Inner Ideals in W^*-Algebras

C. M. EDWARDS & G. T. RÜTTIMANN

1. Introduction

The study of the Jordan structure of operator algebras dates back to the work of Jordan, von Neumann and Wigner [7], who wished to place an appropriate algebraic structure on the set of bounded observables of a quantum mechanical system. Kadison [8] was the first to show that there is an intimate connection between the Jordan structure and the order structure of a C^*-algebra. Much of the study of the Jordan structure of a C^*-algebra A has followed Kadison in investigating the real Jordan algebra of self-adjoint elements of A. This Jordan algebra is an example of a JB-algebra, the properties of which have been extensively investigated over recent years. For a survey of the theory of JB-algebras the reader is referred to the book of Hanche-Olsen and Størmer [5]. In particular, the structure of the set of quadratic ideals in a JB-algebra is discussed in [1].

When investigating the ideal structure of complex Jordan algebras (or, more generally, of Jordan triples), a natural class of ideals which arises is that of inner ideals. In a *tour de force* McCrimmon [10] described algebraically the set of inner ideals in a quadratic Jordan algebra over an arbitrary commutative, associative ring. Some of McCrimmon's results were generalized to Jordan pairs by Loos [9]. The purpose of this paper is to describe the set of weak* closed inner ideals in a W^*-algebra.

Examples of weak* closed inner ideals in a W^*-algebra A are weak* closed left and right ideals of A. Such ideals were studied by Effros [3] and Prosser [13], who showed that every weak* closed left ideal in A is of the form Ae for some unique projection e in A. Moreover, the mapping $e \to Ae$ is an order isomorphism from the complete lattice $P(A)$ of projections in A onto the complete lattice of weak* closed left ideals in A, and the mapping $e \to A_*e$ is an order isomorphism from $P(A)$ onto the complete lattice of norm closed left invariant subspaces of the predual A_* of A. The main result of this paper is that every weak* closed inner ideal in A is of the form eAf for some pair (e,f) of projections in A. In general, this representation is no longer unique but sets up an order isomorphism between the complete

Received August 3, 1988.
Research supported by a grant from the United Kingdom Science and Engineering Research Council.
Michigan Math. J. 36 (1989).

lattice $C(P(A))$ of centrally equivalent pairs of projections in A and the complete lattice of weak* closed inner ideals in A, as well as an anti-order isomorphism between $C(P(A))$ and the complete lattice of norm closed A-bi-invariant subspaces of A_*.

In Section 2 various preliminary results are given whilst Section 3 is devoted to the proof of the main theorem. In Section 4 some of the consequences of the main result are considered. The methods of proof are very much Jordan-theoretic in nature. Indeed, some of them apply equally well when the W^*-algebra A is replaced by a JBW*-triple (for the theory, refer to [2; 4; 6; 11; 16]).

The authors are grateful for the support that their research has received from the United Kingdom Science and Engineering Research Council.

2. Preliminaries

A partially ordered set P is said to be a *lattice* if, for each pair (e,f) of elements of P, the supremum $e\vee f$ and the infimum $e\wedge f$ exist. The partially ordered set P is said to be a *complete lattice* if, for any subset M of P, the supremum $\bigvee M$ and the infimum $\bigwedge M$ exist. A complete lattice has a greatest element and a least element denoted by 1 and 0, respectively. A complete lattice—together with an anti-order morphism $e\to e'$ on P such that, for all elements e and f of P, $e\vee e'=1$, $e''=e$, and $f=e\vee(f\wedge e')$ if $e\le f$—is said to be *orthomodular*. A pair (e,f) of elements of P is said to be *orthogonal*, denoted by $e\perp f$, if $e\le f'$. An element z of P is said to be *central* if, for all elements e in P, $z=(z\wedge e)\vee(z\wedge e')$. The set $Z(P)$ of central elements of the complete orthomodular lattice P contains 0 and 1, and if z lies in $Z(P)$ then so also does z'. With the restricted order and orthocomplementation $z\to z'$, $Z(P)$ forms a subcomplete complete Boolean orthomodular sublattice of P which is said to be the *centre* of P. The *central support* $z(e)$ of an element e of P is defined by

$$z(e)=\bigwedge\{z\in Z(P): e\le z\}.$$

Observe that for each pair (e,f) of elements of P,

$$z(e\wedge z(f))=z(e)\wedge z(f) \tag{2.1}$$

and, for each family (e_j) of elements of P,

$$z(\bigvee e_j)=\bigvee z(e_j). \tag{2.2}$$

When endowed with the product ordering, the Cartesian product $P\times P$ of P with itself is a complete lattice. A pair (e,f) of elements of the complete orthomodular lattice P is said to be *centrally equivalent* if the central supports $z(e)$ and $z(f)$ coincide. Let $C(P)$ be the collection of centrally equivalent pairs of elements of P. It follows from (2.2) that, when endowed with the ordering inherited from $P\times P$, $C(P)$ is a complete lattice with least element $(0,0)$ and greatest element $(1,1)$, and the supremum of a subset of $C(P)$ coincides with its supremum when regarded as a subset of the complete lattice $P\times P$. In general that is not the case for the infimum, though for any element (e,f) of $C(P)$,

$$(e,f)=(z(f),f)\wedge_{C(P)}(e,z(e))=(z(f),f)\wedge_{P\times P}(e,z(e)). \tag{2.3}$$

For details the reader is referred to [14].

Let A be a W^*-algebra. A self-adjoint idempotent e in A is said to be a *projection* in A. For elements e and f in the set $P(A)$ of projections in A, write $e\leq f$ if $ef=e$ and $e'=1-e$. Then, with respect to the order relation $\leq$ and anti-order morphism $e\to e'$, $P(A)$ forms a complete orthomodular lattice. Moreover, the centre $Z(P(A))$ of $P(A)$ consists of the set of projections in the centre $Z(A)$ of A.

A subspace J of the W^*-algebra A is said to be a *left ideal* provided that $AJ\subseteq J$, and is said to be a *right ideal* if $JA\subseteq J$. A subspace J is said to be a *two-sided* ideal if it is both a left ideal and a right ideal. For each weak* closed left ideal J in A there exists a unique projection e in A such that J coincides with Ae. The ideal J is two-sided if and only if e is central.

For each element a in A the unique projection $e(a)$, such that $Ae(a)'=\{b\in A: ba=0\}$, is said to be the *left support* of a. It is the least element of $P(A)$ with the property that $e(a)a=a$. The *right support* $f(a)$ of A is similarly defined. Clearly $e(a)=f(a^*)$, and therefore the left and right supports of a self-adjoint element a of A coincide. This element $s(a)$ is the unit in the sub-W^*-algebra of A generated by a, and is said to be the *support projection* of a.

An element u in A is said to be a *partial isometry* if $uu^*u=u$ or, equiva lently, if either uu^* or u^*u is a projection. Clearly, every nonzero partial isometry is of norm one. A partial isometry u is said to be a *unitary* if uu^* and u^*u are equal to the unit 1 in A. The collection of partial isometries in A is denoted by $U(A)$. The subset of $U(A)$ consisting of unitaries in A forms a group which is denoted by $G(A)$. For each partial isometry u in A, $e(u)=uu^*$ and $f(u)=u^*u$.

For elements p and q in $P(A)$, write $p\lesssim q$ if there exists an element u in $U(A)$ such that $e(u)\leq q$ and $f(u)=p$. Notice that the relation $\lesssim$ is transitive. For each element a of A there exists a unique partial isometry $r(a)$ in A such that $a=r(a)|a|$ and $f(r(a))=s(|a|)$, where $|a|=(a^*a)^{1/2}$. Moreover,

$$r(a^*)=r(a)^*,\qquad f(a)=f(r(a)),\qquad e(a)=e(r(a)) \tag{2.4}$$

and

$$a=r(a)a^*r(a). \tag{2.5}$$

The partial isometry $r(a)$ is said to be the *support* of a. For details of these and related results, the reader is referred to Pedersen [12] and Sakai [15].

3. Inner Ideals

This section is devoted to the statement and proof of the main theorem. Let A be a W^*-algebra. The Jordan triple product $\{a\,c\,b\}$ of elements a, c, and

b of A is defined by

$$\{a\,c\,b\} = \tfrac{1}{2}(ac^*b + bc^*a).$$

A subspace J of A is said to be an *inner ideal* in A if $\{J\,A\,J\} \subseteq J$. Observe that right and left ideals of A are inner ideals. Moreover, for each pair (a, b) of elements of A, aAb is an inner ideal. Since multiplication in A is separately weak* continuous, the weak* closure of an inner ideal is an inner ideal. Because the involution in A is a weak* continuous isometry on A, J^* is a weak* closed inner ideal for every weak* closed inner ideal J in A. Since the intersection of a family of weak* closed inner ideals is a weak* closed inner ideal, when ordered by set inclusion the collection $I(A)$ of weak* closed inner ideals forms a complete lattice. Notice that, for each pair (e, f) of projections in A, the inner ideal eAf is weak* closed.

LEMMA 3.1. *Let u and v be partial isometries in the W^*-algebra A. Then the inner ideal uAv coincides with $e(u)Af(v)$.*

Proof. Since u and v are elements of $U(A)$, for each element a in A,

$$uav = uu^*uavv^*v = e(u)uavf(v),$$

which is contained in $e(u)Af(v)$. Conversely, if a is an element of $e(u)Af(v)$ then

$$a = e(u)af(v) = uu^*av^*v,$$

which is contained in uAv. □

LEMMA 3.2. *Let u be a partial isometry in the W^*-algebra A. For elements a and b in the weak* closed inner ideal uAu of A, define*

$$a \cdot b = au^*b, \qquad a^+ = ua^*u.$$

Then, with respect to the multiplication $(a, b) \to a \cdot b$ and the involution $a \to a^+$, uAu is a W^*-algebra with unit u.

Proof. Since uAu coincides with $e(u)Af(u)$, it is clear that uAu is closed under the two operations and that it forms a weak* closed *-algebra with unit u. Since

$$\|a^+\| = \|ua^*u\| \leq \|u\|\,\|a^*\|\,\|u\| = \|a^*\| = \|a\|$$

and the mapping $a \to a^+$ is an involution, $a \to a^+$ is also an isometry. Clearly, for all elements a and b in uAu,

$$\|a \cdot b\| = \|au^*b\| \leq \|a\|\,\|u^*\|\,\|b\| = \|a\|\,\|b\|.$$

Finally, observe that

$$\begin{aligned}\|(a^+ \cdot a)\|^2 &= \|(a^+ \cdot a)^*(a^+ \cdot a)\| = \|(ua^*uu^*a)^*(ua^*uu^*a)\|\\ &= \|(ua^*a)^*(ua^*a)\| = \|a^*au^*ua^*a\|\\ &= \|(a^*a)^*(a^*a)\| = \|a\|^4\end{aligned}$$

for each element a in uAu; thus uAu is a C^*-algebra which, being weak* closed, is a dual space and hence a W^*-algebra. □

LEMMA 3.3. *Let a be an element of the W^*-algebra A with support $r(a)$. Then, in the W^*-algebra $r(a)Ar(a)$, a is self-adjoint with support projection $r(a)$.*

Proof. It follows from (2.5) that a lies in $r(a)Ar(a)$ and that a is self-adjoint. Let p be a projection in the W^*-algebra $r(a)Ar(a)$ such that $pr(a)^*a=a$. Because

$$pr(a)^*p=p \quad \text{and} \quad r(a)p^*r(a)=p,$$

it follows that

$$pp^*p=pr(a)^*pr(a)^*p=pr(a)^*p=p$$

and hence that p is an element of $U(A)$. Moreover, $p|a|=a$.

Since $r(a)$ is the unit in the W^*-algebra $r(a)Ar(a)$,

$$pf(a)=pf(r(a))=pr(a)^*r(a)=p.$$

Hence

$$p^*p\leq f(r(a))=s(|a|).$$

On the other hand,

$$ap^*p=ar(a)^*pr(a)^*p=ar(a)^*p=a,$$

and therefore

$$s(|a|)\leq p^*p.$$

It follows from the polar decomposition theorem that p and $r(a)$ coincide. □

THEOREM 3.4. *Let A be a W^*-algebra and let a and b be elements of A with supports $r(a)$ and $r(b)$, respectively. Then, the weak* closed inner ideal $r(a)Ar(b)$ in A coincides with the weak* closure $\overline{aAb}^{w^*}$ of the inner ideal aAb in A.*

Proof. By (2.5),

$$aAb=r(a)a^*r(a)Ar(b)b^*r(b)\subseteq r(a)Ar(b),$$

and hence $\overline{aAb}^{w^*}$ is contained in $r(a)Ar(b)$. Conversely, by spectral theory there exists a sequence (q_{2n+1}) of odd real polynomials such that $(q_{2n+1}(|a|))$ converges in the weak* topology to the support projection $s(|a|)$ of $|a|$. Then, using Lemma 3.3, it follows that the sequence $(q_{2n+1}(a))$ converges in the weak* topology to $r(a)$. Since $q_{2n+1}(a)$ is contained in $aAa+\mathbf{R}a$ for each nonnegative integer n, it follows that, for nonnegative integers n and m,

$$q_{2n+1}(a)Aq_{2m+1}(b)\subseteq aAb$$

and $r(a)Ar(b)$ is contained in $\overline{aAb}^{w^*}$ as required. □

COROLLARY 3.5. *Let J be a weak* closed inner ideal in the W^*-algebra A. Then, for each element a in J, the weak* closed inner ideal $r(a)Ar(a)$ in A is contained in J.*

Proof. This is immediate from Theorem 3.4. □

LEMMA 3.6. *Let J be a weak* closed inner ideal in the W^*-algebra A and let z be a central projection in A. Then the weak* closed inner ideal $J \cap zA$ coincides with zJ, and J is the direct sum of the weak* closed inner ideals $J \cap zA$ and $J \cap z'A$.*

Proof. By Corollary 3.5, the support $r(a)$ of each element a in J is contained in J. Therefore,

$$zr(a) = \{r(a)\, zr(a)\, r(a)\} \subseteq J \quad \text{and} \quad za = \{zr(a)\, a\, zr(a)\} \subseteq J.$$

It follows that zJ is contained in $J \cap zA$; the remaining parts of the proof are straightforward. □

LEMMA 3.7. *Let A be a W^*-algebra. Then:*

(i) *for a pair (e, f) of projections in A, the weak* closed inner ideal eAf is zero if and only if $z(e) \perp z(f)$;*
(ii) *the central support $z(e)$ of a projection e in A is the supremum of the set S_e of projections f in A such that $f \precsim e$; and*
(iii) *if $e(a)$ and $f(a)$ are the left and right supports of an element a in A, then the pair $(e(a), f(a))$ is centrally equivalent.*

Proof. The proof of (i) is given in [15, 1.10.7]. Suppose that u is an element of the group $G(A)$ of unitary elements of A and that f lies in S_e. Then there exists an element v in $U(A)$ such that $v^*v = f$ and $vv^* \leq e$. Furthermore, vu lies in $U(A)$ and

$$(vu)^*(vu) = u^*fu, \quad (vu)(vu)^* = vv^* \leq e.$$

Hence u^*fu is contained in S_e.

Let p be the supremum of the elements of S_e. By [12, 2.6.3], the central support $z(f)$ of each element f in S_e is majorized by p. Therefore

$$p = \bigvee S_e \leq \bigvee \{z(f) : f \in S_e\} \leq p,$$

and p is a central projection. It follows that, for each element f in S_e, $z(f) \leq z(e)$ and therefore $p \leq z(e)$. However, because e lies in S_e, $z(e) \leq p$. This completes the proof of (ii). Notice that (iii) follows from (ii) using (2.4). □

The proof of the following lemma is straightforward.

LEMMA 3.8. *Let e, f, g, and h be projections in the W^*-algebra A with $e \leq g$ and $f \leq h$. Then the weak* closed inner ideal eAf is contained in gAh.*

LEMMA 3.9. *Let (e,f) be a pair of projections in the W^*-algebra A and let a be an element of A. Then the following are equivalent:*

(i) $a \in eAf$;
(ii) $r(a) \in eAf$;
(iii) $e(a) \leq e$ *and* $f(a) \leq f$.

Proof. If (i) holds then $r(a)$ lies in eAf by Corollary 3.5. If (ii) holds then $er(a) = r(a)$ and $r(a)f = r(a)$, which implies that

$$ee(a) = er(a)r(a)^* = r(a)r(a)^* = e(a),$$

$$f(a)f = r(a)^*r(a)f = r(a)^*r(a) = f(a),$$

and $e(a) \leq e$ and $f(a) \leq f$ as required. If (iii) holds then, by Lemma 3.8, $e(a)Af(a)$ is contained in eAf. Since a is contained in $e(a)Af(a)$, the proof is complete. □

For each pair (e,f) of projections in A define the pair $(e,f)^\sim$ by

$$(e,f)^\sim = \bigvee\{(e(a),f(a)) : a \in eAf\}, \tag{3.1}$$

the supremum being taken in the complete lattice $P(A) \times P(A)$.

THEOREM 3.10. *For each element (e,f) of the complete lattice $P(A) \times P(A)$ of pairs of projections in the W^*-algebra A, let $J(e,f)$ denote the weak* closed inner ideal eAf and let $(e,f)^\sim$ be the element of $P(A) \times P(A)$ defined by* (3.1). *Then, for elements (e,f) and (g,h) of $P(A) \times P(A)$:*

(i) $(e,f)^\sim \leq (e,f)$;
(ii) $J(e,f)^\sim = J(e,f)$;
(iii) *if* $J(e,f) \subseteq J(g,h)$ *then* $(e,f)^\sim \leq (g,h)^\sim$;
(iv) *if* $(e,f) \leq (g,h)$ *then* $(e,f)^\sim \leq (g,h)^\sim$;
(v) $(e,f)^\sim$ *is a centrally equivalent pair of projections*;
(vi) *if* (e,f) *is a centrally equivalent pair of projections then* $(e,f) = (e,f)^\sim$; *and*
(vii) $(e,f)^\sim = (e,f)^{\sim\sim}$.

Proof. (i)–(iv) The proofs are immediate consequences of (3.1) and Lemmas 3.8 and 3.9.

(v) Observe that, for each element a in A, $(e(a),f(a))$ is centrally equivalent by Lemma 3.7(iii). Since the suprema of any subset of $C(P(A))$ obtained in the complete lattices $C(P(A))$ and $P(A) \times P(A)$ (respectively) coincide, it follows that $(e,f)^\sim$ lies in $C(P(A))$.

(vi) Let e_1 and f_1 be elements of $P(A)$ such that (e_1,f_1) is equal to $(e,f)^\sim$. Since by (i) $(e_1,f_1) \leq (e,f)$ and by (ii) e_1Af_1 coincides with eAf, it follows that

$$e_1A(f-f_1) \oplus (e-e_1)Af_1 \oplus (e-e_1)A(f-f_1) = \{0\}.$$

Therefore, by Lemma 3.7(i),

$$z(e_1)\perp z(f-f_1), \quad z(e-e_1)\perp z(f_1), \quad \text{and} \quad z(e-e_1)\perp z(f-f_1).$$

Thus

$$z(e)=z(f)=z(f_1+(f-f_1))=z(f_1)\vee z(f-f_1)\perp z(e-e_1).$$

Therefore $e\perp e-e_1$, which shows that e and e_1 coincide. Similarly, so also do f and f_1.

(vii) This is now immediate from (v) and (vi). □

COROLLARY 3.11. *For each pair* (e,f) *of projections in the* W^**-algebra* A, *the pair* $(e,f)^{\sim}$ *defined by* (3.1) *is the greatest element of the complete lattice* $C(P(A))$ *of centrally equivalent pairs of projections in* A *such that* $(e,f)^{\sim}\leq(e,f)$.

Proof. This is immediate from Theorem 3.10. □

LEMMA 3.12. *Let* (e,f) *be a centrally equivalent pair of projections in the* W^**-algebra* A. *Then the supremum of a maximal orthogonal subset* M *of the set* $\{f(a): a\in eAf\}$ *coincides with* f.

Proof. Let g be the supremum of the set M. Then, by Lemma 3.9, $g\leq f$. For each element a in $eA(f-g)$, again using Lemma 3.9, $f(a)\leq f-g$ and $f(a)\perp g$. Since by Lemma 3.8 a also lies in eAf, by maximality, $f(a)$ and therefore a is zero. It follows from Lemma 3.7(i) that $z(f)=z(e)\perp z(f-g)$ and hence that f and g coincide. □

LEMMA 3.13. *Let* J *be a weak* closed inner ideal in the* W^**-algebra* A. *Then* J *contains a maximal weak* closed inner ideal of the form* eAf *for* e *and* f *projections in* A.

Proof. Let T be the nonempty set of weak* closed inner ideals in A contained in J and of the form eAf for some pair (e,f) of projections in A partially ordered by set inclusion. Let C be a chain in T. Then, for each element I in C, by Theorem 3.10, there exists a unique centrally equivalent pair (e_I,f_I) such that I coincides with e_IAf_I and such that $(e_I)_{I\in C}$ and $(f_I)_{I\in C}$ are increasing nets in $P(A)$. Let g and h (respectively) be their suprema. Then the union K of the elements of C is an inner ideal contained in J and also contained in gAh. Let a be an element of gAh and let I_1 and I_2 be elements of C such that $I_1\subseteq I_2$. Then

$$e_{I_2}e_{I_1}af_{I_2}f_{I_2}=e_{I_1}af_{I_2}$$

and $e_{I_1}af_{I_2}$ lies in K. Because multiplication by elements of A is separately weak* continuous, and since the increasing nets $(e_I)_{I\in C}$ and $(f_I)_{I\in C}$ converge in the weak* topology to g and h (respectively), it follows that gAh is contained in the weak* closure $\bar{K}^{w^*}$ of K. Since K is contained in gAh it follows that gAh and $\bar{K}^{w^*}$ coincide. Clearly gAh is the least upper bound in T of the chain C, and the result follows. □

The proof of the next lemma is straightforward.

LEMMA 3.14. *Let e and f be projections in the W^*-algebra A and let J be an inner ideal in A. Then J is contained in the weak* closed inner ideal eAf if and only if $e'J$ and Jf' are zero.*

The final lemma is a technical result needed in the proof of the main theorem.

LEMMA 3.15. *Let J be a weak* closed inner ideal in the W^*-algebra A and let (e,f) be a centrally equivalent pair of projections in A such that the weak* closed inner ideal eAf is contained in J. Then $Jf'Af$ is a subset of J and, for all elements c in J such that cf' lies in J, $eAcf'$ is a subset of J.*

Proof. By Lemma 3.12 there exists a nonempty subset M of eAf such that, for distinct elements a_1 and a_2 of M, $f(a_1)$ is orthogonal to $f(a_2)$ and the supremum of the set $\{f(a): a \in M\}$ is f. For each finite subset L of M, let

$$f_L = \sum_{a \in L} f(a).$$

Then, if M^f denotes the set of nonempty finite subsets of M, $(f_L)_{L \in M^f}$ is an increasing net in $P(A)$ converging to f in the weak* topology. By Lemma 3.1 and Lemma 3.9, for each element b in A and each element a in M,

$$bf(a) \in Af(a) = Ar(a) = Aer(a)f \subseteq AeAf.$$

Therefore, for each element c in J,

$$cf'bf(a) \in cf'AeAf \subseteq \{cAf'eAf\},$$

since f is orthogonal to f'. Since J is an inner ideal, it follows that $cf'bf(a)$ is contained in J and, by linearity, the same is true of $cf'bf_L$ for all elements L in M^f. Since J is weak* closed it follows that $cf'bf$ lies in J as required.

Repeating the argument of Lemma 3.12 shows that there is a nonempty subset N of eAf such that, for distinct elements a_1 and a_2 of N, $e(a_1)$ is orthogonal to $e(a_2)$ and the supremum of the set $\{e(a): a \in N\}$ is e. Moreover, denoting by e_L the sum of the elements $e(a)$ for a a member of a finite subset L of N, it follows that $(e_L)_{L \in N^f}$ is an increasing net converging to e in the weak* topology. By Lemma 3.1 and Lemma 3.9, for each element b in A and a in N, $e(a)b$ lies in $eAfA$. If c is an element of J such that cf' lies in J, then

$$e(a)bcf' \in eAfAf' \subseteq \{eAfAfcf'\}.$$

Because J is an inner ideal, $e(a)bcf'$ is contained in J and, as above, it follows that $ebcf'$ lies in J, as required. □

It is now possible to prove the main result of the paper.

THEOREM 3.16. *Let A be a W^*-algebra and let J be a weak* closed inner ideal in A. Then there exist projections e and f in A such that J coincides with eAf.*

Proof. Let T be as in Lemma 3.13 and let eAf be a maximal element of T. By Theorem 3.10, (e,f) may be chosen to be an element of $C(P(A))$. Let z be the common central support of e and f and let a be an element of $z'J$. Since $e(a)$ and $f(a)$ are orthogonal to z it follows that $z(e(a))$ and $z(f(a))$ are orthogonal to z. Furthermore, by Lemmas 3.1 and 3.6 and by Corollary 3.5, $e(a)Af(a)$ is a subset of J. Therefore, by Lemma 3.7(i),

$$eAf \subseteq (e+e(a))A(f+f(a)) = eAf \oplus e(a)Af(a)$$

and a is therefore zero. Hence $z'J$ is zero and, since $eAf = zeAf \subseteq zJ \subseteq zA$, Lemma 3.6 shows that $eAf \subseteq J \subseteq zA$.

Without loss of generality, assume that z is the unit 1 in A, let c be an element of J, and let b denote the element cf'. By Lemma 3.3 and Theorem 3.4 there exists a net (b_j) in A such that the net (bb_jb) converges to b in the weak* topology. Then, by Lemma 3.15,

$$bb_jb \in cf'Acf' \subseteq cf'Ac - cf'Acf \subseteq cAc + cf'Af \subseteq J$$

for all j. It follows that b lies in J. The second part of Lemma 3.15 shows that eAb is a subset of J. By Theorem 3.4, $eAf(b)$ is the weak* closure of eAb and hence $eAf(b)$ is contained in J. Since $f(b)$ is orthogonal to f it follows that

$$eAf \subseteq eA(f+f(b)) = eAf \oplus eAf(b) \subseteq J.$$

By maximality, $eAf(b)$ is zero and, by Lemma 3.7(i), $z(f(b))$ and therefore b itself is also zero. Hence Jf' is zero. Notice that J^* is a weak* closed inner ideal and, since $(e'J)^*$ coincides with J^*e', a similar argument shows that $e'J$ is also zero. The assertion now follows from Lemma 3.13. □

4. The Complete Lattice of Weak* Closed Inner Ideals

Recall that the set of weak* closed inner ideals in the W^*-algebra A, when ordered by set inclusion, forms a complete lattice $I(A)$. The following result is an immediate consequence of Theorem 3.10 and Theorem 3.16.

THEOREM 4.1. *Let A be a W^*-algebra and, for each element (e,f) of the complete lattice $C(P(A))$ of centrally equivalent pairs of projections in A, let $J(e,f)$ denote the weak* closed inner ideal eAf. Then the mapping J is an order isomorphism from $C(P(A))$ onto the complete lattice $I(A)$ of weak* closed inner ideals in A.*

THEOREM 4.2. *Let (e,f) be a centrally equivalent pair of projections in A and let eAf be the corresponding weak* closed inner ideal in A. Then eAf*

is a left ideal in A if and only if e is central, and eAf is a two-sided ideal if and only if both e and f are central.

Proof. Let e be central and let a be an element of A. Then $aeAf = eaAf \subseteq eAf$ and eAf is a left ideal. Conversely, if eAf is a left ideal then there exists a projection g in A such that eAf coincides with Ag. By Theorem 3.10, $f \leq g$. Then, by Lemmas 3.7(ii), 3.9, and Theorem 3.10,

$$\begin{aligned} z(e) = z(f) \leq z(g) &= \vee \{e(u) : u \in U(A), f(u) \leq g\} \\ &\leq \vee \{e(a) : a \in A, f(a) \leq g\} \\ &= \vee \{e(a) : a \in Ag\} = e, \end{aligned}$$

and e is central. The remaining assertions are similarly proved. □

COROLLARY 4.3. *Let A be a W^*-algebra and let (e, f) be a centrally equivalent pair of projections in A. Then the weak* closed inner ideal eAf is the intersection of the weak* closed left ideal $z(f)Af$ and the weak* closed right ideal $eAz(e)$.*

Proof. This follows from (2.3), Theorem 4.1, and Theorem 4.2. □

COROLLARY 4.4. *The following conditions on a W^*-algebra A are equivalent:*

(i) *every weak* closed inner ideal in A is a left ideal;*
(ii) *every weak* closed inner ideal in A is a two-sided ideal;*
(iii) *A is commutative.*

Proof. The equivalence of (i) and (ii) and the implication that (iii) implies (ii) are clear. If (ii) holds and e is a projection in A, then eAe is a weak* closed inner ideal in A and (e, e) is centrally equivalent. Therefore, by Theorem 4.2, e is central. It follows that A coincides with its centre and is therefore commutative. □

THEOREM 4.5. *Let A be a σ-finite Type III W^*-algebra and let J be a weak* closed inner ideal in A. Then there exists a partial isometry u in A such that J coincides with uAu.*

Proof. This follows immediately from Theorem 4.1, Lemma 3.1, and [15, 2.2.14]. □

Let A be a W^*-algebra with predual A_* regarded as being canonically embedded in the dual A^* of A. For each element c in A there exists a mapping W_c from $A \times A$ to A defined, for elements a and b of A, by $W_c(a, b) = \{a\, c\, b\}$.

For each element x in A_*, the sesquilinear functional $x \circ W_c$ on A is jointly strongly continuous on bounded subsets $A \times A$. Let $B(A)$ be the complex vector space of sesquilinear functionals on A which are strongly continuous on bounded subsets of $A \times A$. A linear subspace L of A_* is said to be *A-bi-invariant* [1] if, for all elements x in L and all elements c in A, $x \circ W_c$ lies in the annihilator $(L^0 \times L^0)_0$ of the product $L^0 \times L^0$ of the annihilator of L in A with itself. Clearly, the zero set and A_* are A-bi-invariant, and the set $A(A)$ of norm-closed A-bi-invariant subspaces of A_* (when ordered by set inclusion) forms a complete lattice, the infimum of a family of elements of $A(A)$ coinciding with its intersection.

Let x be an element of A_* and let a and b be elements of A. The linear functional axb on A is defined, for each element c in A, by $axb(c) = x(acb)$. Clearly axb is an element of A_* and, for elements a, b, c, and d of A,

$$c(axb)d = (ac)x(db).$$

THEOREM 4.6. *Let A be a W^*-algebra with predual A_*. For each pair (e, f) of centrally equivalent projections in A, the subset $L(e, f)$ of A defined by*

$$L(e, f) = e'Af \oplus eAf' \oplus e'Af'$$

is a norm closed A-bi-invariant subspace of A_*. Moreover, the mapping L is an anti-order isomorphism from the complete lattice $C(P(A))$ of centrally equivalent pairs of projections in A onto the complete lattice $A(A)$ of norm closed A-bi-invariant subspaces of A_*.

Proof. It follows immediately from the properties of annihilators that the mapping $J \to J_0$ is an anti-order isomorphism from the complete lattice $I(A)$ onto the complete lattice $A(A)$. Because

$$A = eAf \oplus e'Af \oplus eAf' \oplus e'Af'$$

for each centrally equivalent pair (e, f) of projections in A, it is clear that $(eAf)_0$ coincides with $L(e, f)$. □

References

1. C. M. Edwards, *Ideal theory in JB-algebras,* J. London Math. Soc. 16 (1977), 507–513.
2. C. M. Edwards and G. T. Rüttimann, *On the facial structure of the unit balls in a JBW*-triple and its predual,* J. London Math. Soc. 38 (1988), 317–332.
3. E. G. Effros, *Order ideals in a C*-algebra and its dual,* Duke Math. J. 30 (1963), 391–412.
4. Y. Friedman and B. Russo, *Structure of the predual of a JBW*-triple,* J. Reine Angew. Math. 356 (1985), 67–89.
5. H. Hanche-Olsen and E. Størmer, *Jordan operator algebras,* Pitman, London, 1984.
6. L. A. Harris, *A generalization of C*-algebras,* Proc. London Math. Soc. 42 (1981), 331–361.
7. P. Jordan, J. von Neumann, and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism,* Ann. of Math. (2) 35 (1934), 29–64.

8. R. V. Kadison, *Isometries of operator algebras,* Ann. of Math. (2) 54 (1951), 325–338.
9. O. Loos, *Jordan pairs,* Lecture Notes in Math., 460, Springer, Berlin, 1975.
10. K. McCrimmon, *Inner ideals in quadratic Jordan algebras,* Trans. Amer. Math. Soc. 159 (1971), 445–468.
11. E. Neher, *Jordan triple systems by the grid approach,* Lecture Notes in Math., 1280, Springer, Berlin, 1987.
12. G. K. Pedersen, *C*-algebras and their automorphism groups,* Academic Press, London, 1979.
13. R. T. Prosser, *On the ideal structure of operator algebras,* Mem. Amer. Math. Soc. 45 (1963).
14. G. T. Rüttimann, *Non-commutative measure theory,* Habilitationsschrift, Universität Bern, 1980.
15. S. Sakai, *C*-algebras and W*-algebras,* Springer, Berlin, 1971.
16. H. Upmeier, *Symmetric Banach manifolds and Jordan C*-algebras,* North-Holland, Amsterdam, 1985.

C. M. Edwards
The Queen's College
Oxford
England

G. T. Rüttimann
The Queen's College
Oxford
England

Permanent address:
Universität Bern
Bern
Switzerland