SYSTEMS OF OPERATOR EQUATIONS AND PERTURBATION OF SPECTRAL SUBSPACES OF COMMUTING OPERATORS

Alan McIntosh, Alan Pryde, and Werner Ricker*

1. Introduction. Let $\{A_j: 1 \le j \le m\}$ be a commuting set of continuous linear operators on a Banach space X and let $\{B_j: 1 \le j \le m\}$ be a commuting set of continuous linear operators on a Banach space Y. In this paper we address the problem of determining the existence of continuous linear operators $Q: Y \to X$ satisfying the system of equations

$$A_j Q - QB_j = U_j, \quad 1 \le j \le m, \tag{1.1}$$

where $U_j: Y \to X$, $1 \le j \le m$, is a given m-tuple of continuous linear operators, and of finding estimates on $\|Q\|$.

In the case of a single equation ($m=1$) the situation is relatively well understood. The question of the existence of a solution is essentially determined in [14]; the relevant criterion is that the spectra $\sigma(A)$ and $\sigma(B)$ should be disjoint. Finding estimates on $\|Q\|$ is somewhat more involved and requires restrictions on the operators A and B. For instance, if X and Y are Hilbert spaces and A and B are normal operators in X and Y, respectively, then it is known that there exists a (universal) constant $c>0$ such that

$$\|Q\| \le c\delta^{-1}\|U\|, \tag{1.2}$$

where δ is the distance between the disjoint sets $\sigma(A)$ and $\sigma(B)$ [1].

In [9] it was indicated that estimates for $\|Q\|$, where Q is a solution of (1.1), are possible in the case when the commuting m-tuples

$$\underset{\sim}{A} = (A_1, \dots, A_m) \quad \text{and} \quad \underset{\sim}{B} = (B_1, \dots, B_m)$$

generate bounded groups, that is, when $\|e^{i\langle \xi, \underset{\sim}{A}\rangle}\| \le \alpha < \infty$ and $\|e^{i\langle \xi, \underset{\sim}{B}\rangle}\| \le \beta < \infty$ for each $\xi \in \mathbf{R}^m$. Here $\langle \xi, \underset{\sim}{A}\rangle = \sum_{j=1}^m \xi_j A_j$. Each operator A_j and B_j in such an m-tuple necessarily has real spectrum. Furthermore, the m-tuple $\underset{\sim}{A}$ (resp. $\underset{\sim}{B}$) admits an $L_1^\vee(\mathbf{R}^m)$-functional calculus which takes its values in the Banach space $L(X)$ (resp. $L(Y)$) of all continuous linear operators from X (resp. Y) into itself. Here $L_1^\vee(\mathbf{R}^m)$ is the space consisting of all functions which are the inverse Fourier transforms of elements of $L_1(\mathbf{R}^m)$. If $Z = L(Y,X)$, the Banach space of all continuous linear operators from Y into X, and $T_j \in L(Z)$, $1 \le j \le m$, is defined by

$$T_j(Q) = A_j Q - QB_j, \quad Q \in Z, \tag{1.3}$$

then the commuting m-tuple $\underset{\sim}{T} = (T_1, \dots, T_m)$ also has a functional calculus, say $\Phi: L_1^\vee(\mathbf{R}^m) \to L(Z)$, whose support is the set $\mathrm{Sp}(\underset{\sim}{T}) \subseteq \mathrm{Sp}(\underset{\sim}{A}) - \mathrm{Sp}(\underset{\sim}{B})$ where $\mathrm{Sp}(\cdot)$

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denotes the Taylor spectrum. Under the assumption that $\mathrm{Sp}(\underset{\sim}{A}) \cap \mathrm{Sp}(\underset{\sim}{B}) = \varnothing$, a proof was outlined in [9] which showed (via an interesting use of Clifford analysis) that a unique solution Q of (1.1) exists if and only if the compatibility conditions

$$A_j U_k - U_k B_j = A_k U_j - U_j B_k, \quad 1 \leq k < j \leq m, \tag{1.4}$$

are satisfied. Furthermore, there exists a universal constant c_m such that if the distance between $\mathrm{Sp}(\underset{\sim}{A})$ and $\mathrm{Sp}(\underset{\sim}{B})$ is $\delta > 0$, then

$$\|Q\| \leq \alpha \beta c_m \delta^{-1} \|\underset{\sim}{U}\|_{Y \to X^m}, \tag{1.5}$$

where $\|\underset{\sim}{U}\|_{Y \to X^m} = \sup\{(\sum_{j=1}^m \|U_j y\|^2)^{1/2}; y \in Y, \|y\| \leq 1\}$; see [9, Thm. 1].

In this paper we shall give a detailed proof of the results announced in [9]. In doing so we relax the requirement that $\underset{\sim}{A}$ and $\underset{\sim}{B}$ generate bounded groups. It will suffice that the groups grow no faster than $O((1+|\xi|)^s)$, $\xi \in \mathbf{R}^m$, for some $s \geq 0$; see Theorem 2. In this case the constant c_m in (1.5) will also depend on s. The functional calculus needed to treat such m-tuples is briefly described in Section 2; it is systematically developed in [10]. As for the case of bounded groups, such operators A_j and B_j, $1 \leq j \leq m$, necessarily have real spectrum. For certain types of m-tuples $\underset{\sim}{A}$ and $\underset{\sim}{B}$ this restriction can also be relaxed; see Theorem 1.

In Section 3 we consider symmetric norms $|\cdot|$ on subspaces $\mathfrak{M}$ of $L(Y, X)$. In this case, if the data elements U_j, $1 \leq j \leq m$, belong to $\mathfrak{M}$, then the solution Q of (1.1), whenever it exists, is also an element of $\mathfrak{M}$ and estimates for $|Q|$ of the type given in Theorem 1 remain valid with the symmetric norm $|\cdot|$ replacing the operator norm $\|\cdot\|$. The desirability of admitting symmetric norms on subspaces of $L(Y, X)$ is discussed in [1].

Section 4 is devoted to various perturbation results for commuting m-tuples of generalized scalar operators with real spectrum. The idea is that a fixed commuting m-tuple $\underset{\sim}{A}$ of such operators in a Banach space X is given together with an invariant subspace Z. If $\underset{\sim}{A}$ is perturbed within the class of commuting m-tuples consisting of generalized scalar operators with real spectrum, and if Y is an invariant subspace of the perturbed m-tuple, then the basic problem is to estimate how close (in a sense made precise in §4) elements of Y are to the original subspace Z. Such results are used to show that if $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are commuting m-tuples of generalized scalar operators with real spectrum and if $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are sufficiently close, then the Hausdorff distance between their joint Taylor spectrum is also small (cf. Theorem 5). In the case of finite-dimensional spaces more can be stated. For example, suppose that X is a finite-dimensional Hilbert space and that $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are commuting m-tuples of self-adjoint operators. In this case there is a constant c, depending only on m, such that if $\|\underset{\sim}{A} - \underset{\sim}{B}\| < d/c$ then there exists a permutation π of $\{1, \ldots, N\}$ such that $|\mu_i - \lambda_{\pi(i)}| < d$ for every $1 \leq i \leq N$. Here

$$\{\mu_i : 1 \leq i \leq N\} \quad \text{and} \quad \{\lambda_i : 1 \leq i \leq N\}$$

are the joint eigenvalues of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ (counted according to multiplicity) and $N = \dim(X)$. This answers a question of Davis [4].

In the final section the techniques developed earlier in the paper are used to provide estimates for $\|Q\|$, where now Q is the solution of the so-called elementary operator equation

$$\sum_{j=1}^{m} A_j Q B_j = U.$$

Some attention, especially in the Hilbert space setting, has been devoted to finding criteria which guarantee the existence of a solution Q; see [3], for example, and the references therein. However, we are more interested in indicating some reasonable estimates for the solution Q which apply to a large class of the coefficient operators A_j and B_j, $1 \le j \le m$.

2. Solutions of systems of operator equations. We require some further notation and definitions. If $\underset{\sim}{W}$ is a commuting m-tuple of elements from $L(E)$ for some Banach space E, then $\mathrm{Sp}(\underset{\sim}{W}, E)$ or $\mathrm{Sp}(\underset{\sim}{W})$ denotes the Taylor spectrum of $\underset{\sim}{W}$ [16]. The distance $d(K,L)$ between two closed subsets K and L of $\mathbf{C}^m$ is defined by $d(K,L) = \inf\{|x-y| : x \in K, y \in L\}$.

Let E be a Banach space, $\underset{\sim}{W}$ be a commuting m-tuple of elements from $L(E)$, and $\delta > 0$. If there exist constants $\alpha \ge 1$ and $s \ge 0$ such that

$$\|e^{i\langle \xi, \underset{\sim}{W} \rangle}\| \le \alpha(1+\delta|\xi|)^s, \quad \xi \in \mathbf{R}^m, \tag{2.1}$$

then $\underset{\sim}{W}$ is said to be of type (α, s) with respect to δ. In this case $\underset{\sim}{W}$ necessarily consists of generalized scalar operators with real spectrum [2, Ch. 5, Thm. 4.5].

Suppose that E is a Banach space and $\underset{\sim}{W} = (W_1, \ldots, W_m)$ is an m-tuple of elements from $L(E)$. A partition of $\underset{\sim}{W}$ is a $2m$-tuple

$$\mathbf{T}(\underset{\sim}{W}) = (W_{11}, \ldots, W_{m1}, W_{12}, \ldots, W_{m2})$$

of elements from $L(E)$ such that

$$W_j = W_{j1} + iW_{j2}, \quad 1 \le j \le m. \tag{2.2}$$

We say that $\underset{\sim}{W}$ is strongly commuting if there exists a partition $\mathbf{T}(\underset{\sim}{W})$ of $\underset{\sim}{W}$ which is a commuting $2m$-tuple of operators with real spectrum. Examples of classes of operators with the property that any commuting m-tuple of elements from such a class is necessarily strongly commuting include spectral operators, prespectral operators, and regular generalized scalar operators [11], where a generalized scalar operator T is regular if it has a spectral distribution all of whose values lie in the bicommutant of T.

Let $m \ge 1$ be an integer. If $s \ge 0$, then the space $L_1^\vee(s, \mathbf{R}^m)$ consists of the inverse Fourier transforms $f = \check{g}$ of those measurable functions $g : \mathbf{R}^m \to \mathbf{C}$ for which $\int_{\mathbf{R}^m} (1+|\xi|)^s |g(\xi)|\, d\xi$ is finite. We shall write $\hat{f}$ for g. The Fourier inversion formula being used is

$$f(x) = (2\pi)^{-m} \int_{\mathbf{R}^m} e^{i\langle x, \xi \rangle} \hat{f}(\xi)\, d\xi, \quad x \in \mathbf{R}^m.$$

Note that $L_1^\vee(s, \mathbf{R}^m)$ is a Banach algebra with respect to pointwise addition and multiplication [10, §8], where the norm is defined by

$$\|f\| = (2\pi)^{-m} \int_{\mathbf{R}^m} (1+|\xi|)^s |\hat{f}(\xi)|\, d\xi, \quad f \in L_1^\vee(s, \mathbf{R}^m).$$

THEOREM 1. *Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be commuting m-tuples of elements from $L(X)$ and $L(Y)$, respectively, such that $d(\mathrm{Sp}(\underset{\sim}{A}), \mathrm{Sp}(\underset{\sim}{B})) \geq \delta > 0$.*

(i) *If $\underset{\sim}{U}$ is an m-tuple of elements from the Banach space $Z = L(Y, X)$, then there exists a solution $Q \in Z$ of the system*

$$(1.1) \qquad A_j Q - QB_j = U_j, \quad 1 \leq j \leq m,$$

if and only if the compatibility conditions

$$(1.4) \qquad A_j U_k - U_k B_j = A_k U_j - U_j B_k, \quad 1 \leq k < j \leq m$$

are satisfied. In this case Q is unique.

(ii) *Suppose that $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are strongly commuting m-tuples which have commuting partitions $\mathbf{T}(\underset{\sim}{A})$ and $\mathbf{T}(\underset{\sim}{B})$ of type (α_1, s_1) and (α_2, s_2) with respect to δ. If $Q \in Z$ is a solution of the system (1.1), then*

$$(2.3) \qquad \|Q\| \leq \alpha_1 \alpha_2 \delta^{-1} e_{m,s} \|\underset{\sim}{U}\|_{Y \to X^m},$$

where $s = s_1 + s_2$ and

$$(2.4) \qquad e_{m,s} = \inf\left\{(2\pi)^{-2m} \int_{\mathbf{R}^{2m}} (1+|\xi|)^s \left(\sum_{j=1}^m |\hat{f}_j(\xi)|^2\right)^{1/2} d\xi\right\},$$

the infimum being taken over all functions $f: \mathbf{R}^{2m} \to \mathbf{C}^m$ whose components f_j, $1 \leq j \leq m$, are elements of $L_1^\vee(s, \mathbf{R}^{2m})$ and satisfy

$$(2.5) \qquad f_j(x, y) = (x_j - iy_j)\left(\sum_{j=1}^m (x_j^2 + y_j^2)\right)^{-1}$$

whenever $(x, y) \in \mathbf{R}^m \times \mathbf{R}^m$ and $\sum_{j=1}^m (x_j^2 + y_j^2) > (1 - \epsilon)$ for some $0 < \epsilon < 1$.

REMARK 1. (i) The existence of functions satisfying (2.5) and belonging to $L_1^\vee(s, \mathbf{R}^{2m})$ can be established as for the case $m = 1$, $s = 0$ in [1].

(ii) If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are commuting m-tuples of normal operators in Hilbert spaces, then the choices $A_{j1} = \frac{1}{2}(A_j + A_j^*)$ and $A_{j2} = \frac{1}{2}i(A_j^* - A_j)$, $1 \leq j \leq m$, for $\mathbf{T}(\underset{\sim}{A})$ and (similarly) $B_{j1} = \frac{1}{2}(B_j + B_j^*)$ and $B_{j2} = \frac{1}{2}i(B_j^* - B_j)$, $1 \leq j \leq m$, for $\mathbf{T}(\underset{\sim}{B})$ produce commuting partitions (by Fuglede's theorem) of self-adjoint operators, and hence $\alpha_1 = \alpha_2 = 1$ and $s = 0$. The estimate (2.3) then reduces to

$$\|Q\| \leq e_m \delta^{-1} \|\underset{\sim}{U}\|_{Y \to X^m},$$

where $e_m = e_{m,0}$ depends only on m and, in the case of $m = 1$, coincides with the constant c in (1.2).

(iii) It will become apparent from the proof of Theorem 1 that the norm $\|\cdot\|_{Y \to X^m}$ in (2.3) can be replaced by any norm $|||\cdot|||$ in $L(Y, X)^m$ satisfying

$$(2.6) \qquad \|\langle \alpha, \underset{\sim}{U} \rangle\| = \left\| \sum_{j=1}^m \alpha_j U_j \right\| \leq |\alpha|\, |||\underset{\sim}{U}|||, \quad \underset{\sim}{U} \in L(Y, X)^m,$$

for each $\alpha \in \mathbf{C}^m$, where $|\alpha|$ denotes the Euclidean norm of α. For example, we could take $\|\underset{\sim}{U}\| = (\sum_{j=1}^m \|U_j\|^2)^{1/2}$.

(iv) It will also become apparent from the proof of Theorem 1 that if the partitions $\mathbf{T}(\underset{\sim}{A}) = \underset{\sim}{V}$ and $\mathbf{T}(\underset{\sim}{B}) = \underset{\sim}{W}$ are such that $V_{j2} = 0 = W_{j2}$ whenever $0 \le l < j \le m$ and satisfy the estimates $\|e^{i\langle \xi, \underset{\sim}{V}\rangle}\| \le \alpha_1(1+\delta|\xi|)^{s_1}$ and $\|e^{i\langle \xi, \underset{\sim}{W}\rangle}\| \le \alpha_2(1+\delta|\xi|)^{s_2}$, for each $\xi \in \mathbf{R}^m \times \mathbf{R}^l$, then the constant $e_{m,s}$ given by (2.4) can be replaced by one defined using m-tuples of functions $f_j : \mathbf{R}^{m+l} \to \mathbf{C}$, $1 \le j \le m$, from $L_1^\vee(s, \mathbf{R}^{m+l})$ such that

$$f_j(x,y) = \begin{cases} (x_j - iy_j)|(x,y)|^{-2}, & 1 \le j \le l, \\ x_j|(x,y)|^{-2}, & l < j \le m. \end{cases} \tag{2.7}$$

In the special case when $l = 0$, let us denote this constant by $c_{m,s}$. Then we have the following result.

THEOREM 2. *Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be commuting m-tuples of operators with real spectra (in X and Y, respectively) such that $d(\mathrm{Sp}(\underset{\sim}{A}), \mathrm{Sp}(\underset{\sim}{B})) \ge \delta > 0$. Suppose that $\underset{\sim}{A}$ is of type (α_1, s_1) with respect to δ and $\underset{\sim}{B}$ is of type (α_2, s_2) with respect to δ. Let $\underset{\sim}{U} \in L(Y,X)^m$. If $Q \in L(Y,X)$ is the solution of the system* (1.1) and $s = s_1 + s_2$, *then*

$$\|Q\| \le \alpha_1 \alpha_2 \delta^{-1} c_{m,s} \|\underset{\sim}{U}\|_{Y \to X^m}. \tag{2.8}$$

REMARK 2. If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ generate bounded groups, then $s_1 = 0 = s_2$ and $c_m = c_{m,0}$ depends only on m. In this case the estimate (2.8) reduces to (1.5). Still more specifically, if $\underset{\sim}{A}$ and $\underset{\sim}{B}$ consist of Hermitian operators [5, Ch. 4], in which case $s_1 = 0 = s_2$ and $\alpha_1 = 1 = \alpha_2$ [5, Thm. 4.7], then (2.8) reduces to

$$\|Q\| \le c_m \delta^{-1} \|\underset{\sim}{U}\|_{Y \to X^m}$$

which is of the form (1.2) when $m = 1$. It is known [15] that $c_1 = \frac{1}{2}\pi$. We remark that Hermitian operators in Hilbert spaces are just self-adjoint operators [5, Thm. 7.23]. However, in Banach spaces (even reflexive ones) they need not even be spectral operators [5, p. 195].

Theorem 1 is a consequence of the following results from [10].

Let Z be a Banach space and $n \ge 1$ an integer. Let $\underset{\sim}{T}$ be a commuting n-tuple of elements from $L(Z)$ such that

$$\|e^{i\langle \xi, \underset{\sim}{T}\rangle}\| \le M(1+|\xi|)^s, \quad \xi \in \mathbf{R}^n,$$

for some $M \ge 1$ and $s \ge 0$. Then the linear map $\Phi : L_1^\vee(s, \mathbf{R}^n) \to L(Z)$ defined by

$$\Phi(f) = (2\pi)^{-n} \int_{\mathbf{R}^n} \hat{f}(\xi) e^{i\langle \xi, \underset{\sim}{T}\rangle} d\xi, \quad f \in L_1^\vee(s, \mathbf{R}^n), \tag{2.9}$$

is a functional calculus for $\underset{\sim}{T}$ in the sense of [10]. In particular, Φ is a continuous (multiplicative) homomorphism of the Banach algebra $L_1^\vee(s, \mathbf{R}^n)$ into $L(Z)$ [10, §§8, 9]. Furthermore, the support of Φ is precisely $\mathrm{Sp}(\underset{\sim}{T})$. This follows from Theorem 6.2 of [10] and the identity $\mathrm{Sp}(\underset{\sim}{T}) = \gamma(\underset{\sim}{T})$ [10, Cor. 10.2], where

$$\gamma(\underset{\sim}{T}) = \left\{\lambda \in \mathbf{R}^n;\, 0 \in \sigma\left(\sum_{j=1}^{n} (T_j - \lambda_j I)^2\right)\right\}.$$

We remark that the integral (2.9) exists as a Bochner integral in the Banach space $L(Z)$. Indeed, for each $f \in L_1^\vee(s, \mathbf{R}^n)$, the integrand is strongly measurable [6, §3.5] as an $L(Z)$-valued function (using the continuity of $\xi \mapsto e^{i\langle \xi, \underset{\sim}{T}\rangle}$) and

$$\int_{\mathbf{R}^n} \|\hat{f}(\xi) e^{i\langle \xi, \underset{\sim}{T}\rangle}\|\, d\xi \le M(2\pi)^n \|f\| < \infty.$$

Accordingly, $\|\Phi\| \le M$. Furthermore, for each $f \in L_1^\vee(s, \mathbf{R}^n)$ and $z \in Z$ the Z-valued function $\xi \mapsto \hat{f}(\xi) e^{i\langle \xi, \underset{\sim}{T}\rangle} z$ is also Bochner integrable and its integral over $\mathbf{R}^n$ is $(2\pi)^n \Phi(f) z$.

Let $B(\delta)$ denote the open unit ball of $\mathbf{C}^m$ with center zero and radius $\delta > 0$. The following result is Theorem 11.1 of [10].

PROPOSITION 1. *Let $\underset{\sim}{T}$ be a commuting m-tuple of elements from $L(Z)$ such that* $\mathrm{Sp}(\underset{\sim}{T}) \cap B(\delta) = \varnothing$ *for some $\delta > 0$. Let $\underset{\sim}{z} \in Z^m$. Then the system of equations*

$$T_j q = z_j, \quad 1 \le j \le m, \tag{2.10}$$

has a solution $q \in Z$ if and only if $T_j z_k = T_k z_j$ for all j, k. In this case q is unique.

Suppose further that $T_j = M_j + iN_j$, $1 \le j \le m$, where the operators M_j and N_j, $1 \le j \le m$, all commute with each other, and that

$$\|e^{i\langle \lambda, (\underset{\sim}{M}, \underset{\sim}{N})\rangle}\| \le \kappa(1 + \delta|\lambda|)^s, \quad \lambda \in \mathbf{R}^{2m}, \tag{2.11}$$

for some $\kappa \ge 1$ and $s \ge 0$. Then the solution q satisfies the estimate

$$\|q\| \le \kappa e_{m,s} \delta^{-1} |||\underset{\sim}{z}|||, \tag{2.12}$$

where $e_{m,s}$ is given by (2.4) *and $|||\cdot|||$ is any norm in Z^m satisfying an inequality of the type* (2.6).

REMARK 3. An examination of the proof of Proposition 1 (given in [10]) shows that the solution q (when $\delta = 1$) is given by

$$q = (2\pi)^{-2m} \int_{\mathbf{R}^{2m}} e^{i\langle \lambda, (\underset{\sim}{M}, \underset{\sim}{N})\rangle} \sum_{j=1}^{m} \hat{f}_j(\lambda) z_j\, d\lambda = \sum_{j=1}^{m} \Phi(f_j) z_j, \tag{2.13}$$

where $\Phi: L_1^\vee(s, \mathbf{R}^{2m}) \to L(Z)$ is defined by (2.9) with $\underset{\sim}{T} = (\underset{\sim}{M}, \underset{\sim}{N})$ and $n = 2m$, and the functions f_j, $1 \le j \le m$, are elements of $L_1^\vee(s, \mathbf{R}^{2m})$ satisfying (2.5). It is clear that (2.12) follows from (2.13) by properties of the Bochner integral and (2.6). It is also clear from (2.13) that if $N_j = 0$ for each $0 \le l < j \le m$ and if (2.11) holds in this case whenever $\lambda \in \mathbf{R}^m \times \mathbf{R}^l$, then $e_{m,s}$ can be replaced by the constant defined using functions f_j, $1 \le j \le m$, which satisfy (2.7). The case when $\delta \ne 1$ follows by a scaling argument as in [9].

Proof of the Theorems 1 & 2. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ satisfy the hypotheses of Theorem 1 and let $Z = L(Y, X)$. Let $\underset{\sim}{T} = (T_j)$ denote the commuting m-tuple of elements from $L(Z)$ defined by $T_j(Q) = A_j Q - Q B_j$ (cf. (1.3)). The idea of the proof of Theorem 1 is to establish the inclusion

$$\text{(2.14)} \qquad \mathrm{Sp}(\underset{\sim}{T}) \subseteq \mathrm{Sp}(\underset{\sim}{A}) - \mathrm{Sp}(\underset{\sim}{B}),$$

for it then follows from the hypothesis $d(\mathrm{Sp}(\underset{\sim}{A}), \mathrm{Sp}(\underset{\sim}{B})) \geq \delta > 0$ that

$$\mathrm{Sp}(\underset{\sim}{T}) \cap B(\delta) = \varnothing.$$

Theorem 1(i) then follows from the first part of Proposition 1 applied in $Z = L(Y, X)$. Theorem 1(ii) will also follow from Proposition 1 once an estimate of the form (2.11) is established for suitable $(\underset{\sim}{M}, \underset{\sim}{N})$. Finally, Remark 1(iii) and (iv), and hence also Theorem 2, then follow from Remark 3 applied to the present set-up.

So, our first aim is to establish (2.14). Let

$$\mathbf{T}(\underset{\sim}{A}) = (A_{11}, \ldots, A_{m1}, A_{12}, \ldots, A_{m2}) \quad \text{and} \quad \mathbf{T}(\underset{\sim}{B}) = (B_{11}, \ldots, B_{m1}, B_{12}, \ldots, B_{m2})$$

be partitions as in Theorem 1. Define elements L_{jk} and R_{jk} of $L(Z)$, for each $1 \leq j \leq m$ and $k \in \{1, 2\}$, by

$$L_{jk}(Q) = A_{jk} Q \quad \text{and} \quad R_{jk} = Q B_{jk}, \quad Q \in Z.$$

Then define a commuting $2m$-tuple $(\underset{\sim}{M}, \underset{\sim}{N})$ in $L(Z)^{2m}$ by $M_j = L_{j1} - R_{j1}$ and $N_j = L_{j2} - R_{j2}$, $1 \leq j \leq m$, in which case we have $T_j = M_j + iN_j$ for each $j = 1, 2, \ldots, m$. Finally, let $\underset{\sim}{S}$ denote the commuting $4m$-tuple $(\underset{\sim}{L}_1, \underset{\sim}{L}_2, \underset{\sim}{R}_1, \underset{\sim}{R}_2)$ where $\underset{\sim}{L}_1 = (L_{11}, \ldots, L_{m1})$, $\underset{\sim}{L}_2 = (L_{12}, \ldots, L_{m2})$, and $\underset{\sim}{R}_1$ and $\underset{\sim}{R}_2$ are defined similarly.

LEMMA 1. *With the above notation it is the case that*

$$\text{(2.15)} \qquad \gamma(\underset{\sim}{S}) \subseteq \gamma(\mathbf{T}(\underset{\sim}{A})) \times \gamma(\mathbf{T}(\underset{\sim}{B})) = \{(u, v) : u \in \gamma(\mathbf{T}(\underset{\sim}{A})),\ v \in \gamma(\mathbf{T}(\underset{\sim}{B}))\}.$$

If $X = Y$, then this inclusion is actually an equality.

To establish Lemma 1 we will require the following two facts. The first follows from Taylor's spectral mapping theorem [16] applied to the polynomial $\psi : \mathbf{C}^m \to \mathbf{C}$ given by $\psi(z) = \sum_{j=1}^{m} z_j$. The second fact is essentially Theorem 3.1 and Corollary 3.3 of [14]; the proofs given there for $\mathcal{B}$ a Banach algebra can be adapted to the situation where $\mathcal{B}$ is the Banach space $L(Y, X)$ and $T(Q) = AQ - QB$ for $Q \in L(Y, X)$.

Fact 1. Let E be a Banach space and $\underset{\sim}{W}$ be a commuting m-tuple of elements from $L(E)$ such that $\sigma(W_j) \subseteq [0, \infty)$ for each $j = 1, \ldots, m$. Then also $\sigma(\sum_{j=1}^{m} W_j) \subseteq [0, \infty)$.

Fact 2. Let X and Y be Banach spaces, $G \in L(X)$ and $H \in L(Y)$. Then the equation $GQ - QH = U$ is well-posed (i.e., for every $U \in L(Y, X)$ there is a unique $Q \in L(Y, X)$ such that $GQ - QH = U$) whenever $\sigma(G) \cap \sigma(H) = \varnothing$.

REMARK 4. If $X = Y$, then it follows from a result of Kleinecke [8, Thm. 10] that the well-posedness of $GQ - QH = U$ is actually equivalent to

$$\sigma(G) \cap \sigma(H) = \varnothing.$$

Proof of Lemma 1. By definition, an element $(u, v) \in \mathbf{R}^{2m} \times \mathbf{R}^{2m}$ belongs to $\gamma(\underset{\sim}{S})$ if and only if

$$\sum_{j=1}^{m} (L_{j1}-u_j)^2+(L_{j2}-u_{j+m})^2+(R_{j1}-v_j)^2+(R_{j2}-v_{j+m})^2$$

is not invertible in $L(Z)$, that is, if and only if

$$\left(\sum_{j=1}^{m} (A_{j1}-u_j)^2+(A_{j2}-u_{j+m})^2\right)Q + Q\left(\sum_{j=1}^{m} (B_{j1}-v_j)^2+(B_{j2}-v_{j+m})^2\right)=U \tag{2.16}$$

is not well-posed in Z which, by Fact 2, implies that

$$\sigma\left(\sum_{j=1}^{m} (A_{j1}-u_j)^2+(A_{j2}-u_{j+m})^2\right) \cap \sigma\left(-\sum_{j=1}^{m} (B_{j1}-v_j)^2+(B_{j2}-v_{j+m})^2\right)\neq \varnothing. \tag{2.17}$$

However, by Fact 1 this is possible if and only if the intersection in (2.17) is precisely $\{0\}$, that is, if and only if $u\in\gamma(\mathbf{T}(\underset{\sim}{A}))$ and $v\in\gamma(\mathbf{T}(\underset{\sim}{B}))$. This establishes (2.15). In the case where $X=Y$ the ill-posedness of (2.16) is actually equivalent to (2.17) (by Remark 4) and we then have equality in (2.15). □

The proof of (2.14) now follows easily. Indeed, applying Taylor's spectral mapping theorem to the coordinate projections of $\mathbf{C}^{4m}$ onto $\mathbf{C}$ and noting that

$$\sigma(L_{jk})=\sigma(A_{jk}) \quad \text{and} \quad \sigma(R_{jk})=\sigma(B_{jk}) \quad \text{for all } 1\le j\le m,\ 1\le k\le 2,$$

it follows that

$$\mathrm{Sp}(\underset{\sim}{S})\subseteq \prod_{j=1}^{m}\sigma(A_{j1})\times\prod_{j=1}^{m}\sigma(A_{j2})\times\prod_{j=1}^{m}\sigma(B_{j1})\times\prod_{j=1}^{m}\sigma(B_{j2})\subseteq\mathbf{R}^{4m}.$$

Accordingly, $\mathrm{Sp}(\underset{\sim}{S})=\gamma(\underset{\sim}{S})$ by Proposition 10.1 of [10]. It follows from Lemma 1 that

$$\mathrm{Sp}(\underset{\sim}{S})\subseteq\gamma(\mathbf{T}(\underset{\sim}{A}))\times\gamma(\mathbf{T}(\underset{\sim}{B})). \tag{2.18}$$

Noting that $\underset{\sim}{M}=\underset{\sim}{L}_1-\underset{\sim}{R}_1$ and $\underset{\sim}{N}=\underset{\sim}{L}_2-\underset{\sim}{R}_2$ it follows from Taylor's spectral mapping theorem [16, Thm. 4.8] that $\mathrm{Sp}((\underset{\sim}{M},\underset{\sim}{N}))=\mathrm{Sp}(\psi(\underset{\sim}{S}))=\psi(\mathrm{Sp}(\underset{\sim}{S}))$, where ψ: $\mathbf{C}^{4m}\to\mathbf{C}^{2m}$ is defined by $\psi(u,v)=u-v$ for each $(u,v)\in\mathbf{C}^{2m}\times\mathbf{C}^{2m}$. Then (2.18) implies that

$$\mathrm{Sp}((\underset{\sim}{M},\underset{\sim}{N}))\subseteq\gamma(\mathbf{T}(\underset{\sim}{A}))-\gamma(\mathbf{T}(\underset{\sim}{B})). \tag{2.19}$$

If $p:\mathbf{C}^{2m}\to\mathbf{C}^{m}$ is defined by $p(z)=(z_1+iz_{m+1},\dots,z_m+iz_{2m})$ for each $z\in\mathbf{C}^{2m}$, then Theorem 10.8 of [10] states that $p(\gamma(\mathbf{T}(\underset{\sim}{A})))=\mathrm{Sp}(\underset{\sim}{A})$ and $p(\gamma(\mathbf{T}(\underset{\sim}{B})))=\mathrm{Sp}(\underset{\sim}{B})$. Combining these identities with $\underset{\sim}{T}=p(\underset{\sim}{M},\underset{\sim}{N})$ it follows from (2.19), linearity of p, and the spectral mapping theorem that

$$\mathrm{Sp}(\underset{\sim}{T})=p(\mathrm{Sp}((\underset{\sim}{M},\underset{\sim}{N})))\subseteq p(\gamma(\mathbf{T}(\underset{\sim}{A})))-p(\gamma(\mathbf{T}(\underset{\sim}{B})))=\mathrm{Sp}(\underset{\sim}{A})-\mathrm{Sp}(\underset{\sim}{B}),$$

which is the desired inclusion (2.14). □

REMARK 5. If $X=Y$, then we have shown that (2.14) is actually an equality.

Thus, Theorem 1(i) is established. Assume now that $\mathbf{T}(\underset{\sim}{A})$ and $\mathbf{T}(\underset{\sim}{B})$ satisfy the additional hypotheses of Theorem 1(ii). If $\lambda \in \mathbf{R}^{2m}$, then it follows (from the power series expansion of the exponential function, for example) that

$$(e^{i\langle\lambda,(\underset{\sim}{M},\underset{\sim}{N})\rangle})(Q)=e^{i\langle\lambda,\mathbf{T}(\underset{\sim}{A})\rangle}Qe^{-i\langle\lambda,\mathbf{T}(\underset{\sim}{B})\rangle}, \quad Q\in Z. \tag{2.20}$$

It follows from this identity and the assumptions on $\mathbf{T}(\underset{\sim}{A})$ and $\mathbf{T}(\underset{\sim}{B})$ that

$$\|e^{i\langle\lambda,(\underset{\sim}{M},\underset{\sim}{N})\rangle}\|\leq\alpha_1\alpha_2(1+\delta|\lambda|)^{s_1+s_2}, \quad \lambda\in\mathbf{R}^{2m}.$$

This is precisely an estimate of the form (2.11) needed to apply Proposition 1, and so (2.3) follows from (2.12). This completes the proofs of Theorems 1 and 2. □

EXAMPLE 1. It was noted earlier that if $\underset{\sim}{T}$ is a commuting m-tuple of regular generalized scalar operators then $\underset{\sim}{T}$ is strongly commuting. Actually more is true; there exists a commuting partition $\mathbf{T}(\underset{\sim}{T})$ each of whose components is a generalized scalar operator with real spectrum (cf. [2, Ch. 4, Lemma 6.1] and [11, Prop. 4]). It then follows that given any $\delta>0$ there exist constants $\alpha\geq 1$ and $s\geq 0$ such that $\mathbf{T}(\underset{\sim}{T})$ is of type (α, s) with respect to δ [2, Ch. 5, Thm. 4.5]. We remark that spectral operators of finite type, which include all linear operators in finite-dimensional spaces, are regular generalized scalar operators; see Theorem 3.6 and Example 3.12 in Chapter 4 of [2].

3. Symmetric norms. The aim of this section is to establish estimates of the type given in Theorem 1 for symmetric norms. Throughout X and Y are Banach spaces and $m\geq 1$. A subspace $\mathfrak{M}$ of $L(Y,X)$ is a symmetric normed space if it is equipped with a norm $|\cdot|$ (with respect to which $\mathfrak{M}$ is complete) satisfying $\|S\|\leq|S|$ for each $S\in\mathfrak{M}$, and such that $R\in L(X)$, $S\in\mathfrak{M}$, and $T\in L(Y)$ implies $RST\in\mathfrak{M}$ and $|RST|\leq\|R\|\,|S|\,\|T\|$. Examples are the spaces of nuclear operators from Y to X and the absolutely r-summing operators from Y to X, $1\leq r<\infty$, each equipped with their usual norm. If X and Y are Hilbert spaces, then the Schatten p-classes, $1\leq p<\infty$, with their standard norm are also symmetric normed spaces; the cases $p=1$ and $p=2$ correspond to the trace class operators and Hilbert–Schmidt operators, respectively.

THEOREM 3. *Let* $(\mathfrak{M},|\cdot|)$ *be a symmetric normed subspace of* $L(Y,X)$. *Let* $\underset{\sim}{A}$ *and* $\underset{\sim}{B}$ *be commuting m-tuples of elements from* $L(X)$ *and* $L(Y)$, *respectively, such that* $d(\mathrm{Sp}(\underset{\sim}{A}),\mathrm{Sp}(\underset{\sim}{B}))\geq\delta>0$. *Suppose that, for the usual bound norm,* $\mathbf{T}(\underset{\sim}{A})$ *is a commuting partition in* $L(X)^{2m}$ *of type* (α_1,s_1) *with respect to* δ *and that* $\mathbf{T}(\underset{\sim}{B})$ *is a commuting partition in* $L(Y)^{2m}$ *of type* (α_2,s_2) *with respect to* δ. *If* $\underset{\sim}{U}$ *is an m-tuple of elements from* $\mathfrak{M}$, *then there exists a solution* Q *of the system* (1.1), *necessarily belonging to* $\mathfrak{M}$, *if and only if the compatibility conditions* (1.4) *are satisfied. In this case* Q *is unique and*

$$|Q|\leq\alpha_1\alpha_2\delta^{-1}c_{m,s}|||\underset{\sim}{U}|||, \tag{3.1}$$

where $s=s_1+s_2$ *and* $|||\cdot|||$ *is any norm in* $\mathfrak{M}^m$ *satisfying*

$$(3.2)\qquad \sum_{j=1}^{m} |\nu_j|\,|U_j| \le |\nu|\,|\!|\!|\underset{\sim}{U}|\!|\!|, \quad \nu \in \mathbf{C}^m.$$

The proof of Theorem 3 requires the following result.

LEMMA 2. *Let $(\mathfrak{M}, |\cdot|)$ be a symmetric normed subspace of $L(Y,X)$. If $f\colon \mathbf{R}^m \to \mathfrak{M}$ is Bochner integrable in the Banach space $\mathfrak{M}$, then f is also Bochner integrable with respect to the usual bound norm when considered as an $L(Y,X)$-valued function and $\int_{\mathbf{R}^m}^{\mathfrak{M}} f\,d\mu = \int_{\mathbf{R}^m} f\,d\mu$. Here μ is Lebesgue measure in $\mathbf{R}^m$, $\int_{\mathbf{R}^m}^{\mathfrak{M}}$ denotes the integral in $\mathfrak{M}$, and $\int_{\mathbf{R}^m}$ denotes the integral in $L(Y,X)$.*

Proof. Let $P = \{\xi \in \mathbf{R}^m : |f(\xi)| > 0\}$. If $n > 0$ then there exists a decomposition of P into disjoint measurable sets $\{E_k^{(n)}\}_{k=1}^{\infty}$ such that, for arbitrary points $\xi_k^{(n)} \in E_k^{(n)}$, the function f_n defined by $f_n(\xi) = f(\xi_k^{(n)})$ if $\xi \in E_k^{(n)}$ $(k = 1, 2, \ldots)$ and by $f_n(\xi) = 0$ if $\xi \notin P$ is $\mathfrak{M}$-Bochner integrable and satisfies $\int_{\mathbf{R}^m} |f - f_n|\,d\mu < n^{-1}$ [6, p. 81, Corollary]. It is clear that each f_n is strongly measurable as a function with values in the Banach space $L(Y,X)$. Since

$$\int_{\mathbf{R}^m} \|f_n\|\,d\mu \le \int_{\mathbf{R}^m} |f_n|\,d\mu < \infty,$$

it follows that f_n is $L(Y,X)$-Bochner integrable. By disjointness of the sets $E_k^{(n)}$ we have

$$(3.3)\qquad \int_{\mathbf{R}^m}^{\mathfrak{M}} f_n\,d\mu = \sum_{k=1}^{\infty} f(\xi_k^{(n)})\mu(E_k^{(n)}) = \lim_{N\to\infty} \sum_{k=1}^{N} f(\xi_k^{(n)})\mu(E_k^{(n)}),$$

where convergence is with respect to $|\cdot|$. Since f_n is also $L(Y,X)$-Bochner integrable it follows similarly that

$$(3.4)\qquad \int_{\mathbf{R}^m} f_n\,d\mu = \lim_{N\to\infty} \sum_{k=1}^{N} f(\xi_k^{(n)})\mu(E_k^{(n)}),$$

where convergence is with respect to $\|\cdot\|$. Since each partial sum

$$\sum_{k=1}^{N} f(\xi_k^{(n)})\mu(E_k^{(n)}), \quad N = 1, 2, \ldots,$$

belongs to $\mathfrak{M}$, it follows from (3.3), (3.4), and properties of the symmetric norm that $\int_{\mathbf{R}^m}^{\mathfrak{M}} f_n\,d\mu = \int_{\mathbf{R}^m} f_n\,d\mu$ for every $n \ge 1$. Accordingly,

$$\left\| \int_{\mathbf{R}^m}^{\mathfrak{M}} f\,d\mu - \int_{\mathbf{R}^m} f\,d\mu \right\| \le \left| \int_{\mathbf{R}^m}^{\mathfrak{M}} f\,d\mu - \int_{\mathbf{R}^m}^{\mathfrak{M}} f_n\,d\mu \right| + \left\| \int_{\mathbf{R}^m} f\,d\mu - \int_{\mathbf{R}^m} f_n\,d\mu \right\| < \frac{2}{n}$$

for every $n = 1, 2, \ldots$, from which the desired conclusion follows. □

Proof of Theorem 3. The estimates

$$(3.5)\qquad |e^{i\langle\lambda, \mathrm{T}(\underset{\sim}{A})\rangle} U e^{-i\langle\lambda, \mathrm{T}(\underset{\sim}{B})\rangle}| \le \|e^{i\langle\lambda, \mathrm{T}(\underset{\sim}{A})\rangle}\|\,|U|\,\|e^{-i\langle\lambda, \mathrm{T}(\underset{\sim}{B})\rangle}\|,$$

valid for each $\lambda \in \mathbf{R}^{2m}$ and $U \in \mathfrak{M}$, show that the $\mathfrak{M}$-valued function

$$g(\lambda) = e^{i\langle\lambda, \mathrm{T}(\underset{\sim}{A})\rangle} U e^{-i\langle\lambda, \mathrm{T}(\underset{\sim}{B})\rangle}, \quad \lambda \in \mathbf{R}^{2m},$$

is continuous. Accordingly, for each $\psi \in L_1^\vee(s, \mathbf{R}^{2m})$, the $\mathfrak{M}$-valued function $f = g\hat{\psi}$ is weakly measurable (i.e., $\langle f(\cdot), \nu\rangle$ is measurable for each ν in the dual space of $\mathfrak{M}$) and has separable range, and hence is strongly measurable in $\mathfrak{M}$ [6, Thm. 3.5.3]. Then the estimates assumed for $\mathbf{T}(\underset{\sim}{A})$ and $\mathbf{T}(\underset{\sim}{B})$ with respect to the bound norm in $L(Y,X)$, together with (3.5), imply that f is $\mathfrak{M}$-Bochner integrable. Therefore, whenever a solution Q of (1.1) exists (assuming $\delta = 1$) it follows from the substitution of (2.20) into (2.13) and Lemma 2 that

$$Q = (2\pi)^{-2m} \sum_{j=1}^{m} \int_{\mathbf{R}^{2m}}^{\mathfrak{M}} e^{i\langle \lambda, \mathbf{T}(\underset{\sim}{A})\rangle} U_j e^{-i\langle \lambda, \mathbf{T}(\underset{\sim}{B})\rangle} \hat{f}_j(\lambda)\, d\lambda.$$

In particular, $Q \in \mathfrak{M}$. Furthermore, (3.1) then follows from (3.2), (3.5), and the definition of $e_{m,s}$. Since we have shown that whenever a solution Q exists in $L(Y,X)$ it actually belongs to $\mathfrak{M}$ (assuming that $\underset{\sim}{U} \in \mathfrak{M}^m$), the first part of Theorem 3 follows from Theorem 1(i). □

4. Applications to perturbation theory. The purpose of this section is to illustrate the use of the results in Section 2 to obtain information concerning the perturbation of spectra and spectral subspaces of certain classes of commuting m-tuples of operators. See [1] for the case $m = 1$ (in the Hilbert space setting).

We first make some comments about the restriction to invariant subspaces of a commuting m-tuple $\underset{\sim}{T}$ (in a Banach space E) which is of type (α, s) with respect to some $\delta > 0$. Thus, let V be a closed subspace of E which is invariant for each operator T_j, $1 \le j \le m$. Then V is also invariant for each operator $e^{i\langle \lambda, \underset{\sim}{T}\rangle}$, $\lambda \in \mathbf{R}^m$, and hence an argument as in the proof of Lemma 2 shows that V is an invariant subspace for each operator

$$f(\underset{\sim}{T}) = (2\pi)^{-m} \int_{\mathbf{R}^m} e^{i\langle \lambda, \underset{\sim}{T}\rangle} \hat{f}(\lambda)\, d\lambda, \quad f \in L_1^\vee(s, \mathbf{R}^m). \tag{4.1}$$

Denote by $\underset{\sim}{T}_V$ the commuting m-tuple in $L(V)^m$ whose jth component is the restriction of T_j to V, $1 \le j \le m$. Then $\underset{\sim}{T}_V$ is also of type (α, s) with respect to δ and the restriction of $f(\underset{\sim}{T})$ to V, denoted by $f(\underset{\sim}{T})_V$, is precisely the operator

$$f(\underset{\sim}{T}_V) = (2\pi)^{-m} \int_{\mathbf{R}^m} e^{i\langle \lambda, \underset{\sim}{T}_V\rangle} \hat{f}(\lambda)\, d\lambda, \quad f \in L_1^\vee(s, \mathbf{R}^m). \tag{4.2}$$

Accordingly, if $\Phi: L_1^\vee(s, \mathbf{R}^m) \to L(E)$ is the functional calculus $\Phi(f) = f(\underset{\sim}{T})$ defined by (4.1) and $\Phi_V: L_1^\vee(s, \mathbf{R}^m) \to L(V)$ is the functional calculus

$$\Phi_V(f) = f(\underset{\sim}{T}_V) = f(\underset{\sim}{T})_V = \Phi(\underset{\sim}{T})_V$$

defined by (4.2), then $\operatorname{Supp}(\Phi_V) \subseteq \operatorname{Supp}(\Phi)$. Here $\operatorname{Supp}(\Phi)$ denotes the support of Φ: it is the smallest closed set $K \subseteq \mathbf{R}^m$ such that $\Phi(f) = 0$ whenever $f \in L_1^\vee(s, \mathbf{R}^m)$ is a function with compact support contained in $\mathbf{R}^m \setminus K$. The support of Φ_V is defined similarly. Observing that

$$\operatorname{Supp}(\Phi) = \gamma(\underset{\sim}{T}) = \operatorname{Sp}(\underset{\sim}{T}) \quad \text{and} \quad \operatorname{Supp}(\Phi_V) = \gamma(\underset{\sim}{T}_V) = \operatorname{Sp}(\underset{\sim}{T}_V)$$

[10, Thm. 6.2 & Cor. 10.2], it follows that

$$\mathrm{Sp}(\underset{\sim}{T}_V) \subseteq \mathrm{Sp}(\underset{\sim}{T}). \tag{4.3}$$

We now list some notation and conditions which are assumed fixed for the remainder of this section. Let X be a Banach space and $\delta > 0$. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be commuting m-tuples in $L(X)^m$ such that $\underset{\sim}{A}$ (resp. $\underset{\sim}{B}$) is of type (α_1, s_1) (resp. (α_2, s_2)) with respect to δ. Given $d > 0$ define

$$\kappa(d) = \alpha_1 \alpha_2 c_{m,s} d^{-1} \max\{1, \delta^s d^{-s}\},$$

where $s = s_1 + s_2$ and $c_{m,s}$ is as in Theorem 2. If X is a Hilbert space and $\underset{\sim}{A}$ and $\underset{\sim}{B}$ consist of self-adjoint operators, then $\kappa(d) = c_m/d$. The following conditions will be assumed at various stages.

Condition (i). There is a closed subspace Y of X which is invariant for each operator B_j, $1 \leq j \leq m$, and K is a closed set in $\mathbf{R}^m$ such that $\mathrm{Sp}(\underset{\sim}{B}_Y) \subseteq K$.

Condition (ii). There is a closed subspace Z of X which is invariant for each operator A_j, $1 \leq j \leq m$, and L is a closed set in $\mathbf{R}^m$ such that $\mathrm{Sp}(\underset{\sim}{A}_Z) \subseteq L$.

LEMMA 3. *Suppose that conditions* (i) *and* (ii) *hold. Let d satisfy $0 < d \leq d(K, L)$ and let $S \in L(X)$ be an operator commuting with each A_j, $1 \leq j \leq m$, such that $R(S) \subseteq Z$ where $R(S) = \{Sx : x \in X\}$. If S_Y denotes the restriction of S to Y, considered as an element of $L(Y, Z)$, then*

$$\|S_Y\| \leq \kappa(d) \|S\| \|\underset{\sim}{A} - \underset{\sim}{B}\|_{X \to X^m}.$$

Proof. Let $J_Y \in L(Y, X)$ denote the natural inclusion of Y into X. Then the operator $Q = SJ_Y \in L(Y, Z)$ is a solution of the system of operator equations (notation is obvious)

$$\underset{\sim}{A}_Z Q - Q \underset{\sim}{B}_Y = S(\underset{\sim}{A} - \underset{\sim}{B}) J_Y \tag{4.4}$$

and $d(\mathrm{Sp}(\underset{\sim}{A}_Z), \mathrm{Sp}(\underset{\sim}{B}_Y)) \geq d$. Thus, if $d \geq \delta$, then Theorem 2 applied to (4.4) in $L(Y, Z)$ yields

$$\|S_Y\| = \|SJ_Y\| \leq \alpha_1 \alpha_2 c_{m,s} d^{-1} \|S\| \|\underset{\sim}{A} - \underset{\sim}{B}\|_{X \to X^m}.$$

If $d < \delta$, then $\underset{\sim}{A}$ (resp. $\underset{\sim}{B}$) is of type $(\alpha_1 \delta^{s_1} d^{-s_1}, s_1)$ (resp. $(\alpha_2 \delta^{s_2} d^{-s_2}, s_2)$) with respect to d. So, applying Theorem 2 again to (4.4) in $L(Y, Z)$ but now considering $\underset{\sim}{A}$ and $\underset{\sim}{B}$ as having their type with respect to d gives

$$\|S_Y\| = \|SJ_Y\| \leq \alpha_1 \alpha_2 c_{m,s} \delta^s \|\underset{\sim}{A} - \underset{\sim}{B}\|_{X \to X^m} \|S\| / d^{s+1}. \qquad \square$$

If V and W are closed subspaces of X, define

$$\Delta(V, W) = \sup\{\inf\{\|v - w\| : w \in W\};\ v \in V,\ \|v\| = 1\}; \tag{4.5}$$

see [7, p. 197]. Then $0 \leq \Delta(V, W) \leq 1$ and $\Delta(V, W) = 0$ if and only if $V \subseteq W$. If X is a Hilbert space, then $\Delta(V, W) \leq \|(I - P_W) P_V\|$, where P_V (resp. P_W) is the orthogonal projection onto V (resp. W).

THEOREM 4. *Suppose that conditions* (i) *and* (ii) *hold, d satisfies $0 < d \leq d(K, L)$, and $\|\underset{\sim}{A} - \underset{\sim}{B}\|_{X \to X^m} \leq \epsilon$. In addition, assume that there is a continuous projection $E(Z)$ in X with range Z such that $A_j E(Z) = E(Z) A_j$, $1 \leq j \leq m$.*

(a) *If* $E(Z)_Y$ *denotes the restriction of* $E(Z)$ *to* Y, *considered as an element of* $L(Y,Z)$, *then* $\|E(Z)_Y\| \le \epsilon\kappa(d)\|E(Z)\|$.
(b) *If* $\epsilon < \kappa(d)^{-1}\|E(Z)\|^{-1}$, *then* $Z \cap Y = \{0\}$.
(c) *If* $\mathfrak{N}(E(Z)) = \{x \in X; E(Z)x = 0\}$, *then* $\Delta(Y, \mathfrak{N}(E(Z))) \le \epsilon\kappa(d)\|E(Z)\|$.

Proof. (a) This follows from Lemma 3 with $S = E(Z)$.

(b) If $\epsilon < \kappa(d)^{-1}\|E(Z)\|^{-1}$, then (a) implies that $\|E(Z)_Y\| < 1$ and hence $Z \cap Y = \{0\}$.

(c) This follows from the inequality

$$\Delta(Y, \mathfrak{N}(E(Z))) \le \sup\{\|y-(y-E(Z)y)\| : y \in Y, \|y\|=1\} = \|E(Z)_Y\|. \qquad \square$$

If X is a Hilbert space and $\underset{\sim}{T} \in L(X)^m$ is a commuting m-tuple of self-adjoint operators, then associated with $\underset{\sim}{T}$ is its joint resolution of the identity which assigns a projection, necessarily an element of the commutant $\{T_j : 1 \le j \le m\}'$, to each closed subset K of $\mathbf{R}^m$. The range of this projection will be called the spectral subspace of $\underset{\sim}{T}$ with respect to K.

COROLLARY 4.1. *Let* X *be a Hilbert space and* $\underset{\sim}{A} \in L(X)^m$ *be a commuting m-tuple of self-adjoint operators. Let d and* ϵ *be positive numbers and suppose that* $\underset{\sim}{B} \in L(X)^m$ *is a commuting m-tuple of type* (α_2, s_2) *with respect to d such that* $\|\underset{\sim}{A}-\underset{\sim}{B}\|_{X\to X^m} \le \epsilon$. *Suppose K is a closed subset of* $\mathbf{R}^m$ *and Y is a closed subspace of X which is invariant for each operator* B_j, $1 \le j \le m$, *such that* $\mathrm{Sp}(B_Y) \subseteq K$. *Then*

$$\Delta(Y, X_d) \le \alpha_2 \epsilon c_{m,s_2}/d,$$

where X_d *is the spectral subspace of* $\underset{\sim}{A}$ *with respect to the closed set* $K + B(d)$. *In particular, if* $\underset{\sim}{B}$ *is also an m-tuple of self-adjoint operators and Y is the spectral subspace of* $\underset{\sim}{B}$ *with respect to K, then*

$$\Delta(Y, X_d) \le \epsilon c_m/d.$$

REMARK 6. To interpret Corollary 4.1 we should think of $\underset{\sim}{A}$ as given, $\underset{\sim}{B}$ as an approximation to $\underset{\sim}{A}$, and Y as a known "spectral subspace" of $\underset{\sim}{B}$ associated with K. The conclusion is that elements of Y are near the spectral subspace of $\underset{\sim}{A}$ corresponding to a set which is slightly larger than K.

Proof of Corollary 4.1. Let F be the projection in the joint resolution of the identity for $\underset{\sim}{A}$ assigned to the closed set $K+B(d)$. Then $X_d = R(F)$. So, put $Z = R(I-F)$ and $E(Z) = I - F$. Then $\mathrm{Sp}(\underset{\sim}{A}_Z) \subseteq L$ and $d(K,L) = d$, where L is the closure of $\mathbf{R}^m \setminus (K + B(d))$. Thus, with $\delta = d$ all the hypotheses of Theorem 4 are satisfied and hence Theorem 4(c) implies, after noting $\alpha_1 = 1$, $s_1 = 0$, and $\|E(Z)| = 1$, that

$$\Delta(Y, X_d) = \Delta(Y, \mathfrak{N}(E(Z))) \le \epsilon\kappa(d) = \alpha_2 \epsilon c_{m,s_2} d^{-1}.$$

If $\underset{\sim}{B}$ is also self-adjoint, then $\alpha_2 = 1$ and $s_2 = 0$. $\square$

LEMMA 4. *Suppose that condition* (i) *holds and* $\|\underset{\sim}{A}-\underset{\sim}{B}\|_{X\to X^m} \le \epsilon$. *Let* $f \in L_1^\vee(s_1, \mathbf{R}^m)$ *have compact support,* $\mathrm{Supp}(f)$, *in* $\mathbf{R}^m$ *and let* $0 < d \le d(K, \mathrm{Supp}(f))$.

If X_f is the closure of $R(f(\underset{\sim}{A}))$, then $f(\underset{\sim}{A})_Y$, considered as an element of $L(Y, X_f)$, satisfies

$$\|f(\underset{\sim}{A})_Y\| \leq \epsilon\kappa(d)\|f(\underset{\sim}{A})\|.$$

Proof. Let $S = f(\underset{\sim}{A})$, $Z = X_f$, and $L = \operatorname{Supp}(f)$. Then it follows that condition (ii) is satisfied, that is, $\operatorname{Sp}(\underset{\sim}{A}_Z) \subseteq L$. The conclusion follows from Lemma 3. □

Concerning perturbation of the Taylor spectrum as a whole we have the following result.

THEOREM 5. *Let d and ϵ be positive numbers. Let $\underset{\sim}{A}$ (resp. $\underset{\sim}{B}$) be a commuting m-tuple in $L(X)^m$ such that $\underset{\sim}{A}$ (resp. $\underset{\sim}{B}$) is of type (α_1, s_1) (resp. (α_2, s_2)) with respect to d and $\|\underset{\sim}{A} - \underset{\sim}{B}\|_{X \to X^m} \leq \epsilon$. If*

$$\alpha_1\alpha_2\epsilon c_{m,s} < d, \tag{4.6}$$

then $\operatorname{Sp}(\underset{\sim}{A}) \subset \operatorname{Sp}(\underset{\sim}{B}) + B(d)$ *and* $\operatorname{Sp}(\underset{\sim}{B}) \subset \operatorname{Sp}(\underset{\sim}{A}) + B(d)$.

Proof. Let $\delta = d$, $K = \operatorname{Sp}(\underset{\sim}{B})$, and $K_d = K + B(d)$. Let $f \in L_1^\vee(s_1, \mathbf{R}^m)$ be any function with compact support satisfying $\operatorname{Supp}(f) \cap K_d = \varnothing$. By Lemma 4 with $Y = X$ we have $\|f(\underset{\sim}{A})\| \leq \epsilon\kappa(d)\|f(\underset{\sim}{A})\|$. But $\kappa(d) = \alpha_1\alpha_2 c_{m,s}/d$ since $\delta = d$, and so $f(\underset{\sim}{A}) = 0$ by (4.6). It follows that $\operatorname{Supp}(\Phi) \subset K_d$, where Φ is given by (4.1) with $\underset{\sim}{T} = \underset{\sim}{A}$, and hence $\operatorname{Sp}(\underset{\sim}{A}) \subset K_d$. A similar argument establishes the other inclusion. □

Suppose now that $\dim(X) < \infty$. In this case, if V and W are subspaces of X then

$$\Delta(V, W) < 1 \quad \text{implies that} \quad \dim(V) \leq \dim(W); \tag{4.7}$$

see [7, p. 200]. If $\underset{\sim}{T}$ is a commuting m-tuple in $L(X)^m$, then $\operatorname{Sp}(\underset{\sim}{T})$ is necessarily a finite set, say $\{\lambda_1, \ldots, \lambda_r\}$ [11, Prop. 7]. Elements of $\operatorname{Sp}(\underset{\sim}{T})$ are called joint eigenvalues of $\underset{\sim}{T}$. There exists a direct sum decomposition $X = X_1 \oplus \cdots \oplus X_r$ where each subspace X_k is invariant for each operator T_j, $1 \leq j \leq m$, and $\operatorname{Sp}(\underset{\sim}{T}_{X_k}) = \{\lambda_k\}$, $1 \leq k \leq r$ [16, Thm. 4.9]. The dimension of X_k is called the multiplicity of λ_k, $1 \leq k \leq r$. So, counting multiplicity, $\underset{\sim}{T}$ has precisely N joint eigenvalues where $N = \dim(X)$.

We require a further definition. Let X be a Banach space (not necessarily finite-dimensional) and $\underset{\sim}{T} \in L(X)^m$ be a commuting m-tuple of type (α, s) with respect to some $\delta > 0$. For each $\rho > 0$ and each subset $V \subseteq \operatorname{Sp}(\underset{\sim}{T})$ let $\mathfrak{F}(V, \rho)$ denote the collection of all $f \in C_c^\infty(\mathbf{R}^m)$ which are 1 in a neighbourhood of V and satisfy $\operatorname{Supp}(f) \subseteq V + B(\rho)$. Then, for $\rho > 0$, define

$$M_\rho(\underset{\sim}{T}) = \sup\{\inf\{\|f(\underset{\sim}{T})\| : f \in \mathfrak{F}(V, \rho)\}; V \subseteq \operatorname{Sp}(\underset{\sim}{T})\}.$$

If X is a Hilbert space and $\underset{\sim}{T}$ consists of self-adjoint operators, then $M_\rho(\underset{\sim}{T}) = 1$ for every $\rho > 0$. In general, $M_\rho(\underset{\sim}{T})$ depends on the type (α, s) of $\underset{\sim}{T}$ and the geometry of the set $\operatorname{Sp}(\underset{\sim}{T})$.

THEOREM 6. *Let $N = \dim(X)$ be finite, d and ϵ be positive numbers and $\underset{\sim}{A}$ be a commuting m-tuple in $L(X)^m$ of type (α_1, s_1) with respect to d. Let $\underset{\sim}{B}$ be a commuting m-tuple in $L(X)^m$ of type (α_2, s_2) with respect to d such that*

$$\|\underset{\sim}{A}-\underset{\sim}{B}\|_{X\to X^m}\le\epsilon.$$

Suppose that $\mathrm{Sp}(\underset{\sim}{A})=\{\lambda_l: 1\le l\le N\}$ *and* $\mathrm{Sp}(\underset{\sim}{B})=\{\mu_l: 1\le l\le N\}$ *are the joint eigenvalues of* $\underset{\sim}{A}$ *and* $\underset{\sim}{B}$ *repeated according to multiplicity. Then, for any* $\rho>0$ *such that*

$$M_\rho(\underset{\sim}{A})\epsilon\alpha_1\alpha_2 c_{m,s}<d, \tag{4.8}$$

there exists a permutation π *of* $\{1,2,\dots,N\}$ *such that*

$$|\mu_l-\lambda_{\pi(l)}|<(d+\rho),\quad 1\le l\le N.$$

REMARK 7. For X a Hilbert space and $\underset{\sim}{A}$ a commuting m-tuple of self-adjoint operators it follows that there exists a permutation π such that

$$|\mu_l-\lambda_{\pi(l)}|<d,\quad 1\le l\le N, \tag{4.9}$$

whenever $\epsilon\alpha_2 c_{m,s_2}<d$. If $\underset{\sim}{B}$ is also an m-tuple of self-adjoint operators, then the conclusion is that there exists a permutation π such that (4.9) holds whenever $\|\underset{\sim}{A}-\underset{\sim}{B}\|\le d/c_m$. This answers a question of Davis [4] as foreshadowed in [9]; the case $m=1$ is discussed in [1].

If $\underset{\sim}{T}\in L(X)^m$ is a commuting m-tuple of type (α,s) with respect to some $\delta>0$ (it is not assumed that $\dim(X)<\infty$), and if $\mathrm{Sp}(\underset{\sim}{T})=K\cup L$ where K and L are non-empty, disjoint, compact sets, then it follows from the $L_1^\vee(s,\mathbf{R}^m)$-functional calculus for $\underset{\sim}{T}$ (cf. (4.1)) that there exist disjoint commuting projections $E(K)$ and $E(L)$ such that $E(K)+E(L)=I$. The range of $E(K)$ (resp. $E(L)$) will be called the spectral subspace of $\underset{\sim}{T}$ with respect to the component K (resp. L) of $\mathrm{Sp}(\underset{\sim}{T})$.

Proof of Theorem 6. Let $\Lambda=\{1,2,\dots,N\}$. Define a relation $\mathcal{R}$ in $\Lambda\times\Lambda$ by $i\mathcal{R}j$ if and only if $|\mu_i-\lambda_j|<(d+\rho)$. For any subset Γ of Λ let $\#(\Gamma)$ denote the number of elements in Γ and $\mathcal{R}(\Gamma)=\{j\in\Lambda;\ i\mathcal{R}j \text{ for some } i\in\Gamma\}$.

CLAIM. $\#(\Gamma)\le\#(\mathcal{R}(\Gamma))$ *for every* $\Gamma\subseteq\Lambda$.

The conclusion of the theorem will follow directly from this Claim by appealing to a combinatorial result, called the Marriage Theorem [17, Thm. 25A], which states that if $\mathcal{R}$ is a relation on a finite set with the property that $\#(\Gamma)\le\#(\mathcal{R}(\Gamma))$ for every subset Γ, then there exists a one–one mapping of the set onto itself that is a restriction of $\mathcal{R}$.

To establish the Claim, fix a subset Γ of Λ and then define $K=\{\mu_i: i\in\Gamma\}$. Let $V=\{\lambda\in\mathrm{Sp}(\underset{\sim}{A}): d(\lambda,K)\ge\rho+d\}$, in which case $d(K,V+B(\rho))\ge d$. By (4.8) there is $\psi\in\mathfrak{F}(V,\rho)$ such that $\|\psi(\underset{\sim}{A})\|<d/\epsilon\alpha_1\alpha_2 c_{m,s}$. Hence, $d(K,\mathrm{Supp}(\psi))\ge d$, and if Y denotes the spectral subspace of $\underset{\sim}{B}$ with respect to the component K of $\mathrm{Sp}(\underset{\sim}{B})$ then the hypotheses of Lemma 4 are satisfied with $\delta=d$. Hence

$$\|\psi(\underset{\sim}{A})_Y\|\le\epsilon\kappa(d)\|\psi(\underset{\sim}{A})\|<1\quad\text{and so}\quad Y\cap\{x\in X;\ \psi(\underset{\sim}{A})x=x\}=\{0\}.$$

It follows that $\dim(Y)+\dim(\mathfrak{N}(I-\psi(\underset{\sim}{A})))\le N$. Since

$$\dim(Y)=\#(\Gamma)\quad\text{and}\quad\dim(\mathfrak{N}(I-\psi(\underset{\sim}{A})))=N-\#(\mathcal{R}(\Gamma)),$$

the Claim follows. $\square$

Our final result of this section deals with the stability of multiplicity with respect to perturbations.

THEOREM 7. *Let* $N=\dim(X)$ *be finite, d be positive,* $\epsilon\in(0,1]$, *and* $\underset{\sim}{A}$ *be a commuting m-tuple in* $L(X)^m$ *of type* (α_1, s_1) *with respect to d. Let* $\underset{\sim}{B}$ *be a commuting m-tuple in* $L(X)^m$ *of type* (α_2, s_2) *with respect to d and suppose that*

$$\|\underset{\sim}{A}-\underset{\sim}{B}\|_{X\to X^m}\le d\epsilon/\alpha_1\alpha_2 c_{m,s}. \tag{4.10}$$

If $\min\{|\lambda_j-\lambda_k|:\lambda_j\ne\lambda_k\}>2d$ *where* $\mathrm{Sp}(\underset{\sim}{A})=\{\lambda_k: 1\le k\le r\}$, *and* $\epsilon<\min\{\|E_L\|^{-1}: L\subseteq \mathrm{Sp}(\underset{\sim}{A})\}$ *where* E_L *is the spectral projection of* $\underset{\sim}{A}$ *with respect to the component L of* $\mathrm{Sp}(\underset{\sim}{A})$, *then*

$$\dim(X_d(k))=n_k,\quad 1\le k\le r. \tag{4.11}$$

Here n_k *is the multiplicity of* λ_k *and* $X_d(k)$ *is the spectral subspace of* $\underset{\sim}{B}$ *with respect to the component* $(\{\lambda_k\}+B(d))\cap\mathrm{Sp}(\underset{\sim}{B})$ *of* $\mathrm{Sp}(\underset{\sim}{B})$.

REMARK 8. Theorem 7 states that if $\underset{\sim}{A}$ is given and $\underset{\sim}{B}$ is a perturbation of $\underset{\sim}{A}$, then even though a joint eigenvalue λ_j of $\underset{\sim}{A}$ may "split" into multiple joint eigenvalues of $\underset{\sim}{B}$, the dimension of the spectral subspace of $\underset{\sim}{B}$ with respect to the component $(\{\lambda_j\}+B(d))\cap\mathrm{Sp}(\underset{\sim}{B})$ of $\mathrm{Sp}(\underset{\sim}{B})$, which is non-empty by Theorem 5, remains at n_j. In particular, $(\{\lambda_j\}+B(d))\cap\mathrm{Sp}(\underset{\sim}{B})$ contains at most n_j elements. If X is a Hilbert space and $\underset{\sim}{A}$ is an m-tuple of self-adjoint operators, then the assumptions simplify somewhat since $\alpha_1=1$, $s_1=0$, and $\|E_L\|=1$ for every $L\subseteq\mathrm{Sp}(\underset{\sim}{A})$.

Proof of Theorem 7. Fix $k\in\{1,2,\dots,r\}$. Let $K=\{\lambda_k\}$ and $L=\mathrm{Sp}(\underset{\sim}{A})\setminus K$. If X_K (resp. X_L) is the spectral subspace of $\underset{\sim}{A}$ with respect to the component K (resp. L) of $\mathrm{Sp}(\underset{\sim}{A})$, then it follows that

$$\Delta(X_d(k), X_K)\le\sup\{\|u-E_K u\|: u\in X_d(k),\ \|u\|=1\}=\|E_L J_{K(d)}\|,$$

where $J_{K(d)}$ is the natural inclusion of $X_d(k)$ into X and $E_L J_{K(d)}$ is considered an element of $L(X_d(k), X_L)$. But $Q=E_L J_{K(d)}$ is a solution, in $L(X_d(k), X_L)$, of the system of equations $\underset{\sim}{A}_L Q-Q\underset{\sim}{B}_{K(d)}=E_L(\underset{\sim}{A}-\underset{\sim}{B})J_{K(d)}$, where $\underset{\sim}{A}_L$ is the restriction of $\underset{\sim}{A}$ to X_L and $\underset{\sim}{B}_{K(d)}$ is the restriction of $\underset{\sim}{B}$ to $X_d(k)$. Since $d(\underset{\sim}{A}_L, \underset{\sim}{B}_{K(d)})\ge d(L, K+B(d))>d$ it follows from Theorem 2 that

$$\|E_L J_{K(d)}\|\le\alpha_1\alpha_2 c_{m,s} d^{-1}\|\underset{\sim}{U}\|_{X_d(k)\to X_L^m},$$

where $\underset{\sim}{U}=E_L(\underset{\sim}{A}-\underset{\sim}{B})J_{K(d)}$. Then the inequality (4.10) together with

$$\|\underset{\sim}{U}\|_{X_d(k)\to X_L^m}\le\|E_L\|\,\|\underset{\sim}{A}-\underset{\sim}{B}\|_{X\to X^m}$$

shows that $\Delta(X_d(k), X_K)\le\epsilon\|E_L\|<1$. Applying (4.7) we have $\dim(X_d(k))\le\dim(X_K)=n_k$. Since this is for every $1\le k\le r$ and $X=X_d(1)\oplus\cdots\oplus X_d(r)$, the equality (4.11) follows. □

REMARK 9. All the results in this section have natural analogues for m-tuples $\underset{\sim}{A}$ and $\underset{\sim}{B}$ which are strongly commuting with respect to some partitions $\mathbf{T}(\underset{\sim}{A})$ and $\mathbf{T}(\underset{\sim}{B})$ consisting of generalized scalar operators with real spectrum. The arguments are based on Theorem 1 rather than Theorem 2. In particular, this includes the case when $\underset{\sim}{A}$ and $\underset{\sim}{B}$ consist of normal operators in a Hilbert space X.

5. Elementary operators. Let X and Y be Banach spaces and $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be commuting m-tuples in $L(X)^m$ and $L(Y)^m$, respectively. Some attention has been given to the operator $T: L(Y,X) \to L(Y,X)$, called an elementary operator, defined by

$$T(Q) = \sum_{j=1}^{m} A_j Q B_j, \quad Q \in L(Y,X), \tag{5.1}$$

especially in the setting of Hilbert spaces. For example, if $X = Y$ is a Hilbert space, then it is known [3] that

$$\sigma(T) = \{\langle u, v\rangle : u \in \mathrm{Sp}(\underset{\sim}{A}), v \in \mathrm{Sp}(\underset{\sim}{B})\}, \tag{5.2}$$

where $\langle u, v\rangle = \sum_{j=1}^{m} u_j v_j$. Using the methods of this paper we have the following result.

PROPOSITION 2. *Let X, Y be Banach spaces and let $\underset{\sim}{A} \in L(X)^m$ and $\underset{\sim}{B} \in L(Y)^m$ be strongly commuting m-tuples. If $T \in L(L(Y,X))$ is the operator*

$$Q \mapsto \sum_{j=1}^{m} A_j Q B_j,$$

then

$$\sigma(T) \subseteq \{\langle u, v\rangle : u \in \mathrm{Sp}(\underset{\sim}{A}), v \in \mathrm{Sp}(\underset{\sim}{B})\}. \tag{5.3}$$

If $X = Y$, then the inclusion (5.3) *is actually an equality.*

Proof. Let $\mathbf{T}(\underset{\sim}{A})$ and $\mathbf{T}(\underset{\sim}{B})$ be partitions with respect to which $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are strongly commuting. Let $\underset{\sim}{S} = (\underset{\sim}{L}_1, \underset{\sim}{L}_2, \underset{\sim}{R}_1, \underset{\sim}{R}_2)$ be the commuting $4m$-tuple defined in Lemma 1. Since $\mathrm{Sp}(\underset{\sim}{S}) \subseteq \mathbf{R}^{4m}$ (cf. proof of Theorem 1), it follows that $\mathrm{Sp}(\underset{\sim}{S}) = \gamma(\underset{\sim}{S})$ [10, Cor. 10.2]. If $\psi: \mathbf{C}^m \times \mathbf{C}^m \times \mathbf{C}^m \times \mathbf{C}^m \to \mathbf{C}$ is defined by $\psi(u, v, w, z) = \sum_{j=1}^{m} (u_j + iv_j)(w_j + iz_j)$ then $T = \psi(\underset{\sim}{S})$, and so Taylor's spectral mapping theorem implies that

$$\sigma(T) = \psi(\mathrm{Sp}(\underset{\sim}{S})) = \psi(\gamma(\underset{\sim}{S})) \subseteq \psi(\gamma(\mathbf{T}(\underset{\sim}{A})) \times \gamma(\mathbf{T}(\underset{\sim}{B})));$$

see Lemma 1. But $\psi(u, v, w, z) = \langle p(u, v), p(w, z)\rangle$, where $p: \mathbf{C}^{2m} \to \mathbf{C}^m$ is the polynomial defined in the proof of Theorem 1. Accordingly, Theorem 10.8 of [10] implies that

$$\begin{aligned}\psi(\gamma(\mathbf{T}(\underset{\sim}{A})) \times \gamma(\mathbf{T}(\underset{\sim}{B}))) &= \{\langle p(\mu), p(\nu)\rangle : \mu \in \gamma(\mathbf{T}(\underset{\sim}{A})), \nu \in \gamma(\mathbf{T}(\underset{\sim}{B}))\} \\ &= \{\langle \lambda, \xi\rangle : \lambda \in \mathrm{Sp}(\underset{\sim}{A}), \xi \in \mathrm{Sp}(\underset{\sim}{B})\},\end{aligned}$$

which is (5.3). The case of equality, when $X = Y$, follows from the second part of Lemma 1. □

REMARK 10. If $X = Y$ is a Hilbert space, then it is known that the Taylor spectrum in (5.2) can be replaced by the Harte spectrum [3]. The same is true if $X = Y$ is a Banach space and $\underset{\sim}{A}$ and $\underset{\sim}{B}$ satisfy the assumptions of Proposition 2 [10, Thm. 10.8].

COROLLARY 2.1. *Let X and Y be Banach spaces and let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be as in Proposition 2. If*

(5.4) $$\tau(\underset{\sim}{A},\underset{\sim}{B})=\inf\{|\langle u,v\rangle|: u\in \mathrm{Sp}(\underset{\sim}{A}), v\in \mathrm{Sp}(\underset{\sim}{B})\}$$

is positive, then for each $U\in L(Y,X)$ there is a unique solution Q in $L(Y,X)$ of the equation

(5.5) $$\sum_{j=1}^{m} A_j Q B_j = U.$$

Proof. The condition $\tau(\underset{\sim}{A},\underset{\sim}{B})>0$ is equivalent to $0\notin\sigma(T)$, that is, to T being invertible. Here T is the operator (5.1). □

We now consider the problem of finding estimates on $\|Q\|$ where Q is the solution of (5.5). It is assumed henceforth that $\tau(\underset{\sim}{A},\underset{\sim}{B})>0$ and that $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are commuting m-tuples of generalized scalar operators with real spectrum. Let $Z=L(Y,X)$. If L_j and R_j are the elements of $L(Z)$ defined by $L_j: Q\mapsto A_jQ$ and $R_j: Q\mapsto QB_j$, $1\le j\le m$, for each $Q\in Z$, then $T=\sum_{j=1}^m L_jR_j$ (cf. (5.1)) is a generalized scalar operator with real spectrum [12]. Then [2, Ch. 5, Thm. 4.5] guarantees the existence of constants $\alpha\ge 1$ and $s\ge 0$ such that

$$\|e^{itT}\|\le \alpha(1+\tau(\underset{\sim}{A},\underset{\sim}{B})|t|)^s,\quad t\in\mathbf{R}.$$

Accordingly, Proposition 1 applies with $m=1$ and Z and T as above to yield

$$\|Q\|\le \alpha c_{1,s}\tau(\underset{\sim}{A},\underset{\sim}{B})^{-1}\|U\|.$$

This approach is discussed in [12]. However, it may be difficult to apply in practice if the explicit dependence of the type (α,s) of T with respect to $\tau(\underset{\sim}{A},\underset{\sim}{B})$ is required in terms of the type (α_1,s_1) of $\underset{\sim}{A}$ and the type (α_2,s_2) of $\underset{\sim}{B}$ (with respect to $\tau(\underset{\sim}{A},\underset{\sim}{B})$). The purpose of this section is to outline an alternative approach for finding estimates on $\|Q\|$.

Suppose that $\tau(\underset{\sim}{A},\underset{\sim}{B})>0$ and that $\|e^{i\langle\xi,\underset{\sim}{A}\rangle}\|\le\alpha_1(1+|\xi|)^{s_1}$, $\xi\in\mathbf{R}^m$, and $\|e^{i\langle\xi,\underset{\sim}{B}\rangle}\|\le\alpha_2(1+|\xi|)^{s_2}$, $\xi\in\mathbf{R}^m$, for some constants $\alpha_r\ge1$ and $s_r\ge0$, $r\in\{1,2\}$. If L_j and R_j, $1\le j\le m$, are the operators defined above and $\underset{\sim}{S}$ is the commuting $2m$-tuple $(\underset{\sim}{L},\underset{\sim}{R})$, then

$$e^{i\langle\lambda,\underset{\sim}{S}\rangle}Q=\left(\exp\left[i\sum_{j=1}^m\lambda_jA_j\right]\right)Q\exp\left[i\sum_{j=1}^m\lambda_{m+j}B_j\right],\quad \lambda\in\mathbf{R}^{2m},$$

for each $Q\in Z$. It follows that

(5.6) $$\|e^{i\langle\lambda,\underset{\sim}{S}\rangle}\|=\|e^{i\langle\lambda,(\underset{\sim}{L},\underset{\sim}{R})\rangle}\|\le\alpha_1\alpha_2(1+|\lambda|)^s,\quad\lambda\in\mathbf{R}^{2m},$$

where $s=s_1+s_2$, and so $\underset{\sim}{S}$ admits an $L_1^\vee(s,\mathbf{R}^{2m})$-functional calculus given by

(5.7) $$f(\underset{\sim}{S})=f(\underset{\sim}{L},\underset{\sim}{R})=(2\pi)^{-2m}\int_{\mathbf{R}^m\times\mathbf{R}^m}e^{i\langle(u,v),(\underset{\sim}{L},\underset{\sim}{R})\rangle}\hat f(u,v)\,du\,dv$$

for each $f\in L_1^\vee(s,\mathbf{R}^{2m})$. Furthermore, it was noted earlier that

$$\mathrm{Sp}(\underset{\sim}{S})\subseteq\mathrm{Sp}(\underset{\sim}{L})\times\mathrm{Sp}(\underset{\sim}{R})\subseteq\mathrm{Sp}(\underset{\sim}{A})\times\mathrm{Sp}(\underset{\sim}{B}).$$

Now, Corollary 2.1 implies that there exists a unique solution $Q\in L(Y,X)$ to the equation $A_1QB_1+\cdots+A_mQB_m=U$. Using the formula (5.7) it follows that

(5.8) $$\|Q\|\le\alpha_1\alpha_2\nu(\underset{\sim}{A},\underset{\sim}{B})\|U\|,$$

where

$$\nu(\underset{\sim}{A},\underset{\sim}{B})=\inf\left\{(2\pi)^{-2m}\int_{\mathbf{R}^m\times\mathbf{R}^m}(1+|(u,v)|)^s|\hat{f}(u,v)|\,du\,dv\right\},$$

the infimum being taken over all functions f in $L_1^\vee(s,\mathbf{R}^{2m})$ such that $f(x,y)=1/\langle x,y\rangle$ in a neighbourhood of $\mathrm{Sp}(\underset{\sim}{A})\times\mathrm{Sp}(\underset{\sim}{B})$. Indeed, as $T=\langle\underset{\sim}{L},\underset{\sim}{R}\rangle$ we see for such f that $f(\underset{\sim}{L},\underset{\sim}{R})=T^{-1}$ and so $Q=T^{-1}U=f(\underset{\sim}{L},\underset{\sim}{R})U$.

The problem is to understand how $\nu(\underset{\sim}{A},\underset{\sim}{B})$ depends on the geometry of the sets $\mathrm{Sp}(\underset{\sim}{A})$ and $\mathrm{Sp}(\underset{\sim}{B})$. For example, is $\nu(\underset{\sim}{A},\underset{\sim}{B})\le c/\tau(\underset{\sim}{A},\underset{\sim}{B})^{1+s}$ for some constant c depending only on m and s? The answer to this is no, as seen by the following example (where $s=0$).

EXAMPLE 2. Let $X=Y=\mathbf{C}^n$ where n is to be chosen suitably large. Then with $m=2$ define $\underset{\sim}{A}=(D,I)$ and $\underset{\sim}{B}=(D^{-1},I)$ where D is the diagonal matrix with entries $2,2^2,\dots,2^n$. It follows that

$$\tau(\underset{\sim}{A},\underset{\sim}{B})=\inf\{|\langle(2^j,1),(2^{-k},1)\rangle|:1\le j\le n,1\le k\le n\}=(1+2^{-n})>1$$

and the solution Q of the corresponding equation (5.5), namely $DQD^{-1}+Q=U$, is the matrix whose (j,k)th entry is given by $q_{jk}=u_{jk}(1+2^{j-k})^{-1}$. Choose $U=(u_{jk})$ to be the $n\times n$ Toeplitz matrix corresponding to the function $g(\theta)=i(\pi-\theta)$ on $[0,2\pi]$, that is, $u_{jk}=(j-k)^{-1}(1-\delta_{jk})$. Then $\|U\|\le\pi$ and $\|Q\|\ge\frac{1}{2}\log(n)-2$, where the second inequality follows by estimating $\|Qw\|$ with $w=(1,1,\dots,1)$. Accordingly, there can be no constant $c=c(m)$ such that $\nu(\underset{\sim}{A},\underset{\sim}{B})\le c\tau(\underset{\sim}{A},\underset{\sim}{B})^{-1}$.

We require some further definitions. Define

$$d(\underset{\sim}{A})=\inf\{|u|:u\in\mathrm{Sp}(\underset{\sim}{A})\}$$

($d(\underset{\sim}{B})$ is defined similarly) and

$$d(\underset{\sim}{A},\underset{\sim}{B})=\inf\left\{\frac{|\langle u,v\rangle|}{|u|\cdot|v|}:u\in\mathrm{Sp}(\underset{\sim}{A}),v\in\mathrm{Sp}(\underset{\sim}{B})\right\}.$$

Then

$$d(\underset{\sim}{A})d(\underset{\sim}{B})d(\underset{\sim}{A},\underset{\sim}{B})\le\tau(\underset{\sim}{A},\underset{\sim}{B})\quad\text{and}\quad\tau(\underset{\sim}{A},\underset{\sim}{B})>0$$

if and only if $d(\underset{\sim}{A})d(\underset{\sim}{B})d(\underset{\sim}{A},\underset{\sim}{B})>0$.

THEOREM 8. *Let $\underset{\sim}{A}$ (resp. $\underset{\sim}{B}$) be a commuting m-tuple in $L(X)^m$ (resp. $L(Y)^m$) such that $\|e^{i\langle\xi,\underset{\sim}{A}\rangle}\|\le\alpha_1(1+|\xi|)^{s_1}$ and $\|e^{i\langle\xi,\underset{\sim}{B}\rangle}\|\le\alpha_2(1+|\xi|)^{s_2}$ for each $\xi\in\mathbf{R}^m$. If $\tau(\underset{\sim}{A},\underset{\sim}{B})>0$, then the unique solution $Q\in L(Y,X)$ of the equation*

$$A_1QB_1+\cdots+A_mQB_m=U$$

satisfies the estimate

$$\|Q\|\le\alpha_1\alpha_2\max\{1,d(\underset{\sim}{A})^{-s_1}d(\underset{\sim}{B})^{-s_2}\}c(m,s)\|U\|\frac{1+|\log d(\underset{\sim}{A},\underset{\sim}{B})|}{d(\underset{\sim}{A})d(\underset{\sim}{B})d(\underset{\sim}{A},\underset{\sim}{B})^{2m+s}},\tag{5.9}$$

where $s=s_1+s_2$ and $c(m,s)$ is a constant depending only on m and s.

Proof. Let $\underset{\sim}{V}=d(\underset{\sim}{A})^{-1}\underset{\sim}{A}$ and $\underset{\sim}{W}=d(\underset{\sim}{B})^{-1}\underset{\sim}{B}$, in which case $d(\underset{\sim}{V})=1=d(\underset{\sim}{W})$. Then the inequalities

$$\|e^{i\langle\xi, \underset{\sim}{V}\rangle}\| \le \alpha_1 \max\{1, d(\underset{\sim}{A})^{-s_1}\}(1+|\xi|)^{s_1}, \quad \xi \in \mathbf{R}^m,$$

and

$$\|e^{i\langle\xi, \underset{\sim}{W}\rangle}\| \le \alpha_2 \max\{1, d(\underset{\sim}{B})^{-s_2}\}(1+|\xi|)^{s_2}, \quad \xi \in \mathbf{R}^m,$$

are valid. Since the solution Q of (5.5) also satisfies the equation

$$\sum_{j=1}^{m} V_j Q W_j = d(\underset{\sim}{A})^{-1} d(\underset{\sim}{B})^{-1} U, \tag{5.10}$$

it follows from (5.8) applied in the context of equation (5.10) that

$$\|Q\| \le \alpha_1\alpha_2 \max\{1, d(\underset{\sim}{A})^{-s_1} d(\underset{\sim}{B})^{-s_2}\} \frac{\nu(\underset{\sim}{V}, \underset{\sim}{W})\|U\|}{d(\underset{\sim}{A})d(\underset{\sim}{B})}. \tag{5.11}$$

Noting that $d(\underset{\sim}{A}, \underset{\sim}{B}) = d(\underset{\sim}{V}, \underset{\sim}{W})$, the desired estimate (5.9) follows from (5.11) and the following lemma. □

LEMMA 5. *Let $\underset{\sim}{V}$ (resp. $\underset{\sim}{W}$) be a commuting m-tuple in $L(X)^m$ (resp. $L(Y)^m$) such that $\underset{\sim}{V}$ is of type (β_1, s_1) with respect to* 1, *$\underset{\sim}{W}$ is of type (β_2, s_2) with respect to* 1, *and $d(\underset{\sim}{V}) = 1 = d(\underset{\sim}{W})$. Let $s = s_1 + s_2$. Then there exists a constant $c(m, s)$, depending only on m and s, such that*

$$\nu(\underset{\sim}{V}, \underset{\sim}{W}) \le c(m,s) \frac{1+|\log d(\underset{\sim}{V}, \underset{\sim}{W})|}{d(\underset{\sim}{V}, \underset{\sim}{W})^{2m+s}}.$$

Proof. We proceed via a Littlewood–Paley argument. Let $\psi \in C^\infty(\mathbf{R})$ be a real-valued function supported in $\{t \in \mathbf{R}; 3/4 \le |t| \le 3\}$ such that

$$\|\psi\|_\infty \le 1 \quad \text{and} \quad \sum_{k=0}^{\infty} \psi_k(t) = 1$$

whenever $|t| \ge 7/8$, where $\psi_k(t) = \psi(2^{-k}t)$, $t \in \mathbf{R}$. Let $h \in C^\infty(\mathbf{R})$ be a real-valued odd function such that $h(t) = 1/t$ whenever $|t| \ge 1/2$, and let h_d denote the function $t \mapsto d^{-1}h(t/d)$, $t \in \mathbf{R}$, where $d = d(\underset{\sim}{V}, \underset{\sim}{W})$. Define $g: \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$ by

$$g(\lambda, \mu) = \psi(|\lambda|)\psi(|\mu|)h_d(\langle\lambda, \mu\rangle), \quad \lambda, \mu \in \mathbf{R}^m,$$

and

$$g_{jk}(\lambda, \mu) = g(2^{-j}\lambda, 2^{-k}\mu), \quad \lambda, \mu \in \mathbf{R}^m,$$

for each $j, k \in \{0, 1, 2, \ldots\}$. Let $L_1^s(m)$ denote the Banach space $L^1(\rho)$ where ρ is the measure $(1+|(u,v)|)^s\,du\,dv$ in $\mathbf{R}^m \times \mathbf{R}^m$. The norm will be denoted by $\|\cdot\|_{1,s}$. Using the inequalities

$$(1+|(u,v)|) = (1+2^{-j-k}|(2^j u, 2^k v)|) \le 1+|(2^j u, 2^k v)|, \quad u, v \in \mathbf{R}^m,$$

it can be shown that $\|\hat{g}_{jk}\|_{1,s} \le \|\hat{g}\|_{1,s}$ for all j and k.

Define $f: \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$ by

$$f(\lambda, \mu) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-j-k} g_{jk}(\lambda, \mu),$$

in which case it follows that $f(\lambda, \mu) = 1/\langle\lambda, \mu\rangle$ whenever $|\lambda| \ge 7/8$, $|\mu| \ge 7/8$, and $|\langle\lambda, \mu\rangle| \cdot |\lambda|^{-1}|\mu|^{-1} \ge 8d/9$. That is,

$$f(\lambda, \mu) = 1/\langle \lambda, \mu \rangle$$

in a neighbourhood of $\mathrm{Sp}(\underset{\sim}{V}) \times \mathrm{Sp}(\underset{\sim}{W})$. Now, the series for f converges in $\mathcal{S}'(\mathbf{R}^{2m})$, where $\mathcal{S}(\mathbf{R}^{2m})$ is the Schwartz space of rapidly decreasing functions; thus also

$$\hat{f} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-j-k} \hat{g}_{jk} \tag{5.12}$$

with convergence in $\mathcal{S}'(\mathbf{R}^{2m})$. Suppose for the moment that

$$(2\pi)^{-2m} \|\hat{g}\|_{1,s} \leq c(m, s) \frac{1 + |\log d|}{d^{2m+s}}. \tag{5.13}$$

In particular then, $\|\hat{g}\|_{1,s} < \infty$, and so the series for $\hat{f}$ converges absolutely in $L_1^s(m)$ since

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-j-k} \|\hat{g}_{jk}\|_{1,s} \leq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-j-k} \|\hat{g}\|_{1,s} = 4\|\hat{g}\|_{1,s}. \tag{5.14}$$

Since the inclusions $L_1^s(m) \hookrightarrow L^1(\mathbf{R}^{2m}) \hookrightarrow \mathcal{S}'(\mathbf{R}^{2m})$ are continuous it follows that $\hat{f} \in L_1^s(m)$. That is, $f \in L_1^\vee(s, \mathbf{R}^{2m})$ and satisfies $f(\lambda, \mu) = 1/\langle \lambda, \mu \rangle$ in a neighbourhood of $\mathrm{Sp}(\underset{\sim}{V}) \times \mathrm{Sp}(\underset{\sim}{W})$. Furthermore, (5.12), (5.13), and (5.14) imply that

$$(2\pi)^{-2m} \int_{\mathbf{R}^{2m}} (1 + |(u, v)|)^s |\hat{f}(u, v)| \, du \, dv \leq 4c(m, s) \frac{1 + |\log d|}{d^{2m+s}}.$$

So, to prove the lemma it remains to establish (5.13).

Fix $l \geq 1$. By direct computation it can be established that

$$\max\{\|\partial^l g(\lambda, \mu)/\partial \lambda_j^l\|_1, \|\partial^l g(\lambda, \mu)/\partial \mu_j^l\|_1\} \leq \gamma_{m,l} d^{-l}, \tag{5.15}$$

for each $1 \leq j \leq m$, where $\gamma_{m,l}$ depends only on m, l and on the choice of ψ and h. The case $l = 0$ yields

$$\|g\|_1 \leq \gamma_{m,0}(1 + |\log d|). \tag{5.16}$$

It should be remarked that in establishing (5.15) and (5.16) we have made use of the fact that $0 < d \leq 1$ and, in estimating the various Jacobians and integrals along the way when variables are changed from rectangular to spherical polar coordinates, the radial symmetry of $(\lambda, \mu) \mapsto \psi(|\lambda|)\psi(|\mu|)$ plays a crucial role. It follows from (5.15), (5.16), and the identity $|(p^{(l)})^\wedge(t)| = |t|^l |\hat{p}(t)|$ (valid for suitably smooth functions $p: \mathbf{R} \to \mathbf{C}$) applied to the partial derivatives of g that

$$|\hat{g}(u, v)| \leq \gamma_{m,0}(1 + |\log d|), \quad u, v \in \mathbf{R}^m,$$

and

$$(|u|^2 + |v|^2)^{l/2} |\hat{g}(u, v)| \leq \gamma'_{m,l} d^{-l}, \quad u, v \in \mathbf{R}^m,$$

for suitable constants $\gamma'_{m,l}$. Using these inequalities and decomposing the integral

$$\int_{\mathbf{R}^m \times \mathbf{R}^m} (1 + |(u, v|)^s |\hat{g}(u, v)| \, du \, dv$$

into integrals over the disjoint regions $K = \{(u, v) \colon |u|^2 + |v|^2 \leq d^{-2}\}$ and $\mathbf{R}^{2m} \backslash K$, the estimate (5.13) follows upon choosing $l = 2m + s + 1$. This completes the proof of the lemma and also of Theorem 8. $\square$

REMARK 11. (i) If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are as in the statement of Theorem 8, then it follows from Lemma 6 that

$$\nu(d(\underset{\sim}{A})^{-1}\underset{\sim}{A}, d(\underset{\sim}{B})^{-1}\underset{\sim}{B}) \leq c(m,s)\frac{1+|\log d(\underset{\sim}{A},\underset{\sim}{B})|}{d(\underset{\sim}{A},\underset{\sim}{B})^{2m+s}}.$$

Furthermore, if β and γ are arbitrary positive constants, then it follows from the definition of $\nu(\cdot)$ that

$$\nu(\underset{\sim}{A},\underset{\sim}{B}) \leq \max\{1, \beta^s, \gamma^s\}\nu(\gamma\underset{\sim}{A}, \beta\underset{\sim}{B}).$$

Combining these two estimates (with the choice $\gamma = d(\underset{\sim}{A})^{-1}$ and $\beta = d(\underset{\sim}{B})^{-1}$) yields the inequality

$$\nu(\underset{\sim}{A},\underset{\sim}{B}) \leq c(m,s)\max\{1, d(\underset{\sim}{A})^{-s}, d(\underset{\sim}{B})^{-s}\}\frac{1+|\log d(\underset{\sim}{A},\underset{\sim}{B})|}{d(\underset{\sim}{A},\underset{\sim}{B})^{2m+s}}, \tag{5.17}$$

which gives some indication, via the notion $d(\underset{\sim}{A},\underset{\sim}{B})$ (a measure of how close the sets $\mathrm{Sp}(\underset{\sim}{A})$ and $\mathrm{Sp}(\underset{\sim}{B})$ are to being "orthogonal"), of the dependence of $\nu(\underset{\sim}{A},\underset{\sim}{B})$ on the geometry of the sets $\mathrm{Sp}(\underset{\sim}{A})$ and $\mathrm{Sp}(\underset{\sim}{B})$.

(ii) The computations used in the proof of Lemma 6 are somewhat crude and can probably be improved. This suggests the question of whether the estimate (5.17) can be sharpened. Suppose, for example, that $s_1 = 0 = s_2$. Is it the case that

$$\nu(\underset{\sim}{A},\underset{\sim}{B}) \leq c_m/d(\underset{\sim}{A})d(\underset{\sim}{B})d(\underset{\sim}{A},\underset{\sim}{B})$$

or, perhaps,

$$\nu(\underset{\sim}{A},\underset{\sim}{B}) \leq c_m(1+|\log d(\underset{\sim}{A},\underset{\sim}{B})|)/d(\underset{\sim}{A})d(\underset{\sim}{B})d(\underset{\sim}{A},\underset{\sim}{B})?$$

(iii) Theorem 8 can be extended to m-tuples $\underset{\sim}{A}$ and $\underset{\sim}{B}$ which have commuting partitions $\mathbf{T}(\underset{\sim}{A})$ and $\mathbf{T}(\underset{\sim}{B})$ consisting of generalized scalar operators with real spectrum. In particular, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ could be commuting m-tuples of normal operators in Hilbert spaces.

(iv) We have restricted ourselves to a consideration of the single elementary operator equation (5.5) rather than a linear system of such equations, say

$$\sum_{j=1}^{m} A_{jk}QB_{jk} = U_k, \quad 1 \leq k \leq n.$$

The reason is that the compatibility conditions required to guarantee solutions of such systems (together with some estimates of a different nature to those of this section and those in [12]) are presented in the recent paper [13].

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School of Mathematics and Physics
Macquarie University NSW 2109
Australia

Department of Mathematics
Monash University
Clayton VIC 3168
Australia

Centre for Mathematical Analysis
Australian National University
Canberra ACT 2601
Australia