \tilde{p}_n$ in $C_1^{\infty}(\gamma W)$,

$$\forall n\, \exists m: \tilde{p}_n(f) \le \tilde{p}_{4n,1+1/(4n)}(f) \le C_1 p^*_{m,1+1/(4n)}(f) \le C_1 p^*_{4n}(f).$$

Thus $N_P \cap C_\Gamma^\infty(W)^s$ and $N_P \cap (\Gamma^\epsilon(W)'_b)^s$ are t-isomorphic and (a) follows from the proof of Theorem 4.2.

(b) Let $R_+ : N_Q \cap (\Gamma^\epsilon(W)'_b)^r \to (\Gamma^\epsilon(W)'_b)^s$ be a right inverse of $P(D)$ (from Theorem 4.2). Then $(\mathrm{Id} - R \circ P(D))$ is a continuous projection in $(\Gamma^\epsilon(W)'_b)^s$ onto $N_P \cap (\Gamma^\epsilon(W)'_b)^s = N_P \cap C_\Gamma^\infty(W)^s$. So, $\pi := (\mathrm{Id} - R \circ P(D)) \,|\, _{C_\Gamma^\infty(W)^s}$ is a continuous projection in $C_\Gamma^\infty(W)^s$ onto $N_P \cap C_\Gamma^\infty(W)^s$. This proves (b). □

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