AN EMBEDDING THEOREM FOR THE FELL TOPOLOGY

Gerald Beer

1. Introduction. Let 2^X (resp. $\mathrm{CL}(X)$) denote the closed (resp. closed and nonempty) subsets of a metric space $\langle X, d\rangle$. The fundamental topology on $\mathrm{CL}(X)$ is the *Hausdorff metric topology*, induced by the infinite-valued metric on $\mathrm{CL}(X)$ defined by

$$h_d(E,F) = \sup(\{d(x,E): x \in F\} \cup \{d(x,F): x \in E\}).$$

If we replace d by the metric $\rho = \min\{d, 1\}$, then h_ρ is finite-valued and determines the same topology on $\mathrm{CL}(X)$. Most importantly [8], the map $E \to \rho(\cdot, E)$ is an isometry of $\langle \mathrm{CL}(X), h_\rho\rangle$ into the bounded continuous real functions on X, equipped with the usual uniform metric.

A somewhat weaker topology on 2^X, which agrees with the Hausdorff metric topology on $\mathrm{CL}(X)$ if and only if X is compact, is the *Fell topology* [10], also called the *topology of closed convergence* [12]. To describe this topology, we introduce the following notation: if $A \subset X$, then

$$A^- = \{E \in 2^X : E \cap A \neq \varnothing\} \quad \text{and} \quad A^+ = \{E \in 2^X : E \subset A\}.$$

The Fell topology τ_F has as a subbase all sets of the form V^-, where V is an open subset of X and $(K^C)^+$, where K is a compact subset of X. Obviously, the Fell topology is similar in spirit to the stronger *Vietoris topology* ([12], [14]). With respect to convergence notions, it is known (cf. [3, Lemma 1.0] or [11, p. 353]) that a sequence $\langle E_n\rangle$ in 2^X converges in the Fell topology to a closed set E if and only if

$$E = \mathrm{Li}\, E_n = \mathrm{Ls}\, E_n,$$

where $\mathrm{Li}\, E_n$ (resp. $\mathrm{Ls}\, E_n$) consists of all points x each neighborhood of which meets $\langle E_n\rangle$ eventually (resp. frequently). In the literature, this form of convergence is usually called *Kuratowski convergence*, but sometimes it goes by the name *topological convergence* [15]. In an arbitrary Hausdorff space (not necessarily metrizable), Kuratowski convergence is stronger than convergence with respect to the Fell topology; also, Kuratowski convergence of *nets* of sets determines the Fell topology if and only if X is locally compact (cf. [7] or [12]). The Fell topology and Kuratowski convergence of sets have been particularly important in the study of the convergence of lower semicontinuous functions and their minima ([1], [4], [5], [16]), but these notions also arise in probability theory ([13], [19]), mathematical economics [12], and the study of C^*-algebras [9].

In this note we show that a locally compact metrizable space X admits a metric d for which $E \to d(\cdot, E)$ is a topological embedding of $\langle \mathrm{CL}(X), \tau_F\rangle$ into the continuous real functions on X with the *compact-open topology* (*the topology of uniform convergence on compacta*), τ_{CO}, that can be extended to 2^X.

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2. Notation. Since $[(A \cup B)^C]^+ = (A^C)^+ \cap (B^C)^+$, a base for the Fell topology consists of all sets of the form

$$\left(\bigcap_{i=1}^{n} V_i^-\right) \cap (K^C)^+,$$

where $V_1, V_2, \ldots, V_n$ are open, K is compact, and $K \cap V_i = \varnothing$ for each i. We denote such a basic open set by $[V_1, \ldots, V_n; K]$. Thus, $F \in [V_1, \ldots, V_n; K]$ means that F hits each V_i and F misses K. If a fixed metric on X is understood, then $B[x; \alpha]$ (resp. $S[x; \alpha]$) will denote the closed (resp. open) ball about $x \in X$ of radius α. Also, if $E \in \mathrm{CL}(X)$, we write $B[E; \alpha]$ for $\bigcup \{B[x; \alpha] : x \in E\}$. We denote the continuous functions from X to R by $C(X, R)$. If $f \in C(X, R)$ and K is a compact subset of X, then $N(f, K, \epsilon)$ will denote

$$\{g \in C(X, R) : \text{for each } x \in K,\ |f(x) - g(x)| < \epsilon\}.$$

Such sets form a local base for the compact-open topology τ_{CO} on $C(X, R)$ at f.

3. Results. A key notion in the sequel was introduced in [3].

DEFINITION. A metric space $\langle X, d \rangle$ is said to have *nice closed balls* if each noncompact closed ball in X is the entire space.

This class of spaces includes the metric spaces in which closed and bounded sets are compact, and the zero-one metric spaces. Theorem 2.3 of [3] says in part that a metric space has nice closed balls if and only if the Fell topology on $\mathrm{CL}(X)$ coincides with the stronger *ball topology,* having as a subbase all sets of the form $S[x; \alpha]^-$ and $(B[x; \alpha]^C)^+$. Our first result follows from this fact, but we prefer a self-contained presentation.

THEOREM 1. *Let $\langle X, d \rangle$ be a metric space. The following are equivalent:*

(a) *the map $E \to d(\cdot, E)$ is an embedding of $\langle \mathrm{CL}(X), \tau_F \rangle$ into $\langle C(X, R), \tau_{\mathrm{CO}} \rangle$;*
(b) *the space $\langle X, d \rangle$ has nice closed balls.*

Proof. (a) $\Rightarrow$ (b). Suppose $\langle X, d \rangle$ fails to have nice closed balls. Then there exist x and y in X and $\alpha > 0$ such that $B[x; \alpha]$ is noncompact and $d(x, y) > \alpha$. Let $\langle x_n \rangle$ be a sequence in $B[x; \alpha]$ with no cluster point. Then $\langle \{x_n, y\} \rangle$ τ_F-converges to $\{y\}$, but $\langle d(x, \{x_n, y\}) \rangle$ does not converge to $d(x, \{y\})$. Thus, the map $E \to d(\cdot, E)$ cannot be an embedding of $\langle \mathrm{CL}(X), \tau_F \rangle$ into $\langle C(X, R), \tau_{\mathrm{CO}} \rangle$.

(b) $\Rightarrow$ (a). Let d be a metric for X with nice closed balls, and let $F \in \mathrm{CL}(X)$ be fixed. We first establish continuity of $E \to d(\cdot, E)$ at F. To this end, let K be a nonempty compact subset of X and let $\epsilon > 0$. We will produce a τ_F-neighborhood of F mapped into $N(f, K, 3\epsilon)$, where f is the function $d(\cdot, F)$. Let $\{x_1, \ldots, x_n\} \subset K$ be an ϵ-net for K, that is, $K \subset \bigcup \{S[x_k; \epsilon] : 1 \le k \le n\}$. For each $k \in \{1, \ldots, n\}$, write $\alpha_k = d(x_k, F)$. We partition $\{1, \ldots, n\}$ as follows:

$$J_1 = \{k : \alpha_k \le \epsilon\} \quad \text{and} \quad J_2 = \{k : \alpha_k > \epsilon\}.$$

Since proper d-balls in X are compact, for each $k \in \{1, \ldots, n\}$ we can choose $y_k \in F$ with $d(x_k, y_k) = \alpha_k$. Now for each $k \in J_2$, $y_k \notin B[x_k; \alpha_k - \epsilon]$. Again by the compactness of proper d-balls, the set

$$C = \bigcup_{k \in J_2} B[x_k; \alpha_k - \epsilon]$$

is compact. By construction, $C \cap F = \varnothing$ and for each $k \in \{1, \ldots, n\}$, $S[y_k; \epsilon] \cap C = \varnothing$. Because $\{y_1, \ldots, y_n\} \subset F$,

$$\Sigma = [S[y_1; \epsilon], \ldots, S[y_n; \epsilon]; C] \cap \mathrm{CL}(X)$$

is a τ_F-neighborhood of F in $\mathrm{CL}(X)$. We claim that $E \to d(\cdot, E)$ maps Σ into $N(f, K, 3\epsilon)$. Let $E \in \Sigma$ and let $x \in K$ be arbitrary. We must show that

$$|d(x, E) - d(x, F)| < 3\epsilon.$$

Choose $k \in \{1, \ldots, n\}$ with $d(x, x_k) < \epsilon$, and choose $y \in E \cap S[y_k; \epsilon]$. We have

$$d(x, E) \leq d(x, y) \leq d(x, x_k) + d(x_k, y_k) + d(y_k, y) < \alpha_k + 2\epsilon.$$

Now if $k \in J_1$ then $d(x, E) < \epsilon + 2\epsilon = 3\epsilon$, and since

$$d(x, F) \leq d(x, x_k) + d(x_k, y_k) < 2\epsilon$$

we have $|d(x, E) - d(x, F)| < 3\epsilon$.

If $k \in J_2$, then $d(x, E) \geq \alpha_k - 2\epsilon$ because $E \cap B[x_k; \alpha_k - \epsilon] = \varnothing$. Thus,

$$|d(x, E) - \alpha_k| = |d(x, E) - d(x_k, F)| \leq 2\epsilon.$$

Since $z \to d(z, F)$ is Lipschitz with constant 1, we obtain

$$|d(x, E) - d(x, F)| \leq |d(x, E) - d(x_k, F)| + |d(x_k, F) - d(x, F)| < 3\epsilon.$$

To show that $E \to d(\cdot, E)$ is open is easier. It suffices to show that the image of each basic τ_F-neighborhood of a closed set F contains a neighborhood of $f \equiv d(\cdot, F)$ in the relative topology of the function space. Let $[V_1, \ldots, V_n; K] \cap \mathrm{CL}(X)$ be such a τ_F-neighborhood of F (note that K can be empty). Choose $x_1, \ldots, x_n$ in F and $\epsilon > 0$ such that, for $k = 1, \ldots, n$,

$$S[x_k; \epsilon] \subset V_k \quad \text{and} \quad B[F; \epsilon] \cap K = \varnothing.$$

We claim that $N(f, \{x_1, \ldots, x_n\} \cup K, \epsilon) \cap \{d(\cdot, E) : E \in \mathrm{CL}(X)\}$ is in the image of $[V_1, \ldots, V_n; K] \cap \mathrm{CL}(X)$. Suppose $d(\cdot, E) \in N(f, \{x_1, \ldots, x_n\} \cup K, \epsilon)$. Since

$$|d(x_k, E) - d(x_k, F)| < \epsilon \quad \text{and} \quad d(x_k, F) = 0,$$

we have $E \cap S[x_k; \epsilon] \neq \varnothing$ for each index k. Thus, E hits V_k for each k. If K is empty, we are done. Otherwise, for every $x \in K$ we have both $d(x, F) \geq \epsilon$ and $|d(x, F) - d(x, E)| < \epsilon$. As a result, $d(x, E) > 0$ so that $E \cap K = \varnothing$. Thus, $E \in [V_1, \ldots, V_n; K]$. □

An alternate proof of Theorem 1 can be constructed using the equivalence of the topology of pointwise convergence with the compact-open topology for distance functions (see [21, Thm. 43.14]). The proof of continuity is simplified exactly to the extent that the proof of openness is made more complicated. We also note that Theorem 1 above says more than the equivalence of conditions (1) and (3) of Theorem 2.3 of [3], because sequences do not in general determine the Fell topology. In fact, if X is not separable then the topology cannot be first countable. We find it worthwhile to single out as a lemma the (perhaps known) equivalence of first countability of $\langle \mathrm{CL}(X), \tau_F \rangle$, second countability of $\langle \mathrm{CL}(X), \tau_F \rangle$, and separability of X in the locally compact case. (Actually, each is equivalent to *metrizability* of the hyperspace; see Theorem 4 below.)

LEMMA 1. *Let $\langle X, d\rangle$ be a locally compact metric space. The following are equivalent:*

(a) *X is separable*;
(b) *$\langle \mathrm{CL}(X), \tau_F\rangle$ is second countable*;
(c) *$\langle \mathrm{CL}(X), \tau_F\rangle$ is first countable.*

Proof. (a) ⇒ (b). Let $\{x_n : n \in Z^+\}$ be a countable dense subset of X. For each n choose $\delta_n > 0$ such that $B[x_n; \delta_n]$ is compact. Then all sets of the form $S[x_n; 1/k]^-$ and $(B[x_n; \delta_n/k]^C)^+$, where n and k are positive integers, is a countable subbase for the hyperspace topology. Thus, $\langle \mathrm{CL}(X), \tau_F\rangle$ is second countable.

(b) ⇒ (c). This is trivial.

(c) ⇒ (a). If X is not separable, there exists for some $\epsilon > 0$ an uncountable subset F of X such that $d(x, y) > \epsilon$ whenever $\{x, y\} \subset F$. Now a local base for τ_F at F consists of all sets of the form

$$[S[x_1; \delta], S[x_2; \delta], \ldots, S[x_n; \delta]; K],$$

where $\{x_1, \ldots, x_n\}$ is a finite subset of F, K is a compact subset of X, and $\delta < \epsilon$. Now if $\{\Sigma_j : j \in Z^+\}$ is any countable subcollection of this local base, where

$$\Sigma_j = [S[x_{j1}; \delta_j], \ldots, S[x_{jn_j}; \delta_j]; K_j],$$

then we can find $x \in F - \{x_{ji} : j \in Z^+ \text{ and } i \le n_j\}$. Clearly, $S[x; \epsilon]^-$ is a τ_F-neighborhood of F which fails to contain any Σ_j. □

We now come to the main result of the paper.

THEOREM 2. *Let X be a metrizable space. The following are equivalent:*

(a) *X is locally compact*;
(b) *X admits a metric d bounded by 1 such that each closed ball of radius less than 1 is compact*;
(c) *X admits a metric d with nice closed balls*;
(d) *X admits a metric d such that $E \to d(\cdot, E)$ is an embedding of $\langle \mathrm{CL}(X), \tau_F\rangle$ into $\langle C(X, R), \tau_{CO}\rangle$.*

Proof. Conditions (c) and (d) are equivalent by Theorem 1, and the implications (b) ⇒ (c) and (c) ⇒ (a) are trivial. It remains to establish (a) ⇒ (b). This follows from general results on refinements of open covers of a metrizable space (see [6, p. 196]); we provide a simple direct proof here. Let ρ be a fixed metric for X. Let $\{V_i : i \in I\}$ be an open cover of X such that for each $i \in I$, cl V_i is compact. By paracompactness, there exists a locally finite open cover $\{W_i : i \in I\}$ of X with $W_i \subset V_i$ for each $i \in I$. By paracompactness and regularity, there exists a locally finite closed cover $\{F_i : i \in I\}$ such that $F_i \subset W_i$ for each $i \in I$. For each i such that F_i is nonempty, define $g_i \in C(X, R)$ by

$$g_i(x) = \frac{2\rho(x, F_i)}{\rho(x, F_i) + \rho(x, W_i^c)}.$$

By local finiteness and continuity of each g_i,

$$\rho^*(x,y) \equiv \rho(x,y) + \sum_{i \in I} |g_i(x) - g_i(y)|$$

is a metric on X equivalent to ρ. For each $x \in X$ there exists $i \in I$ for which $x \in F_i$. By construction, $\rho^*(x,y) \leq 1$ forces $y \in W_i$. This means that $B_{\rho^*}[x;1]$ lies in V_i. Since cl V_i is compact, the ball is compact. The desired metric d is $\min\{\rho^*, 1\}$. □

THEOREM 3. *Let X be a locally compact metrizable space. Then $\langle 2^X, \tau_F \rangle$ is homeomorphic to a closed subspace of $\langle C(X,R), \tau_{CO} \rangle$.*

Proof. Let d be a metric on X as described in condition (b) of Theorem 2. We extend $E \to d(\cdot, E)$ from $\mathrm{CL}(X)$ to 2^X by mapping $\varnothing$ to the constant function $g \equiv 1$ on X. If X is compact then $\varnothing$ is an isolated point of the hyperspace, because $\{\varnothing\} = (X^C)^+$. Clearly, $N(g, X, \frac{1}{2})$ contains no distance function, because each such function assumes the value zero somewhere. Thus, g is also isolated in the relative topology for

$$\Omega = \{g\} \cup \{d(\cdot, E) : E \in \mathrm{CL}(X)\},$$

and the extension to 2^X remains a homeomorphism. If X is noncompact then all sets of the form $(K^C)^+$, where K is a nonempty (proper) compact subset of X, form a local base for τ_F at $\varnothing$. Since K is compact, for each ϵ in $(0,1)$ we have $d(x,K) \leq \epsilon$ if and only if $x \in B[K;\epsilon]$. Also, by the choice of the metric, for each such ϵ the generalized ball $B[K;\epsilon]$ is compact. As a result, for each such K and ϵ, the open set $(B[K; 1-\epsilon]^C)^+$ is mapped onto $\Omega \cap N(g, K, \epsilon)$, and the extension to 2^X again remains a homeomorphism.

To show that Ω is closed in $C(X,R)$, suppose a net $\langle d(\cdot, F_\lambda) \rangle$ converges uniformly on compacta to $h \in C(X,R)$. Set $F = \{x : h(x) = 0\}$. If K is a compact subset of X for which $K \cap F = \varnothing$, then $\langle F_\lambda \rangle$ must miss K eventually, because K is compact and h vanishes at any point of $\mathrm{Ls}\, F_\lambda \cap K$. On the other hand, if F hits an open set V, choose $x \in F$ and $\epsilon > 0$ with $S[x;\epsilon] \subset V$. Now $h(x) = 0$ implies that $d(x, F_\lambda) < \epsilon$ eventually, so $\langle F_\lambda \rangle$ hits the subset $S[x;\epsilon]$ of V eventually. Thus, $\langle F_\lambda \rangle$ τ_F-converges to F, whence $\langle d(\cdot, F_\lambda) \rangle$ converges to $d(\cdot, F)$ or to g (if F is empty) under the embedding. Since limits are unique in $C(X,R)$, we have $h \in \Omega$. □

From Theorem 3, we immediately get the two most important facts about the Fell topology, which can be established using the natural embedding of $\langle 2^X, \tau_F \rangle$ into $\mathrm{CL}(X^*)$ with the Vietoris topology, where X^* is the one-point compactification of X [12]. For a direct proof, consult [1].

THEOREM 4. *Let X be a locally compact metrizable space. Then $\langle 2^X, \tau_F \rangle$ is a compact Hausdorff space, and $\langle 2^X, \tau_F \rangle$ is metrizable if and only if X is separable.*

Proof. The family $\Omega = \{g\} \cup \{d(\cdot, E) : E \in \mathrm{CL}(X)\}$ in the proof of Theorem 3 is equi-Lipschitzian and thus equicontinuous. Also, for each $x \in X$, $\mathrm{cl}\{f(x) : f \in \Omega\}$, as a subset of $[0,1]$, is compact. Since Ω is closed, by the Ascoli theorem (cf. [21, Thm. 43.15]), Ω is compact with respect to the compact-open topology.

By Lemma 1, separability of X is necessary for first countability of the hyperspace, and thus for metrizability. Conversely, if X is separable, then X admits a metric d for which closed and bounded sets are compact (cf. [2] or [18]). Fix x_0 in X; then a countable collection of seminorms determining the (locally convex) compact-open topology on $C(X, R)$ is

$$p_n(f) = \sup\{|f(x)|: d(x, x_0) \leq n\} \quad (n \in Z^+),$$

whence the space $\langle C(X, R), \tau_{CO}\rangle$ is metrizable [17]. By Theorem 3, the hyperspace $\langle 2^X, \tau_F\rangle$ is metrizable, too. □

In this author's view, the natural way to extend the notion of distance function for nonempty closed sets in a metric space $\langle X, d\rangle$ to include $\varnothing$ is to identify $\varnothing$ with $g: X \to [0, \infty]$ defined by

$$g(x) = \sup\{d(x, z): z \in X\}.$$

If d is a bounded metric, then $\{g\} \cup \{d(\cdot, E): E \in \mathrm{CL}(X)\}$ is an equicontinuous family. In the literature, a different program has been used, following [20]. Independent of d, one sets $d(x, \varnothing) = \infty$ for each x in X, and a net $\langle F_\lambda\rangle$ in 2^X is declared *Wijsman convergent* [11] to $F \in 2^X$ provided $\langle d(\cdot, F_\lambda)\rangle$ converges pointwise to $d(\cdot, F)$. Wijsman convergence is compatible with a uniformizable topology on 2^X, and it is metrizable when X is separable (see [11, §4]). If d is a bounded metric, then the Wijsman topology, of course, agrees with the compact-open topology on $\mathrm{CL}(X)$, but the two topologies differ in the way they treat the empty set: In the Wijsman topology, the empty set must be an isolated point; whereas under our suggested correspondence, it need not be.

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Department of Mathematics
California State University
Los Angeles, CA 90032