OPERATORS WITH COMMUTATIVE COMMUTANTS

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It is a well-known consequence of the Putnam–Fuglede theorem that if a normal operator N is a quasiaffine transform of a normal operator M, then M and N are unitarily equivalent. Problem 199 of [4] shows that the result does not remain true if M and N are merely subnormal, even if they are quasisimilar. In the present paper we show that if M (resp. N) is the direct sum of k (resp. m) copies of an operator A having a commutative commutant, where m and k are countable cardinalities, and if N is a quasiaffine transform of M, then $k=m$ (see Theorem 1). In the special case where A is the simple unilateral shift, this extends a result of Hoover [5], who shows that quasisimilar isometries are unitarily equivalent. In fact, using a result of Fan [3], we show that if an isometry N is a quasiaffine transform of an isometry M, and if the unitary part of the Wold decomposition of N has a singular scalar-valued spectral measure, then M and N are unitarily equivalent (see Theorem 2).

We conclude the paper with a result about multiplications M_z by $g(z)\equiv z$ on function spaces $R^2(X,\mu)$, where μ is a positive measure supported on a compact subset X of $\mathbf{C}$; every nonscalar operator commuting with $(M_z)^{(n)}$ has a hyperinvariant subspace if $(M_z)^*$ has an eigenvalue and $n<\infty$ (see Theorem 3). This generalizes a result of Sz.-Nagy and Foiaş [8, p. 191] and Nordgren [6] about the unilateral shift (see also [7, p. 149]).

Let us here fix some notations and definitions. For the commutant of an operator A we use the usual notation $\{A\}'$. If A is an operator on a Hilbert space $\mathcal{H}$, then $A^{(k)}$ denotes the direct sum of k copies of A acting on the direct sum $\mathcal{H}^{(k)}$ of k copies of $\mathcal{H}$, where k is any cardinality; if $k=0$ then $\mathcal{H}^{(0)}=\{0\}$. A bounded linear transformation between two Banach spaces is called a quasiaffinity if it is injective and has dense range; an operator N is a quasiaffine transform of an operator M if $CM=NC$ for some quasiaffinity C. The operators M and N are quasisimilar if $C_1M=NC_1$ and $MC_2=C_2N$ for some quasiaffinities C_1 and C_2.

For a compact subset X of $\mathbf{C}$, $\operatorname{Rat}(X)$ denotes the algebra of all rational functions with poles off X. If μ is a positive Borel measure supported on X, then $R^2(X,\mu)$ denotes the closure of $\operatorname{Rat}(X)$ in $\mathcal{L}^2(\mu)$.

An operator A on $\mathcal{H}$, with spectrum contained in X, is called $\operatorname{Rat}(X)$-cyclic if there exists a vector e in $\mathcal{H}$ such that the linear manifold $\{r(A)e : r\in\operatorname{Rat} X\}$ is dense in $\mathcal{H}$.

THEOREM 1. *Let $A\in B(\mathcal{H})$ and assume $\{A\}'$ is commutative. Let C be a bounded linear transformation such that $CA^{(k)}=A^{(m)}C$ for some finite or countable cardinalities k and m. Then*

(a) *$k\le m$ if C is injective, and*

(b) *$k\ge m$ if C has dense range.*

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Proof. Represent $C: \mathcal{H}^{(k)} \to \mathcal{H}^{(m)}$ by the $m \times k$ operator matrix $((C_{ij}))$, where $C_{ij}: \mathcal{H} \to \mathcal{H}$. Since $CA^{(k)} = A^{(m)}C$, $C_{ij}A = AC_{ij}$ and hence $C_{ij} \in \{A\}'$ for all pairs (i, j).

To prove (a), assume that C is injective and, if possible, that $k > m$. Then m is finite. Let $n \leq m$ be the largest natural number such that the (operator-valued) determinant of at least one $n \times n$ submatrix B of $((C_{ij}))$ is nonzero. Since C is injective, such a number exists. By rearranging the direct summands, we can assume, without loss of generality, that $B = ((C_{ij}: 1 \leq i \leq n,\ 1 \leq j \leq n))$ has nonzero determinant. For an arbitrary $(n+1)$-tuple $Z = (Z_j: 1 \leq j \leq n+1)$ of operators in $\{A\}'$, let D_Z be the $(n+1) \times (n+1)$ matrix whose ith row is $(C_{ij}: 1 \leq j \leq n+1)$ for $1 \leq i \leq n$ and whose $(n+1)$th row is Z. Let

$$\det D_Z = Z_1 X_1 + \cdots + Z_n X_n + Z_{n+1} \det B$$

be the expansion of $\det D_Z$ in terms of the last row (so that $X_1, \ldots, X_n$ are independent of Z). Now, if $Z = (C_{ij}: 1 \leq j \leq n+1)$ for any fixed $i \in \{1, 2, \ldots, m\}$, then $\det D_Z = 0$; for either D_Z has two identical rows or it is an $(n+1) \times (n+1)$ submatrix of $((C_{ij}: 1 \leq i \leq m,\ 1 \leq j \leq k))$. Let $X: \mathcal{H} \to \mathcal{H}^{(k)}$ be a linear transformation represented by the $k \times 1$ (column) matrix with entries $X_1, \ldots, X_n, \det B, 0, 0, \ldots$. It follows that $CX = 0$. Since $X \neq 0$, C is not injective, a contradiction. Thus $k \leq m$.

The proof of (b) follows from the fact that C^* is injective, $C^* A^{*(m)} = A^{*(k)} C^*$, and $\{A^*\}'$ is commutative. □

The above theorem holds for any cardinalities k and m if $\mathcal{H}$ is separable; this reduces to the given case via a dimensionality argument. For definiteness we shall henceforth assume the underlying space to be separable.

COROLLARY 1. *Let S be a* Rat(X)*-cyclic subnormal operator for some compact set $X \supseteq \sigma(S)$. Assume $CS^{(k)} = S^{(m)}C$ for some quasiaffinity C and cardinalities k and m. Then $k = m$.*

Proof. By Yoshino's theorem [9], the commutant of S is commutative ([2, pp. 146–147]). Thus $k = m$. □

COROLLARY 2. *Let the completely nonunitary isometry $V \in B(\mathcal{H})$ be a quasiaffine transform of an isometry $U \in B(\mathcal{H})$. Then U and V are unitarily equivalent.*

Proof. Let $U \cong W \oplus S^{(k)}$ be the Wold decomposition of U, where W is a unitary (possibly acting on the zero subspace), S is the simple unilateral shift, and k is some finite or countable cardinality. (If $k = 0$, then $S^{(0)}$ is the zero operator acting on the zero subspace.) Let $\mathcal{M}$ be the domain of W, and let C be the quasiaffinity such that $CU = VC$. It is easy to see that $(C\mathcal{M})^- \in \operatorname{Lat} V$, $V \mid (C\mathcal{M})^-$ is subnormal, and (as in [6])

$$(C \mid \mathcal{M})W = [V \mid (C\mathcal{M})^-](C \mid \mathcal{M}).$$

By [1], W and $V \mid (C\mathcal{M})^-$ are unitarily equivalent and hence $V \mid (C\mathcal{M})^-$ is normal. Since V is completely nonnormal, $C\mathcal{M} = \{0\}$ and thus $\mathcal{M} = \{0\}$. Now the equivalence of U and V follows from Theorem 1 and the fact that V is also unitarily equivalent to $S^{(m)}$ for some finite or infinite cardinality m. □

The following example shows that an isometry V may be a quasiaffine transform of an isometry U without being unitarily equivalent to it.

Let $\mathbf{T}$ be the unit circle and let U be the simple bilateral shift defined on $L^2(\mathbf{T})$ by $(Uf)(z)=zf(z)$. Partition $\mathbf{T}$ into disjoint arcs $\Gamma_1,\ldots,\Gamma_n$ $(n\geq 1)$ and let $\psi_j \in L^\infty(\Gamma_j)$ be a cyclic vector for $U \mid L^2(\Gamma_j)$ $(1\leq j\leq n)$. Extend ψ_j to all of $\mathbf{T}$ by defining it to be zero on $\mathbf{T}\backslash\Gamma_j$. Define $C\colon (H^2)^{(n)}\to L^2(\mathbf{T})$ by $C(f_1\oplus\cdots\oplus f_n)=\psi_1 f_1+\cdots+\psi_n f_n$, where H^2 is the Hardy space of the unit disc. The linear transformation C is bounded and injective, and has dense range. (Note that if $f\in H^2$ and $f(e^{i\theta})=0$ on a set of nonzero 1-dimensional Lebesgue measure, then $f\equiv 0$.) It is easy to see that $CS^{(n)}=UC$, where S is the unilateral shift $U\mid H^2$. Clearly, $S^{(n)}$ and U are not even similar. □

The following theorem is a sharpening of Corollary 2. For the definition of a scalar-valued spectral measure see [2, p. 91].

THEOREM 2. *Let $V=V_s\oplus V_a\oplus S^{(k)}$ be an isometry such that V_s (resp. V_a) is a unitary operator with a singular (resp. absolutely continuous) scalar-valued spectral measure. S is the simple unilateral shift and k is any cardinality. Assume V is a quasiaffine transform of a contraction operator T. Then $T=T_1\oplus T_2$, where V_s is unitarily equivalent to T_1, and $V_a\oplus S^{(k)}$ is a quasiaffine transform of T_2. Moreover, if $V_a=0$ and T is an isometry, then V and T are unitarily equivalent.*

Proof. Assume without loss of generality that V and T act on the same Hilbert space H. Let $C=PW$ be the polar decomposition of the quasiaffinity C satisfying $CT=VC$. (Here W is unitary and P is an injective positive operator.) Then $PWTW^*=VP$. Let $M=VP$. Since WTW^* is a contraction operator, it follows that

$$\begin{aligned}0&\leq \|Px\|^2-\|WT^*W^*Px\|^2\\&=\|Px\|^2-\|PV^*x\|^2=(P^2x,x)-(VP^2V^*x,x)\\&=([P^2-VP^2V^*]x,x)=([M^*M-MM^*]x,x)\end{aligned}$$

for all $x\in H$. Thus M is hyponormal. Let H_s be the domain of V_s. By [3, Thm. 2], H_s is a reducing invariant subspace of M and $M\mid H_s$ is normal. Hence H_s reduces P and $P_sV_s=V_sP_s$, where $P_s=P\mid H_s$. Therefore, $WTW^*=V_s\oplus P_0^{-1}(V_a\oplus S^{(k)})P_0$, where $P_0=P\mid H_s^\perp$. (Note that $P^{-1}VP$ and consequently $P_0^{-1}(V_a\oplus S^{(k)})P_0$ are well-defined bounded operators.) This shows that $T=T_1\oplus T_2$, where T_1 is unitarily equivalent to V_s.

Next, assume that T is an isometry and $V_a=0$. Then $P_0^{-1}S^{(k)}P_0$ is an isometry and $P_0(P_0^{-1}S^{(k)}P_0)=S^{(k)}P_0$. It follows from Corollary 2 that $P_0^{-1}S^{(k)}P_0$ and $S^{(k)}$ are unitarily equivalent. Hence T and V are also unitarily equivalent. □

REMARK. Let V be a unitary operator with an absolutely continuous scalar-valued spectral measure. Then V is unitarily equivalent to a direct sum $\sum_{i\in\Lambda}\oplus U_i$, where each U_i is the multiplication by $g(z)\equiv z$ on $L^2(E_i)$ for some $E_i\subset\mathbf{T}$ [2, p. 92]. (Here the measure on $\mathbf{T}$ is the normalized 1-dimensional Lebesgue measure.) Assume with no loss of generality that E_i has positive measure and that $\bar{E}_i\neq\mathbf{T}$ for every i. Let $A_i\colon H^2\to L^2(E_i)$ be the map sending $\phi\in H^2$ to $\phi\mid E_i$ $(i\in\Lambda)$. It is easy to see that $A=\Sigma\oplus A_i$ is a quasiaffinity and $AS^{(k)}=(\Sigma\oplus U_i)A$, where k is the cardinality of Λ. Thus the unitary operator V is a quasiaffine transform of the pure isometry $S^{(k)}$.

Our final result is about the hyperinvariant subspaces of operators commuting with $(M_z)^{(n)}$, where $n<\infty$ and M_z is the multiplication by $g(z)\equiv z$ in some $R^2(X,\mu)$.

THEOREM 3. *Let $S=M_z$ be the multiplication by $g(z)\equiv z$ in $R^2(X,\mu)$, where μ is a positive measure supported on a compact subset X of $\mathbf{C}$. Assume S^* has an eigenvalue. Then any nonscalar operator commuting with $S^{(n)}$ has a hyperinvariant subspace if $n<\infty$.*

Proof. We prove more: if T_1 and T_2 commute with $S^{(n)}$, then some nontrivial linear combination $a_1T_1+a_2T_2$ has nondense range. (If T commutes with $S^{(n)}$, then $\{T\}'$ leaves invariant the range of any linear combination of, say, T and T^2.)

Note that by Yoshino's theorem [9] any operator commuting with $S^{(n)}$ has an $n\times n$ matrix representation $((\phi_{ij}))$ with ϕ_{ij} in $L^\infty(\mu)\cap R^2(X,\mu)$.

Let $\bar{\lambda}$ be an eigenvalue of S^*. Then λ is a bounded point evaluation for $R^2(X,\mu)$; that is, the linear functional e_λ defined on $\operatorname{Rat}(X)$ by $e_\lambda(r)=r(\lambda)$ has a bounded extension to $R^2(X,\mu)$ [2, p. 169]. It is not hard to see that if $\psi\in L^\infty(\mu)\cap R^2(X,\mu)$ and $f\in R^2(X,\mu)$ then $\psi f\in R^2(X,\mu)$ and $e_\lambda(\psi f)=e_\lambda(\psi)e_\lambda(f)$. Let $((\phi_{ij}))$ and $((\psi_{ij}))$ be $n\times n$ matrices with entries in $\{S\}'=L^\infty(\mu)\cap R^2(X,\mu)$. If one of the numerical $n\times n$ matrices $((e_\lambda\phi_{ij}))$ and $((e_\lambda\psi_{ij}))$, say $((e_\lambda\psi_{ij}))$, is not invertible then let $((\theta_{ij}))=((\psi_{ij}))$. Otherwise, let η be an eigenvalue of the matrix $((e_\lambda\phi_{ij}))((e_\lambda\psi_{ij}))^{-1}$, and let $((\theta_{ij}))=((\phi_{ij}-\eta\psi_{ij}))$. In either case $\theta_{ij}\in L^\infty(\mu)\cap R^2(X,\mu)$ and $((e_\lambda\theta_{ij}))$ is not invertible.

Let $0\neq x\in\mathbf{C}^{(n)}$ be such that $(((e_\lambda\theta_{ij}))y\mid x)=0$ for all $y\in\mathbf{C}^{(n)}$. Define $Lf=e_\lambda(f(z)\mid x)$ for $f\in[R^2(X,\mu)]^{(n)}$, where $(f(z)\mid x)$ is the inner product of $f(z)$ and $x\in\mathbf{C}^{(n)}$. Observe that

$$|Lf|\le M\left[\int|(f(z)\mid x)|^2\,d\mu\right]^{1/2}\le M\|x\|\cdot\|f\|,$$

where $M=\|e_\lambda\|$ and $\|f\|$ denotes the norm of f in $[R^2(X,\mu)]^{(n)}$. Thus L is a bounded linear functional on $[R^2(X,\mu)]^{(n)}$ and $Lx=\|x\|^2\neq 0$, where x is now regarded as a constant function in $[R^2(X,\mu)]^{(n)}$. On the other hand,

$$\begin{aligned}L[((\theta_{ij}))g]&=e_\lambda(((\theta_{ij}(z)))g(z)\mid x)\\&=(((e_\lambda\theta_{ij}))(e_\lambda g)\mid x)=0\end{aligned}$$

for all $g\in[R^2(X,\mu)]^{(n)}$. This shows that the range of $((\theta_{ij}))$ is not dense. $\square$

The proof of Theorem 3 suggests the following corollary. We first need some terminology. Let $\mathcal{H}$ be a separable Hilbert space and let μ be a positive measure supported on a compact subset X of $\mathbf{C}$. Let $L^2_{\mathcal{H}}(\mu)$ denote the Hilbert space of all measurable functions $f:\mathbf{C}\to\mathcal{H}$ such that $\|f\|^2=:\int\|f(z)\|^2\,d\mu<\infty$. Let $R^2_{\mathcal{H}}(X,\mu)$ be the closure in $L^2_{\mathcal{H}}(\mu)$ of all functions of the form $r_1x_1+\cdots+r_kx_k$, where $k\in\mathbf{N}$,

$\{r_1, \ldots, r_k\} \subset \mathrm{Rat}(X)$, and $\{x_1, \ldots, x_k\} \subset \mathcal{K}$. Let T be the multiplication by $g(z) \equiv z$ in $R^2_{\mathcal{K}}(X, \mu)$ and let $A \in \{T\}'$. Then $T \cong S^{(n)}$ and $A \cong ((\psi_{ij}))_{n \times n}$ for some family of functions $\psi_{ij} \in L^\infty(\mu) \cap R^2(X, \mu)$, where $n = \dim \mathcal{K}$ and S is the multiplication by $g(z) \equiv z$ in $R^2(X, \mu)$. (Note that $\mathcal{K}$ may be finite- or infinite-dimensional.)

COROLLARY 3. *Let $\mathcal{K}$, μ, X, T, A, S, n, and $((\psi_{ij}))$ be as defined above. Assume A has dense range. Then the numerical matrix $((e_\lambda \psi_{ij}))$ as an operator on $\mathcal{K}$ has dense range for all bounded point evaluations λ on $R^2(X, \mu)$.*

REMARK. In the case where S is the simple unilateral shift and $n < \infty$, more is proved by Sz.-Nagy and Foiaş [8, p. 191]. They show that if $X = \mathbf{D}$ and μ is the normalized 1-dimensional Lebesgue measure on $\mathbf{T} = \partial \mathbf{D}$, then the numerical $n \times n$ matrix $((\phi_{ij}(z)))$ has dense range for all z in the interior of $\mathbf{D}$ and for almost all z on $\mathbf{T}$ if $A = ((\phi_{ij}))$, as an operator acting on $[R^2(X, \mu)]^{(n)}$, has dense range.

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