A STABLE SPLITTING FOR THE MAPPING CLASS GROUP

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Introduction. Let $\Gamma_{g,r}^{s}$ denote the mapping class group of a surface of genus g with r boundary components and s punctures. Because of their close connection with moduli spaces of algebraic curves, the homological and homotopy-theoretic properties of these groups are particularly interesting. Many of these properties are independent of the genus g, providing g is sufficiently large. It is therefore convenient to define a limit group $\Gamma = \varinjlim \Gamma_{g,1}^{0}$ (see §1 for details). By a theorem of Harer [7], the homology of Γ is the same as that of $\Gamma_{g,r}^{0}$ for any r, in degrees $i \ll g$. Moreover, Γ is a perfect group; hence we can form a simply connected space $B\Gamma^{+}$ with the same homology as Γ. The space $B\Gamma^{+}$ has a natural H-space structure and there is a natural H-map $B\Gamma^{+} \to B\mathrm{GL}(\mathbf{Z})^{+}$. Using this map, one can derive homotopy properties of $B\Gamma^{+}$ from those of $B\mathrm{GL}(\mathbf{Z})^{+}$.

In particular, Quillen ([12], [13]) showed that the maps $B\mathrm{GL}(\mathbf{Z})^{+} \to B\mathrm{GL}(\mathbf{F}_q)^{+}$ induced by reduction mod q split when localized at appropriate primes p; or, in other words, that $B\mathrm{GL}(\mathbf{Z})^{+}$ splits as a product of spaces

$$B\mathrm{GL}(\mathbf{Z})^{+} \simeq \mathrm{Im}\, J_{(1/2)} \times ?,$$

where

$$\mathrm{Im}\, J_{(1/2)} = \prod_{p \text{ odd}} B\mathrm{GL}(\mathbf{F}_q)^{+}_{(p)}$$

(cf. §2). In [3], Charney and Lee prove that the composite map

$$B\tau : B\Gamma^{+} \to B\mathrm{GL}(\mathbf{Z})^{+} \to \mathrm{Im}\, J_{(1/2)}$$

induces a split surjection on homology. It is natural to ask, therefore, whether this composite also splits on the space level. This question remains unanswered, but in this paper we prove that there is a *stable* splitting of $B\tau$ (Theorem 3.1) and hence a splitting of spaces (Corollary 3.2),

$$\Omega^{\infty}\Sigma^{\infty}B\Gamma \simeq \Omega^{\infty}\Sigma^{\infty}\,\mathrm{Im}\, J_{(1/2)} \times ?.$$

Analogous splitting theorems for $B\Gamma_{g,r}^{s}$ have been proved by the second author in [4] in some special cases. For example, a stable splitting of $B\Gamma_{0,0}^{4}$ is given in terms of the symmetric group on four letters and the Steinberg idempotent for $\mathrm{GL}_2(\mathbf{F}_2)$. In another example, it is shown that there exists a homomorphism θ: $\mathbf{Z}/5 \to \Gamma_{2,0}^{0}$ which induces an isomorphism on the 5-primary component of homology, and in [1] Benson uses these results to give a complete calculation of $H_*(\Gamma_{2,0}^{0})$.

The first two sections of this paper contain definitions and some well-known facts about mapping class groups and general linear groups. Section 3 contains

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the statement and proof of the main theorem, and Sections 4 and 5 contain some observations on $H_*(B\Gamma^+;\mathbf{Z}/2)$ and $\pi_*(B\Gamma^+;\mathbf{Z}/p)$.

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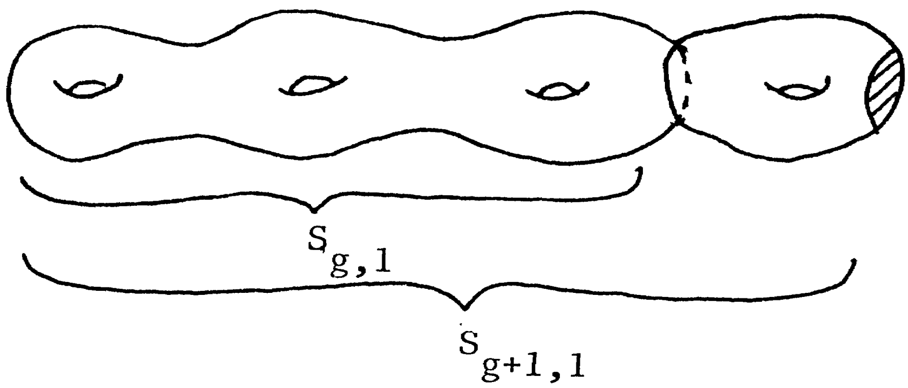
1. Mapping class groups. In this section we establish some notation and review some known facts about mapping class groups. Let $S_{g,r}$ denote an oriented topological surface of genus g with r boundary components. ($S_{g,r}$ is obtained from a g-holed torus by removing r open disks.) Let $\Gamma_{g,r}$ denote the mapping class group of $S_{g,r}$. That is, $\Gamma_{g,r}$ is the group of isotopy classes of orientation-preserving homeomorphisms of $S_{g,r}$ which pointwise fix $\partial S_{g,r}$. (As punctures will play no role in what follows, we write $\Gamma_{g,r}$ instead of $\Gamma^0_{g,r}$.)

For $r=1$, we can identify $S_{g,1}$ with a subspace of $S_{g+1,1}$ as shown below.



Extending a homeomorphism of $S_{g,1}$ via the identity to a homeomorphism of $S_{g+1,1}$ gives rise to an inclusion $\Gamma_{g,1}\hookrightarrow\Gamma_{g+1,1}$. Using these inclusions we define the *stable mapping class group*:

$$\Gamma=\varinjlim_{g}\Gamma_{g,1}.$$

Similarly, identifying $S_{g,r}$ with $(S_{g,0}-\coprod_r D^2)\subset S_{g,0}$ gives rise to a natural map $\Gamma_{g,r}\to\Gamma_{g,0}$.

The following facts about mapping class groups will be used repeatedly in later sections.

1.1 For $g\geq 3$, $H_1(\Gamma_{g,r})=0$ ([6], [11]) and so the groups $\Gamma_{g,r}$ and Γ are perfect. We can therefore apply Quillen's plus-construction to the classifying spaces $B\Gamma_{g,r}$ and $B\Gamma$ to get simply connected spaces $B\Gamma^+_{g,r}$ and $B\Gamma^+$ whose homology is that of $\Gamma_{g,r}$ and Γ, respectively.

1.2 By a theorem of Harer [7], the natural maps $\Gamma_{g,0}\leftarrow\Gamma_{g,1}\rightarrow\Gamma$ induce isomorphisms on H_i for $g\geq 3i+1$. It follows that the induced maps $B\Gamma^+_{g,0}\leftarrow B\Gamma^+_{g,1}\rightarrow B\Gamma^+$ are i-connected for $g\geq 3i+2$. Moreover, Harer [8] shows that $B\Gamma_{g,1}$ has the homotopy type of a finite complex. It follows that $B\Gamma^+$ is of finite type.

1.3 Let A be a commutative ring with unit. Then for $\epsilon = 0, 1$, $H_1(S_{g,\epsilon}; A)$ is a free A-module of rank $2g$ equipped with a natural symplectic pairing (the intersection pairing). An element $\gamma \in \Gamma_{g,\epsilon}$ induces a well-defined isomorphism γ_* of $H_1(S_{g,\epsilon}; A)$ which preserves this pairing. Choosing a symplectic basis for $H_1(S_{g,\epsilon}; A)$ over A, we thus obtain a homomorphism

$$\Gamma_{g,\epsilon} \to \mathrm{Sp}_{2g}(A) \subset \mathrm{GL}_{2g}(A).$$

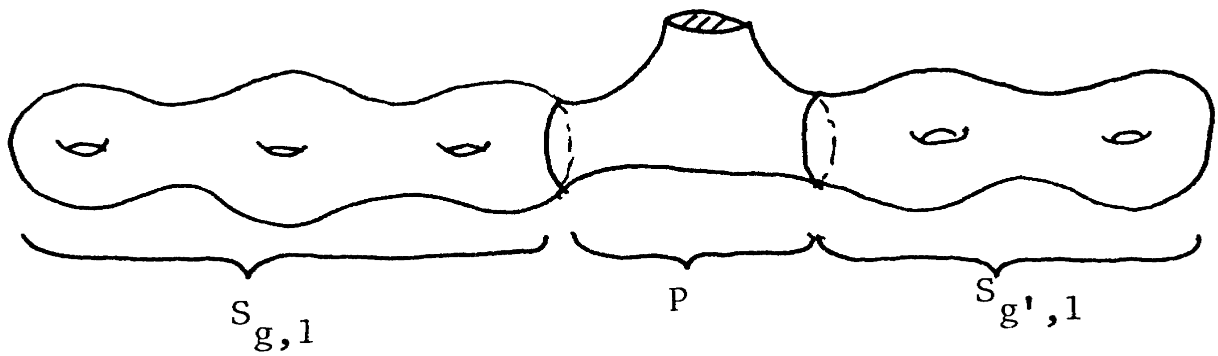
For $\epsilon = 1$, the bases can clearly be chosen so that these maps commute with the natural inclusions $\Gamma_{g,1} \hookrightarrow \Gamma_{g+1,1}$ and $\mathrm{Sp}_{2g}(A) \hookrightarrow \mathrm{Sp}_{2(g+1)}(A)$. Hence they give rise to a homomorphism $\tau_A^{\mathrm{Sp}}: \Gamma \to \mathrm{Sp}(A)$, or forgetting the symplectic pairing,

$$\tau_A: \Gamma \to \mathrm{GL}(A).$$

1.4 As observed by Miller [10], one can define a sum operation

$$\Gamma_{g,1} \times \Gamma_{g',1} \to \Gamma_{g+g',1}$$

by viewing $S_{g+g',1}$ as the union $S_{g+g',1} = S_{g,1} \cup P \cup S_{g',1}$, where P is a sphere with three disks removed (a "pair of pants"):



A pair of homeomorphisms h, h' representing (respectively) elements of $\Gamma_{g,1}$ and $\Gamma_{g',1}$ extends via the identity on P to a homeomorphism of $S_{g+g',1}$. This defines the sum operation and induces a monoid structure on $\coprod_{g \geq 0} B\Gamma_{g,1}$. The space $B\Gamma^+$ may be identified with $\Omega_0 B(\coprod B\Gamma_{g,1})$, the zero component of the group completion of this monoid. In particular, $B\Gamma^+$ is a loop space. (Miller [10] has in fact shown that $B\Gamma^+$ is a double loop space.) The sum operation defined above is clearly compatible with the block matrix sum operations

$$\mathrm{Sp}_{2g}(A) \times \mathrm{Sp}_{2g'}(A) \to \mathrm{Sp}_{2(g+g')}(A),$$

$$\mathrm{GL}_{2g}(A) \times \mathrm{GL}_{2g'}(A) \to \mathrm{GL}_{2(g+g')}(A).$$

Identifying $B\mathrm{Sp}(A)^+$ and $B\mathrm{GL}(A)^+$ with the group completions

$$\Omega_0 B(\coprod B\mathrm{Sp}_{2g}(A)) \quad \text{and} \quad \Omega_0 B(\coprod B\mathrm{GL}_{2g}(A)),$$

it follows that the maps $B\Gamma^+ \to B\mathrm{Sp}(A)^+$ and $B\Gamma^+ \to B\mathrm{GL}(A)^+$ induced by τ_A^{Sp} and τ_A are loop maps.

2. The space Im $J_{(1/2)}$. From now on we make the following assumption on p and q.

ASSUMPTION 2.1. p is an odd prime and q is a prime such that $q \bmod p^2$ generates the multiplicative group of units in $\mathbf{Z}/p^2$.

It follows from this assumption that $q^i - 1 \not\equiv 0 \bmod p$ for $0 < i < p-1$, and that p divides $q^{p-1}-1$ exactly once. We remark also that, for any p, there exist infinitely many q satisfying the required properties.

Let $\mathbf{F}_q$ denote the field with q elements. Then for fixed p, the homotopy type of the space $B\mathrm{GL}(\mathbf{F}_q)^+_{(p)}$ (where $X_{(p)}$ denotes the p-localization of the space X) is independent of the choice of q [12]. We may therefore define (up to homotopy) spaces

$$\operatorname{Im} J_{(p)} = B\mathrm{GL}(\mathbf{F}_q)^+_{(p)},$$
$$\operatorname{Im} J_{(1/2)} = \prod_{p \text{ odd}} \operatorname{Im} J_{(p)}.$$

Here, $\prod$ denotes the weak infinite product. In §1.3 we defined homomorphisms $\tau_q = \tau_{\mathbf{F}_q} \colon \Gamma \to \mathrm{GL}(\mathbf{F}_q)$. These induce maps

$$B\Gamma^+ \xrightarrow{B\tau_q} B\mathrm{GL}(\mathbf{F}_q)^+ \xrightarrow{\ell_{(p)}} \operatorname{Im} J_{(p)}$$

where $\ell_{(p)}$ is the localization map. Together these define a map

$$B\tau \colon B\Gamma^+ \to \operatorname{Im} J_{(1/2)}.$$

The proof of our main theorem will involve the p-Sylow subgroups of the finite groups $\mathrm{GL}_n(\mathbf{F}_q)$. These are well known. A lucid reference is Chapter VIII of Fiedorowicz–Priddy [5].

For any group G, the wreath product $\mathbf{Z}/p \int G$ is defined by

$$\mathbf{Z}/p \textstyle\int G = (\underbrace{G \times \cdots \times G}_{p \text{ copies}}) \times \mathbf{Z}/p,$$

with $\mathbf{Z}/p$ acting on $G \times \cdots \times G$ by permuting the factors. We write

$$\mathbf{Z}/p \textstyle\int^k G = \underbrace{\mathbf{Z}/p \int (\mathbf{Z}/p \int (\cdots (\mathbf{Z}/p}_{k \text{ times}} \int G)) \cdots)$$

and define groups

$$\pi_0 = \mathbf{Z}/p,$$
$$\pi_k = \mathbf{Z}/p \textstyle\int \pi_{k-1} = \mathbf{Z}/p \int^k \mathbf{Z}/p.$$

We also define inclusions

$$\alpha_k \colon \pi_k \hookrightarrow \mathrm{GL}_{p^k(p-1)}(\mathbf{F}_q),$$

$$\alpha_0 \colon \pi_0 \to \mathrm{GL}_{(p-1)}(\mathbf{F}_q), \quad 1 \mapsto \begin{pmatrix} 0 & & & & -1 \\ 1 & 0 & & & \vdots \\ & 1 & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & -1 \end{pmatrix},$$

$$\alpha_k\colon \pi_k = \mathbf{Z}/p \wr \pi_{k-1} \xrightarrow{\mathbf{Z}/p \wr \alpha_{k-1}} \mathbf{Z}/p \wr \mathrm{GL}_{p^{k-1}(p-1)} \subset \mathrm{GL}_{p^k(p-1)},$$

where $\mathbf{Z}/p \wr \mathrm{GL}_n(\mathbf{F}_q)$ is viewed as the subgroup of $\mathrm{GL}_{pn}(\mathbf{F}_q)$ generated by the block matrices

$$\begin{bmatrix} \boxed{\mathrm{GL}_n} & & & 0 \\ & \boxed{\mathrm{GL}_n} & & \\ & & \ddots & \\ 0 & & & \boxed{\mathrm{GL}_n} \end{bmatrix}$$

and the order-p permutation matrix

$$\begin{bmatrix} I_n & 0 & & I_n \\ & I_n & \ddots & \\ & & \ddots & 0 \\ & & I_n & 0 \end{bmatrix}.$$

Using the fact that

$$|\mathrm{GL}_n(\mathbf{F}_q)| = \prod_{i=1}^{n} (q^i - 1) q^{i-1},$$

one can calculate the p-adic valuation of $|\mathrm{GL}_n(\mathbf{F}_q)|$. From this one easily verifies that α_k maps π_k isomorphically onto a p-Sylow subgroup of $\mathrm{GL}_{p^k(p-1)}(\mathrm{F}_q)$.

3. The main theorem. We adopt the following standard notation and terminology. Let X and Y be based CW-complexes and assume that all maps are basepoint preserving. The group of homotopy classes of stable maps $\Sigma^\infty X \to \Sigma^\infty Y$ is denoted by $\{X, Y\}$. The localization of X at p is denoted $X_{(p)}$.

An element $f \in \{X, Y\}$ is an *equivalence* if the following equivalent conditions hold:

(i) $\exists g \in \{Y, X\}$ such that $f \circ g = \text{identity}$ in $\{Y, Y\}$ and $g \circ f = \text{identity}$ in $\{X, X\}$;

(ii) f induces isomorphisms $\pi_*^s(X) \to \pi_*^s(Y)$ (where $\pi_*^s(\cdot) = \{S^n, \cdot\}$ denotes stable homotopy);

(iii) f induces isomorphisms $H_*(X) \to H_*(Y)$.

An element $f \in \{X, Y\}$ is *m-connected* if the following equivalent conditions hold:

(i) the maps $\pi_i^s(X) \to \pi_i^s(Y)$ induced by f are isomorphisms (resp. surjective) for $i < m$ (resp. $i \leq m$);

(ii) the maps $H_i(X) \to H_i(Y)$ induced by f are isomorphisms (resp. surjective) for $i < m$ (resp. $i \leq m$).

We say that $f \in \{X, Y\}$ is a *p-local equivalence* if the induced map

$$f_{(p)} \in \{X_{(p)}, Y_{(p)}\}$$

is an equivalence, and that f is *p-locally m-connected* if $f_{(p)}$ is m-connected.

We now state and prove our main theorem. The conditions of §2.1 on p and q are assumed throughout.

THEOREM 3.1. *The map $B\tau: B\Gamma^+ \to \operatorname{Im} J_{(1/2)}$ has a stable section. That is, there is a stable map $\theta \in \{\operatorname{Im} J_{(1/2)}, B\Gamma^+\}$ such that $\Sigma^\infty B\tau \circ \theta$ is an equivalence.*

REMARK. The use of the plus-construction is actually superfluous here since, for any connected space X which has a plus-construction, the map $X \to X^+$ becomes a homotopy equivalence as soon as it is suspended once. However, it will be more convenient to work with X^+ since, for example, $B\Gamma^+$ and $B\mathrm{GL}(\mathbf{F}_q)^+$ are H-spaces and $B\Gamma_{g,1}^+$ $(g \geq 3)$ is simply connected.

The following corollaries are immediate consequences of Theorem 3.1. Let QX denote $\varinjlim_k \Omega^k \Sigma^k X$.

COROLLARY 3.2. *There exist spaces Y and Z such that*

$$QB\Gamma \simeq Q \operatorname{Im} J_{(1/2)} \times Y \simeq \operatorname{Im} J_{(1/2)} \times \mathbf{Z}.$$

COROLLARY 3.3. *The map $B\tau: B\Gamma^+ \to \operatorname{Im} J$ induces split surjections on any homology theory.*

Proof of Theorem 3.1. For any finite groups $H \subset G$, there exists a stable transfer map $\mathrm{tr} \in \{BG, BH\}$ such that the induced map on homology is the ordinary transfer (see [9]). In particular, let $\pi_k = \mathbf{Z}/p \int^k \mathbf{Z}/p$ and $\alpha_k: \pi_k \to \mathrm{GL}_n(\mathbf{F}_q)$, $n = p^k(p-1)$, be as in §2, so α_k is an isomorphism of π_k onto a p-Sylow subgroup of $\mathrm{GL}_n(\mathbf{F}_q)$. Then the composite

$$\Sigma^\infty B\mathrm{GL}_n(\mathbf{F}_q) \xrightarrow{\mathrm{tr}} \Sigma^\infty B\pi_k \xrightarrow{\Sigma^\infty B\alpha_k} \Sigma^\infty B\mathrm{GL}_n(\mathbf{F}_q)$$

induces multiplication by $r = |\mathrm{GL}_n(\mathbf{F}_q)|/|\pi_k|$ on homology. Since r is prime to p, this composite is a p-local equivalence.

We would like to show that the above composite, followed by the natural map $\Sigma^\infty B\mathrm{GL}_n(\mathbf{F}_q) \to \Sigma^\infty B\mathrm{GL}(\mathbf{F}_q)^+$ factors through $\Sigma^\infty B\Gamma^+$. To do this, we must restrict to finite skeleta. For a CW-complex X, let $X^{(m)}$ denote the m-skeleton of X.

LEMMA 3.4. *For any m and any $s \gg m$, there exist maps $(B\pi_k)^{(m)} \xrightarrow{\beta_k^s} B\Gamma^+$ such that the composite*

$$(B\pi_k)^{(m)} \xrightarrow{\beta_k^s} B\Gamma^+ \xrightarrow{B\tau_q} B\mathrm{GL}(\mathbf{F}_q)^+$$

is homotopic to the composite

$$(B\pi_k)^{(m)} \subset B\pi_k \xrightarrow{B\alpha_k} B\mathrm{GL}_n(\mathbf{F}_q) \xrightarrow{i_n} B\mathrm{GL}(\mathbf{F}_q)^+ \xrightarrow{\times s} B\mathrm{GL}(\mathbf{F}_q)^+,$$

where the last map, $\times s$, is multiplication by s in the H-space structure induced by $\oplus$.

Let us assume the lemma for the moment and complete the proof of the theorem. By the lemma, for any m and any $s \gg m$, we have a commutative diagram of stable maps

$$\begin{tikzcd}
\Sigma^\infty B\mathrm{GL}_n(\mathbf{F}_q) \arrow[r, "\mathrm{tr}"] & \Sigma^\infty B\pi_k \arrow[r, "\Sigma^\infty B\alpha_k"] & \Sigma^\infty B\mathrm{GL}_n(\mathbf{F}_q) \arrow[r, "\Sigma^\infty i_n"] & \Sigma^\infty B\mathrm{GL}(\mathbf{F}_q)^+ \arrow[d, "\Sigma^\infty(\times s)"] \\
\Sigma^\infty B\mathrm{GL}_n(\mathbf{F}_q)^{(m)} \arrow[u, phantom, "\cup"] \arrow[r, "\mathrm{tr}"] & \Sigma^\infty B\pi_k^{(m)} \arrow[u, phantom, "\cup"] \arrow[r, "\Sigma^\infty \beta_k^s"] & \Sigma^\infty B\Gamma^+ \arrow[r, "\Sigma^\infty B\tau_q"] & \Sigma^\infty B\mathrm{GL}(\mathbf{F}_q)^+.
\end{tikzcd}$$

If we assume, in addition to $s \gg m$, that s is prime to p and $n \gg m$, then $\Sigma^\infty(\times s)$ is a p-local equivalence and $\Sigma^\infty i_n$ is m-connected. It follows that the bottom horizontal composite, $\Sigma^\infty B\tau_q \circ \Sigma^\infty \beta_k^s \circ \mathrm{tr}$, is p-locally m-connected.

Now the homotopy groups of $B\mathrm{GL}(\mathbf{F}_q)^+$ are finite, and hence $B\mathrm{GL}(\mathbf{F}_q)^+$ can be obtained as a homotopy limit of a directed system $(X_t, i_t)_{t \in \mathbf{N}}$ of *finite* complexes X_t with finite homotopy groups:

$$B\mathrm{GL}(\mathbf{F}_q)^+ = \underrightarrow{\mathrm{holim}}\, X_t.$$

We may choose these such that the maps $i_t: X_t \to X_{t+1}$ are t-connected.

Consider the groups $G_t = \{X_t, B\Gamma^+\}$. We claim that G_t is finite for every t. To see this, note that X_t can be constructed from Moore spaces, $M_k^n = S^{n-1} \cup_k e^n$, via a finite sequence of cofibrations. That is, there exists a sequence of spaces $Y_0, Y_1, \ldots, Y_r = X_t$ such that Y_0 is a Moore space and Y_{i+1} is the cofiber of some map $M_k^n \to Y_i$. This gives exact sequences of groups

$$\cdots \leftarrow \{M_k^n, B\Gamma^+\} \leftarrow \{Y_i, B\Gamma^+\} \leftarrow \{Y_{i+1}, B\Gamma^+\} \leftarrow \{M_k^{n+1}, B\Gamma^+\} \leftarrow \cdots.$$

Now $B\Gamma^+$ is of finite type, hence likewise $QB\Gamma^+$ $(= \Omega^\infty \Sigma^\infty B\Gamma^+)$. Thus

$$\{M_k^n, B\Gamma^+\} = \pi_n(QB\Gamma^+; \mathbf{Z}/k)$$

is finite for all n and k. By induction on i, it follows that $\{Y_i, B\Gamma^+\}$ is finite for each i. In particular, $G_t = \{Y_r, B\Gamma^+\}$ is finite.

Next define subsets $S_t \subset G_t$ such that

$$S_t = \{\gamma \in G_t \mid (\Sigma^\infty B\tau) \circ \gamma \text{ is } p\text{-locally } t\text{-connected}\}.$$

The maps $i_t: X_t \to X_{t+1}$ induce homomorphisms $\tilde{i}_t: G_{t+1} \to G_t$ which clearly take S_{t+1} into S_t. We claim that S_t is nonempty for every t. To see this, note that since X_t is a finite complex, the natural map

$$X_t \to \underrightarrow{\mathrm{holim}}\, X_t = B\mathrm{GL}(\mathbf{F}_q)^+$$

factors through $B\mathrm{GL}_n(\mathbf{F}_q)^{+(m)}$ for any sufficiently large m and n. Since $X_t \to B\mathrm{GL}(\mathbf{F}_q)^+$ is t-connected, so is the factorization $X_t \to B\mathrm{GL}_n(\mathbf{F}_q)^{+(m)}$. Letting β_k^s be the map guaranteed by Lemma 3.4, it follows that the composite

$$\Sigma^\infty X_t \longrightarrow \Sigma^\infty B\mathrm{GL}_n(\mathbf{F}_q)^{+(m)} \xrightarrow{\mathrm{tr}} \Sigma^\infty B\pi_k^{(m)} \xrightarrow{\Sigma^\infty \beta_k^s} \Sigma^\infty B\Gamma^+$$

lies in S_t. Thus $(S_t, \tilde{i}_t)$ is an inverse system of nonempty, finite sets. Any such system has a nonempty inverse limit, so $\underleftarrow{\lim}\, S_t \neq \varnothing$. Now the natural map

$$\{B\mathrm{GL}(\mathbf{F}_q)^+, B\Gamma^+\} \to \underleftarrow{\lim}\, G_t$$

is surjective, so in particular there exists $\theta^q \in \{B\mathrm{GL}(\mathbf{F}_q)^+, B\Gamma^+\}$ whose image lies in $\underleftarrow{\lim}\, S_t$. In other words, the composites

$$\Sigma^\infty X_t \longrightarrow \Sigma^\infty B\mathrm{GL}(\mathbf{F}_q)^+ \xrightarrow{\theta^q} \Sigma^\infty B\Gamma^+ \xrightarrow{\Sigma^\infty B\tau_q} \Sigma^\infty B\mathrm{GL}(\mathbf{F}_q)^+$$

are p-locally t-connected for every t. We conclude that $\Sigma^\infty B\tau_q \circ \theta^q$ is a p-local equivalence.

Finally, we observe that since $B\mathrm{GL}(\mathbf{F}_q)^+$ and $\mathrm{Im}\, J_{(1/2)}$ are H-spaces with finite homotopy groups, they decompose into the wedge of their primary parts,

$$BGL(\mathbf{F}_q)^+ \simeq \bigvee_{r \text{ prime}} BGL(\mathbf{F}_q)^+_{(r)},$$

$$\operatorname{Im} J_{(1/2)} \simeq \bigvee_{p \text{ odd prime}} \operatorname{Im} J_{(p)}.$$

Thus $\{\operatorname{Im} J_{(1/2)}, X\} = \prod_p \{\operatorname{Im} J_{(p)}, X\}$ for any X, and similarly for $BGL(\mathbf{F}_q)^+$. In particular, letting ${}^p\theta \in \{\operatorname{Im} J_{(p)}, B\Gamma^+\} = \{BGL(\mathbf{F}_q)^+_{(p)}, B\Gamma^+\}$ be the restriction of θ^q (for some appropriate choice of q), we obtain a stable map

$$\theta = \prod_p {}^p\theta \in \{\operatorname{Im} J_{(1/2)}, B\Gamma^+\}.$$

By construction, the p-localization of θ satisfies $\theta_{(p)} = {}^p\theta_{(p)} = \theta^q_{(p)}$, and hence $\Sigma^\infty B_T \circ \theta$ is a p-local equivalence as required by the theorem. It remains only to prove Lemma 3.4.

Proof of Lemma 3.4. Let $S_{g,0}$ be a surface obtained as a p-fold branched cover of a 2-sphere with $s+2$ branch points, $s \geq 1$. By the Riemann-Hurwicz formula, the genus of $S_{g,0}$ is $g = \frac{1}{2}s(p-1)$. The group of covering transformations on $S_{g,0}$ gives a map $\mathbf{Z}/p \xrightarrow{\rho} \Gamma_{g,0}$. The composite map

$$\text{(3.4.1)} \qquad \mathbf{Z}/p \xrightarrow{\rho} \Gamma_{g,0} \longrightarrow GL_{s(p-1)}(\mathbf{F}_q)$$

is determined up to conjugacy by the $\mathbf{Z}/p$-module structure on $H_1(S_{g,0}; \mathbf{F}_q)$, or equivalently by the class of $H_1(S_{g,0}; \mathbf{F}_q)$ in the representation ring $R_{\mathbf{F}_q}(\mathbf{Z}/p)$ of $\mathbf{Z}/p$ over $\mathbf{F}_q$.

Choose a $\mathbf{Z}/p$-equivariant triangulation of $S_{g,0}$ and let C_i be the free $\mathbf{F}_q$-module on the i-simplices of $S_{g,0}$. Let $H_i = H_i(S_{g,0}; \mathbf{F}_q)$. Then as elements in $R_{\mathbf{F}_q}(\mathbf{Z}/p)$, we have

$$H_0 - H_1 + H_2 = C_0 - C_1 + C_2.$$

Now H_0 and H_2 are trivial $\mathbf{Z}/p$-modules while C_1 and C_2 are the free $\mathbf{F}_q[\mathbf{Z}/p]$-modules on the 1-simplices and 2-simplices (respectively) of $S_{g,0}/\mathbf{Z}/p = S^2$. As for C_0, it contains a trivial $\mathbf{Z}/p$-module for each branch point and a copy of $\mathbf{F}_q[\mathbf{Z}/p]$ for each of the remaining 0-simplices of S^2. It follows that

$$\begin{aligned} C_0 - C_1 + C_2 &= \chi(S^2)\mathbf{F}_q[\mathbf{Z}/p] + (s+2)(\mathbf{F}_q - \mathbf{F}_q[\mathbf{Z}/p]) \\ &= -s\mathbf{F}_q[\mathbf{Z}/p] + (s+2)\mathbf{F}_q, \end{aligned}$$

where $\chi(S^2) = 2$ is the Euler characteristic of S^2. Hence

$$\begin{aligned} H_1 &= H_0 + H_2 - C_0 + C_1 - C_2 \\ &= s(\mathbf{F}_q[\mathbf{Z}/p] - \mathbf{F}_q), \end{aligned}$$

under our assumption on p and q, $\mathbf{F}_q[\mathbf{Z}/p]$ splits as a direct sum of $\mathbf{Z}/p$-modules

$$\mathbf{F}_q[\mathbf{Z}/p] \cong \mathbf{F}_q \oplus \tilde{\mathbf{F}}_q[\mathbf{Z}/p],$$

where $\tilde{\mathbf{F}}_q[\mathbf{Z}/p] = \mathbf{F}_q[\mathbf{Z}/p]/\langle 1 + t + \cdots + t^{p-1}\rangle$ and where t is a generator of $\mathbf{Z}/p$ (viewed multiplicatively). We conclude that in the representation ring $R_{\mathbf{F}_q}(\mathbf{Z}/p)$,

$H_1 = s\tilde{\mathbf{F}}_q[\mathbf{Z}/p]$. Choosing $\{1, t, \dots, t^{p-1}\}$ as a basis for $\tilde{\mathbf{F}}_q[\mathbf{Z}/p]$ over $\mathbf{F}_q$, $\tilde{\mathbf{F}}_q[\mathbf{Z}/p]$ may be identified with $\mathbf{F}_q^{(p-1)}$, with $\mathbf{Z}/p$ acting via the representation

$$\alpha_0 : \mathbf{Z}/p \to \mathrm{GL}_{(p-1)}(\mathbf{F}_q), \quad 1 \mapsto \begin{pmatrix} 0 & & & & -1 \\ 1 & 0 & & & \vdots \\ & 1 & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & -1 \end{pmatrix}.$$

Note that this α_0 agrees with the homomorphism α_0 defined in §2. It follows that, up to conjugacy, the composite (3.4.1) above is given by

$$1 \mapsto \begin{pmatrix} \alpha_0(1) & & 0 \\ & \ddots & \\ 0 & & \alpha_0(1) \end{pmatrix} \in \mathrm{GL}_{s(p-1)}(\mathbf{F}_q).$$

Now consider the diagram

$$\begin{array}{ccccc} B\pi_0 = B\mathbf{Z}/p & \xrightarrow{B\rho} & B\Gamma^+_{g,0} \longleftarrow B\Gamma^+_{g,1} & \longrightarrow & B\Gamma^+ \\ {\scriptstyle B\alpha_0}\downarrow & & \downarrow & & \downarrow{\scriptstyle B\tau_q} \\ B\mathrm{GL}_{(p-1)}(\mathbf{F}_q) & \xrightarrow{B\oplus^s} & B\mathrm{GL}_{s(p-1)}(\mathbf{F}_q)^+ & \longrightarrow & B\mathrm{GL}^+(\mathbf{F}_q) \\ {\scriptstyle i_{(p-1)}}\downarrow & & & & \| \\ B\mathrm{GL}(\mathbf{F}_q)^+ & & \xrightarrow{\times s} & & B\mathrm{GL}^+(\mathbf{F}_q). \end{array}$$

This diagram is homotopy commutative since conjugation induces the identity on $B\mathrm{GL}_{s(p-1)}(\mathbf{F}_q)^+$. The map $B\Gamma^+_{g,1} \to B\Gamma^+_{g,0}$ is m-connected providing $s \gg m$ (and hence $g \gg m$), and hence it splits on the m-skeleton. That is, there exists a map $(B\Gamma^+_{g,0})^{(m)} \to B\Gamma^+_{g,1}$ such that the composite

$$(B\Gamma^+_{g,0})^{(m)} \to B\Gamma^+_{g,1} \to B\Gamma^+_{g,0}$$

is homotopic to the natural inclusion. Thus, restricting to m-skeleta, the top vertical arrows in the diagram give rise to a map

$$\beta_0^s : (B\pi_0)^{(m)} \to B\Gamma^+,$$

which satisfies the conditions of the lemma.

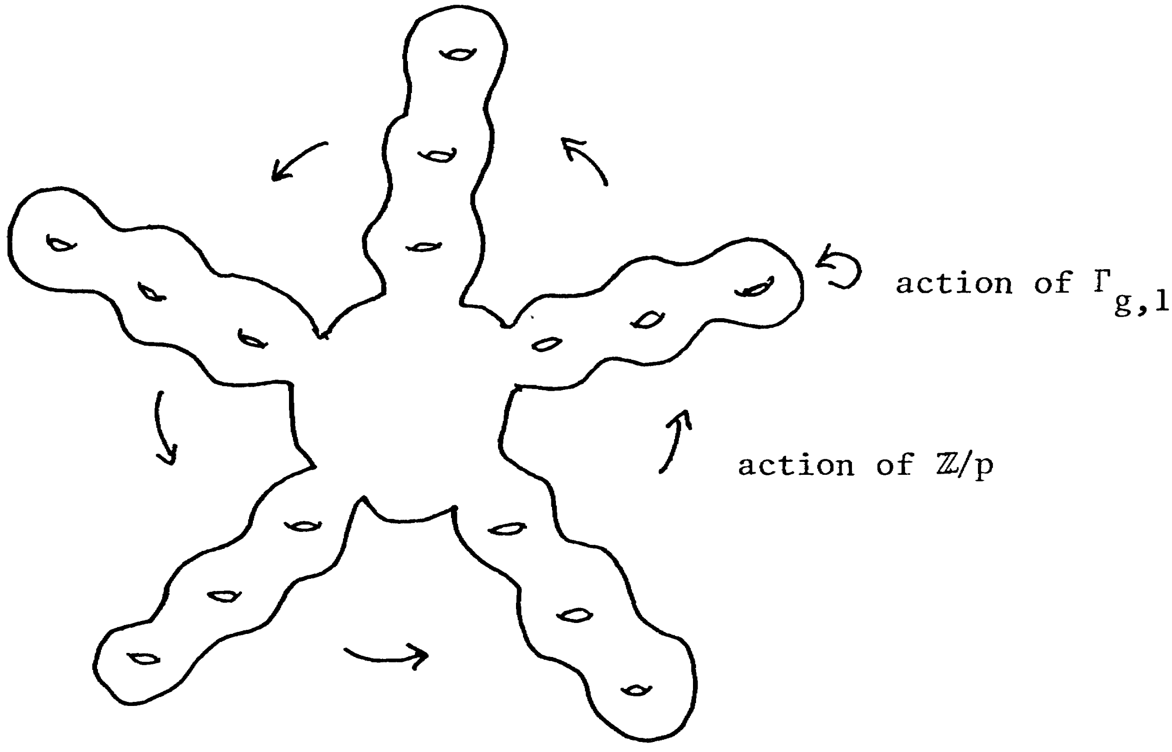
To define β_k^s for $k \geq 1$ we observe that, for any g, there exists a homomorphism

$$\mathbf{Z}/p \wr \Gamma_{g,1} \xrightarrow{\omega} \Gamma_{pg,0}$$

defined as follows. Let $X = S^2 - \coprod_p D^2$ be a 2-sphere minus p open disks arranged symmetrically about the equator. View $S_{pg,0}$ as the surface obtained by gluing a copy of $S_{g,1}$ to the boundary of each disk. Then

$$\underbrace{\Gamma_{g,1} \times \cdots \times \Gamma_{g,1}}_{p}$$

acts on $S_{pg,0}$ leaving X fixed, and $\mathbf{Z}/p$ acts by rotation.



It is easy to see that the diagram

$$\begin{array}{ccc} \mathbf{Z}/p\int\Gamma_{g,1} & \xrightarrow{\omega} & \Gamma_{pg,0} \\ \downarrow & & \downarrow \\ \mathbf{Z}/p\int \mathrm{GL}_{2g}(\mathbf{F}_q) & \subset & \mathrm{GL}_{2pg}(\mathbf{F}_q) \end{array}$$

commutes.

Let $g=\frac{1}{2}s(p-1)$ and consider the diagram on the opposite page.

Taking classifying spaces and applying the plus-construction (with respect to $\prod^{p^i}\Gamma_{n,\epsilon}\triangleleft\mathbf{Z}/p\int^i\Gamma_{n,\epsilon}$), the vertical maps $\mathbf{Z}/p\int^{k-i}\Gamma_{p^ig,1}\to\mathbf{Z}/p\int^{k-i}\Gamma_{p^ig,0}$ become m-connected providing $g\gg m$. Thus, restricting to m-skeleta, these maps split and we obtain a map $\beta_k^s\colon(B\pi_k)^{(m)}\to B\Gamma^+$. That this map satisfies the requirements of the lemma follows from the commutativity (up to conjugacy) of the diagram (opposite page). □

4. Some observations on $p=2$. The techniques of the previous sections can be used to obtain partial results at the prime 2. Recall that $H_1(S_{g,1},\mathbf{F}_q)$ comes equipped with a symplectic pairing, the intersection pairing, which is preserved by the action of $\Gamma_{g,1}$ on $S_{g,1}$, so the image of $\Gamma_{g,1}\to\mathrm{GL}_{2g}(\mathbf{F}_q)$ lies in $\mathrm{Sp}_{2g}(\mathbf{F}_q)$ and the image of $\tau_q\colon\Gamma\to\mathrm{GL}(\mathbf{F}_q)$ lies in $\mathrm{Sp}(\mathbf{F}_q)$.

Consider, in particular, $\mathrm{Sp}_2(\mathbf{F}_3)$. The 2-Sylow subgroup of $\mathrm{Sp}_2(\mathbf{F}_3)$ is the quarternion group Q_8 generated by

$$\alpha=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \beta=\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix},\quad \gamma=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

$$
\begin{tikzcd}
 & & & & & & & \Gamma_{p^k g,1} \arrow[r] \arrow[d] & \Gamma \arrow[ddddd] \\
 & & & & & & & \Gamma_{p^k g,0} \arrow[dddd] & \\
 & & & \mathbf{Z}/p\wr^{k-1}\Gamma_{pg,1} \arrow[r, "\mathbf{Z}/p\wr^{k-2}\omega"] \arrow[d] & & \mathbf{Z}/p\wr^{k-2}\Gamma_{p^2g,0} \arrow[ddd] & \iddots & & \\
 & \mathbf{Z}/p\wr^{k}\Gamma_{g,1} \arrow[rr, "\mathbf{Z}/p\wr^{k-1}\omega"] \arrow[d] & & \mathbf{Z}/p\wr^{k-1}\Gamma_{pg,0} \arrow[dd] & & & & & \\
\mathbf{Z}/p\wr^{k}\mathbf{Z}/p \arrow[r, "\mathbf{Z}/p\wr^{k}\beta_0^s"] \arrow[d, equal] & \mathbf{Z}/p\wr^{k}\Gamma_{g,0} \arrow[d] & & & & & & & \\
\pi_k \arrow[r, "\mathbf{Z}/p\wr^{k}\alpha_0^s"] & \mathbf{Z}/p\wr^{k}\mathrm{GL}_{2g} \arrow[rr, phantom, "\subset"] & & \mathbf{Z}/p\wr^{k-1}\mathrm{GL}_{2pg} \arrow[rr, phantom, "\subset"] & & \mathbf{Z}/p\wr^{k-2}\mathrm{GL}_{2p^2g} \arrow[rr, phantom, "\subset\cdots\subset"] & & \mathrm{GL}_{2p^kg} \arrow[r, phantom, "\subset"] & \mathrm{GL}(\mathbf{F}_q)
\end{tikzcd}
$$

The center of this group is $\mathbf{Z}/2$, generated by

$$c = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is well known that the inclusion of this center $\mathbf{Z}/2 \hookrightarrow Q_8$ induces an isomorphism

$$H_{4j}(\mathbf{Z}/2; \mathbf{F}_2) \xrightarrow{\cong} H_{4j}(Q_8; \mathbf{F}_2)$$

and hence a surjection

$$H_{4j}(\mathbf{Z}/2; \mathbf{F}_2) \twoheadrightarrow H_{4j}(\mathrm{Sp}_2(\mathbf{F}_3); \mathbf{F}_2).$$

Next consider $H_*(Sp(\mathbf{F}_3); \mathbf{F}_2)$. As a Hopf algebra over the Steenrod algebra, it is a symmetric algebra on $\tilde{H}_*(\mathrm{Sp}_2(\mathbf{F}_3); \mathbf{F}_2)$. Namely,

$$H_*(\mathrm{Sp}(\mathbf{F}_3); \mathbf{F}_2) \cong \mathbf{F}_2[x_{4j}] \otimes \Lambda\, [x_{4j-1}], \quad j \geq 1,$$

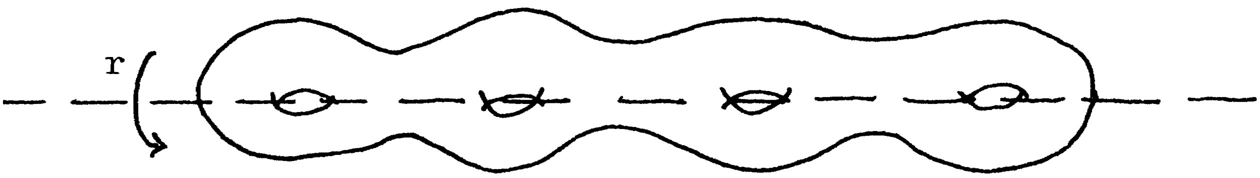
where x_{4j} is the image of a generator of $H_{4j}(\mathbf{Z}/2; \mathbf{F}_2)$ (see [5]).

THEOREM 4.1. *The homomorphism* $\tau_3 : \Gamma \to \mathrm{Sp}(\mathbf{F}_3)$ *induces an epimorphism of algebras*

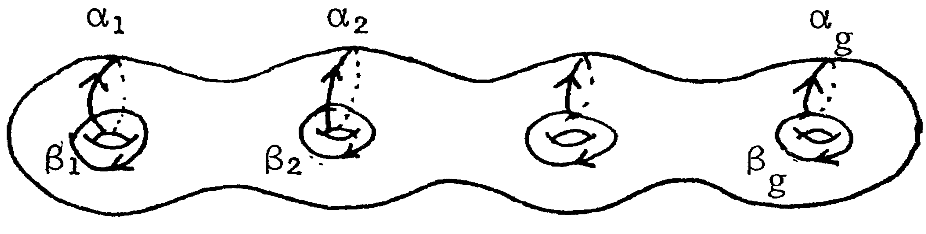
$$H_*(\Gamma; \mathbf{F}_2) \to \mathbf{F}_2[x_{4j}],$$

where $\mathbf{F}_2[x_{4j}]$ *is the polynomial algebra with a generator in dimension* $4j$ *for every* $j \geq 1$.

Proof. Let $r \in \Gamma_{g,0}$ be the element of order 2 which rotates $S_{g,0}$ 180° about a central axis as shown below:



Choosing as basis for $H_1(S_{g,0}; \mathbf{F}_3)$ the oriented curves $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$,



we see that the induced map r_* on $H_1(S_{g,0}; \mathbf{F}_3)$ takes $[\alpha_i]$ to $-[\alpha_i]$ and $[\beta_i]$ to $-[\beta_i]$. In other words, the composite

$$\begin{gathered} \mathbf{Z}/2 \to \Gamma_{g,0} \to \mathrm{Sp}_{2g}(\mathbf{F}_3) \\ 1 \mapsto r \mapsto r_* \end{gathered}$$

takes 1 to

$$\oplus^g c = \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}.$$

This gives a commutative diagram

$$\begin{tikzcd} B\mathbf{Z}/2 \arrow[r] \arrow[d] & B\Gamma_{g,0}^{+} \arrow[d] & B\Gamma_{g,1}^{+} \arrow[l] \arrow[r] & B\Gamma^{+} \arrow[d] \\ B\mathrm{Sp}_2(\mathbf{F}_3) \arrow[r, "B\oplus^g"] \arrow[d] & B\mathrm{Sp}_{2g}(\mathbf{F}_3)^{+} \arrow[rr] & & B\mathrm{Sp}(\mathbf{F}_3)^{+} \arrow[d, equal] \\ B\mathrm{Sp}(\mathbf{F}_3)^{+} \arrow[rrr, "\times g"] & & & B\mathrm{Sp}(\mathbf{F}_3)^{+}. \end{tikzcd}$$

For g odd, $\times g$ induces an isomorphism on $H_*(\mathrm{Sp}(\mathbf{F}_3); \mathbf{F}_2)$. For $g \gg i$, $B\Gamma_{g,0}^{+} \leftarrow B\Gamma_{g,1}^{+}$ induces isomorphisms on $H_i(\ ; \mathbf{F}_2)$. We conclude that $x_{4j} \in H_{4j}(\mathrm{Sp}(\mathbf{F}_3); \mathbf{F}_2)$ is in the image of $H_{4j}(\Gamma; \mathbf{F}_2)$. Since $B\Gamma^{+} \to B\mathrm{Sp}(\mathbf{F}_3)^{+}$ is a map of H-spaces, this completes the proof of the theorem. □

5. Mod p homotopy. We conclude with some general remarks about the mod p homotopy of $B\Gamma^{+}$. In particular, we prove the following.

PROPOSITION 5.1. *The map $B\tau: B\Gamma^{+} \to \mathrm{Im}\, J_{(1/2)}$ induces epimorphisms on $\pi_*(\ ; \mathbf{F}_p)$ for any odd prime p.*

The proof of the proposition is based on some techniques of Browder in [2] (cf. [3]). We remark that these techniques cannot be extended, in general, to yield integral results, as can be seen from the following example. Let $\Omega_1^n S^n$ denote the component of degree-one maps in $\Omega^n S^n$. Define $\rho: \Omega_1^3 S^3 \to \mathrm{Im}\, J$ as the composite of the stabilization map $\Omega_1^3 S^3 \to \Omega_1^\infty S^\infty$ with the standard map $\Omega_1^\infty S^\infty \to \mathrm{Im}\, J_{(1/2)}$ (see [14]). Using the techniques of this section, one can show that ρ induces epimorphisms on $\pi_*(\ ; \mathbf{F}_p)$ for every odd prime p, but a comparison of the groups $\pi_*(\Omega_1^3 S^3)$ and $\pi_*(\mathrm{Im}\, J_{(1/2)})$ shows that ρ cannot induce surjections integrally.

Before proving the proposition we give a brief sketch of Browder's arguments. Given an H-space X and a map

$$m: Y_1 \times \cdots \times Y_\ell \to X$$

of based CW-complexes, Browder defines an induced map

$$\mu_\ell(m): Y_1 \wedge \cdots \wedge Y_\ell \to X$$

using a difference construction, which gives rise to a product

$$\pi_{i_1}(Y_1; \mathbf{F}_p) \times \cdots \times \pi_{i_\ell}(Y_\ell; \mathbf{F}_p) \to \pi_{i_1 + \cdots + i_\ell}(X; \mathbf{F}_p).$$

In particular, applying this construction to the tensor product map

$$B\mathrm{GL}_{i_1}(A) \times \cdots \times B\mathrm{GL}_{i_\ell}(A) \xrightarrow{\otimes} B\mathrm{GL}_{i_1 \cdots i_\ell}(A) \longrightarrow B\mathrm{GL}(A)^{+}$$

and then stabilizing, one obtains maps

$$BGL(A)^+ \wedge \cdots \wedge BGL(A)^+ \xrightarrow{\mu_\ell} BGL(A)^+$$

and thus a product structure on $\pi_*(BGL(A)^+; \mathbf{F}_p)$. (By definition, we take it that $\pi_1(BGL(A)^+; \mathbf{F}_p) = \pi_1(BGL(A)^+) \otimes \mathbf{F}_p$.) In the case where A is a finite field with t elements, one obtains the following.

THEOREM 5.2 ([2, p. 52]). *Suppose* $p \mid (t^d - 1)$ *but* $p \nmid (t^{d-1} - 1)$. *Then*

$$\pi_*(BGL(\mathbf{F}_t)^+; \mathbf{F}_p) \cong \bigwedge [x] \otimes \mathbf{F}_p[y],$$

where $\bigwedge [x]$ *is an exterior algebra on a generator* x *of degree* $2d-1$ *and* $\mathbf{F}_p[y]$ *is a polynomial algebra on a generator* y *of degree* $2d$.

Now suppose p and q satisfy Assumption 2.1 and let $t = q^{p-1}$. Then $p \mid (t-1)$, so in this case the generators x, y of $\pi_*(BGL(\mathbf{F}_t)^+; \mathbf{F}_p)$ are in degrees 1 and 2, respectively. Also in this case, $\mathbf{F}_t$ contains a pth root of unity, so we have an inclusion

$$\tilde{\alpha}: \mathbf{Z}/p \subset \mathbf{F}_t^* = GL_1(\mathbf{F}_t) \subset GL(\mathbf{F}_t).$$

Browder considers the induced map $B\tilde{\alpha}: B\mathbf{Z}/p \to BGL(\mathbf{F}_t)^+$ and observes that $B\tilde{\alpha}$ gives rise to isomorphisms on $\pi_i(\ ; \mathbf{F}_p)$ for $i = 1, 2$. Applying the difference construction to

$$m: B\mathbf{Z}/p \times \cdots \times B\mathbf{Z}/p \xrightarrow{\text{mult}} B\mathbf{Z}/p \xrightarrow{\text{inc}} \Omega\Sigma B\mathbf{Z}/p,$$

one obtains a commutative diagram

$$\begin{array}{ccc} B\mathbf{Z}/p \wedge \cdots \wedge B\mathbf{Z}/p & \xrightarrow{\mu_\ell(m)} & \Omega\Sigma B\mathbf{Z}/p \\ \Big\downarrow{\scriptstyle B\tilde{\alpha} \wedge \cdots \wedge B\tilde{\alpha}} & & \downarrow{\scriptstyle \Omega\Sigma B\tilde{\alpha}} \\ & & \Omega\Sigma BGL(\mathbf{F}_t)^+ \\ & & \downarrow{\scriptstyle \text{ev}} \\ BGL(\mathbf{F}_t)^+ \wedge \cdots \wedge BGL(\mathbf{F}_t)^+ & \xrightarrow{\mu_\ell} & BGL(\mathbf{F}_t)^+, \end{array}$$

where ev is the evaluation map. Setting $\tilde{\gamma} = \text{ev} \circ \Omega\Sigma B\tilde{\alpha}$, it follows that $\tilde{\gamma}$ induces surjections on $\pi_i(\ ; \mathbf{F}_p)$ for *all* i.

To pass to $GL(\mathbf{F}_q)$, we consider the transfer map $\text{tr}: BGL(\mathbf{F}_t)^+ \to BGL(\mathbf{F}_q)^+$. This is the map induced by choosing a basis for $\mathbf{F}_t$ over $\mathbf{F}_q$ and thereby viewing $GL_n(\mathbf{F}_t)$ as a subgroup of $GL_{n(p-1)}(\mathbf{F}_q)$. In the other direction, the canonical inclusion $\mathbf{F}_q \hookrightarrow \mathbf{F}_t$ defines a map $i: BGL(\mathbf{F}_q)^+ \to BGL(\mathbf{F}_t)^+$, and the composite $\text{tr} \circ i$ induces multiplication by $p-1$ on homotopy. Thus tr induces surjections on $\pi_*(\ ; \mathbf{F}_p)$. Combining this with the previous paragraph we conclude:

LEMMA 5.3 ([2]). *The composite*

$$\Omega\Sigma B\mathbf{Z}/p \xrightarrow{\tilde{\gamma}} BGL(\mathbf{F}_t)^+ \xrightarrow{\text{tr}} BGL(\mathbf{F}_q)^+$$

induces a split epimorphism on mod-p homotopy groups.

Proof of Proposition 5.1. Recall the inclusions

$$\alpha_0^s\colon \mathbf{Z}/p \to \mathrm{GL}_{s(p-1)}(\mathbf{F}_q) \to \mathrm{GL}(\mathbf{F}_q)$$

defined in §2, and consider the map γ^s defined as the composite

$$\gamma^s\colon \Omega\Sigma B\mathbf{Z}/p \xrightarrow{\Omega\Sigma B\alpha_0^s} \Omega\Sigma B\mathrm{GL}(\mathbf{F}_q)^+ \xrightarrow{\mathrm{ev}} B\mathrm{GL}(\mathbf{F}_q)^+.$$

We claim that γ^s induces epimorphisms on $\pi_*(\ ;\mathbf{F}_p)$ providing s is prime to p. For $s=1$, the homomorphism α_0^1 is the composite of the map $\tilde{\alpha}\colon \mathbf{Z}/p \to \mathrm{GL}(\mathbf{F}_t)$ defined above (with $t=q^{p-1}$) and the inclusion $\mathrm{GL}(\mathbf{F}_t) \to \mathrm{GL}(\mathbf{F}_q)$, which defines the transfer. Since the transfer commutes with the evaluation map, γ^1 is precisely the composite $\mathrm{tr}\circ\tilde{\gamma}$ of Lemma 5.3. Thus, for $s=1$, the claim follows from Lemma 5.3. For $s>1$, $\alpha_0^s = \oplus^s \alpha_0^1$ and hence γ^s is homotopic to $s\cdot\gamma^1$, where multiplication by s is defined via the direct sum H-space structure on $B\mathrm{GL}(\mathbf{F}_q)^+$. It follows that whenever s is prime to p, γ^s induces epimorphisms on $\pi_*(\ ;\mathbf{Z}/p)$.

Now recall from the proof of Lemma 3.4 that α_0^s factors through a map $\rho\colon \mathbf{Z}/p \to \Gamma_{g,0}$, $g=\frac{1}{2}s(p-1)$. Hence we have a commutative diagram

$$\begin{tikzcd}
\gamma^s\colon \Omega\Sigma B\mathbf{Z}/p \arrow[r] \arrow[d] & \Omega\Sigma B\mathrm{GL}_{2g}(\mathbf{F}_q)^+ \arrow[r] & \Omega\Sigma B\mathrm{GL}(\mathbf{F}_q)^+ \arrow[r] & B\mathrm{GL}(\mathbf{F}_q)^+ \\
\Omega\Sigma B\Gamma_{g,0}^+ \arrow[ur] & \Omega\Sigma B\Gamma_{g,1}^+ \arrow[l] \arrow[u] \arrow[r] & \Omega\Sigma B\Gamma^+ \arrow[u] \arrow[r] & B\Gamma^+. \arrow[u, "B\tau"']
\end{tikzcd}$$

For $s \gg i$, the map $\Omega\Sigma B\Gamma_{g,0}^+ \leftarrow \Omega\Sigma B\Gamma_{g,1}^+$ is i-connected and so (choosing s prime to p) the surjection

$$\gamma_*^s\colon \pi_i(\Omega\Sigma B\mathbf{Z}/p;\mathbf{F}_p) \twoheadrightarrow \pi_i(B\mathrm{GL}(\mathbf{F}_q)^+;\mathbf{F}_p)$$

factors through $B\tau_*$. This proves the proposition. □

Added in proof: A complete calculation of $H^*(\Gamma_{0,0}^n;\mathbf{F}_p)$ will appear in work of the second author, C.-F. Bödigheimer, and M. Peim.

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