ON UNIFORM APPROXIMATION BY HARMONIC FUNCTIONS

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1. Introduction. Let X be a compact set in $\mathbf{R}^2$ and let $R(X)$ denote the uniform closure on X of functions analytic in a neighborhood of X. The following concept of analytic content defined by

$$\lambda(X) \stackrel{\text{def}}{=} \inf_{\phi \in R(X)} \|\bar{z} - \phi(z)\|_\infty \tag{1}$$

has been introduced in [7] and studied in [5], [7], [9], and [10] ($\|\cdot\|_\infty$ stands for the supremum norm in the space of continuous functions $C(X)$ on X). As easily follows from the Stone–Weierstrass theorem, $\lambda(X) = 0$ if and only if $R(X) = C(X)$. This simple observation allows us to view $\lambda(X)$ as a certain measure of solvability of the problem of uniform approximation by analytic functions on X. As it turns out ([1] and [7]; see also [5], [9], and [10]), $\lambda(X)$ admits simple estimates in terms of basic geometric quantities of X such as area and perimeter. More precisely,

$$\left(\frac{A}{\pi}\right)^{1/2} \geq \lambda(X) \geq \frac{2A}{P}, \tag{2}$$

where A = area of X, P = perimeter of X (if X has a finite perimeter; otherwise, $P = \infty$); see [3, Ch. IV] and [8]. We mention that the inequality in the left-hand side of (2) was observed by Alexander [1] and the second inequality is due to the author [7]. We refer the reader to [5], [9], and [10] for a detailed discussion of these inequalities and related isoperimetric problems.

The purpose of this note is to develop a similar concept for $H(X)$, the uniform closure of the space of functions harmonic in a neighborhood of a compact set $X \subset \mathbf{R}^n$, $n \geq 2$. By similarity with (1) we define the harmonic content $\Lambda(X)$ of a compact set $X \subset \mathbf{R}^n$ to be

$$\Lambda(X) \stackrel{\text{def}}{=} \inf_{u \in H(X)} \||x|^2 - u\|_\infty, \tag{3}$$

where $|x|^2 = \sum_{i=1}^n x_i^2$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. The analogy with (1) can be seen if one observes the correspondence between $H(X)$ = uniform closure of the kernel of Δ on X and $R(X)$ = uniform closure of the kernel of $\partial/\partial\bar{z}$, $|x|^2 : \Delta(|x|^2) \equiv 2n \equiv \text{const} \neq 0$ in $\mathbf{R}^n$ and $\bar{z} : (\partial/\partial\bar{z})(\bar{z}) \equiv 1 \neq 0$ in $\mathbf{R}^2$. (We use the standard notation: Δ denotes the Laplacian $\sum_{i=1}^n (\partial^2/\partial x_i^2)$ and $\partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + i\,\partial/\partial y)$, $z = x + iy$.) However, in this case the equivalence

$$\Lambda(X) = 0 \Leftrightarrow H(X) = C(X) \tag{4}$$

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is no longer trivial, since $H(X)$ is not an algebra and, therefore, one cannot directly appeal to the Stone–Weierstrass theorem. The major goal of this paper is to show that (4) still holds.

The rest of the paper is organized as follows. In Section 2 we prove (4). It should be mentioned that in our proof we are making use of an idea due to Huber [6]. On the other hand, as a corollary from our theorem we also obtain one of Huber's theorems [6, Thm. 1] by a nonconstructive argument which is simpler than the original proof in [6].

Finally, in Section 3 we obtain some geometric estimates of $\Lambda(X)$ which are similar to (2), and discuss related problems.

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2. The main theorem. Let $X\subset \mathbf{R}^n$ be a compact set contained in a certain ball B. Define $f_0\in C_0^\infty$ to be equal to $(1/2n)|x|^2$ on X and $\equiv 0$ outside of B. It is clear that $\Lambda(X)=0 \Leftrightarrow \operatorname{dist}_{C(X)}(f_0, H(X))=0$.

Before proceeding with our Theorem let us discuss some of the notation we shall be using in the paper.

As usual, $C_0^\infty(\mathbf{R}^n)$ denotes the space of C^∞-functions in $\mathbf{R}^n$ with compact support. For a given X one can find a decreasing sequence of smoothly bounded finitely connected compact sets $\{X_l\}$ such that $\bigcap_{l=1}^\infty X_l = X$ and the Dirichlet problem is solvable on ∂X_l for all l (see [11, Ch. IV, §2]).

If X is a smoothly bounded compact set in $\mathbf{R}^n$ so that the Dirichlet problem is solvable on X, then

$$g(x,y)=g_X(x,y): X\times X \to \mathbf{R}_+$$

denotes the Green function on X (see [11, Ch. IV]). Also, we shall use the notation

$$k(x,y)=k_n(x,y)=\begin{cases} -\dfrac{1}{2\pi}\ln|x-y|, & n=2 \\ \dfrac{1}{\omega_n(n-2)}\cdot\dfrac{1}{|x-y|^{n-2}}, & n\geq 3\end{cases}$$

to denote the fundamental solution of the Laplacian taken with an opposite sign. (Here, ω_n stands for the area of the surface of the unit sphere in $\mathbf{R}^n$.) For μ, a finite, compactly supported Borel measure in $\mathbf{R}^n$,

$$U^\mu(x)=\int_{\mathbf{R}^n} k_n(x,y)\,d\mu(y)$$

denotes the potential of μ. It is well known that U^μ is defined almost everywhere with respect to Lebesgue measure $m_n(y)$ in $\mathbf{R}^n$, and is locally integrable there. It is also clear that $U^\mu(x)$ is harmonic outside of the support of μ (see [11]).

THEOREM. *$H(X)=C(X)$ if and only if $f_0\in H(X)$.*

Proof. Necessity is obvious. To prove sufficiency let us assume that $f_0 \in H(X)$. We will need the following two lemmas.

LEMMA 1 [6]. *Let $\{X_l\}_1^\infty$ be as above and let $g_l(x,y) = g_{X_l}(x,y)$ denote the Green functions of X_l respectively. Define*

$$c_l = \max_{x \in X_l} \int_{X_l} g_l(x,y)\, dm_n(y).$$

Then, $\lim_{l\to\infty} c_l = 0$ implies that $H(X) = C(X)$.

Proof of Lemma 1. At first we observe that, since for each l and each $x \in X_l$

$$g_l(x,y) = k(x,y) + h_x^l(y) \quad (h_x^l(y) \in H(X_l)) \tag{5}$$

is an integrable function, the integral

$$\int_{X_l} g_l(x,y)\, dm_n(y)$$

is a continuous function in X_l. So c_l are well defined and finite.

To prove the lemma it suffices to show that every measure annihilating $H(X)$ is identical zero. So let μ be a measure orthogonal to $H(X)$, that is (see [11, Ch. IV]),

$$U^\mu(x) \equiv 0 \quad \text{in } \mathbf{R}^n \setminus X.$$

It is convenient to separate the following.

ASSERTION. *For all l,*

$$\int_X |U^\mu(x)|\, dm_n(x) \le c_l \|\mu\|. \tag{6}$$

Proof of the Assertion. Let $X_0 = \{x \in X : U^\mu(x) \text{ is not defined}\}$. Then $m_n(X_0) = 0$. Then, for all l and all $x \in X_l \setminus X_0$, we have

$$\int_{X_l} g_l(x,y)\, d\mu(y) = \int_{X_l} k_n(x,y)\, d\mu(y) + \int_{X_l} h_x^l(y)\, d\mu(y) = U^\mu(x). \tag{7}$$

(As $h_x^l(y) \in H(X_l) \subset H(X)$ and $\mu \perp H(X)$, the second integral in (7) vanishes.) In particular, (7) holds almost everywhere on X. Therefore, applying Fubini's theorem, we obtain ($g_l \ge 0$) that

$$\begin{aligned}\int_{X_l} |U^\mu(x)|\, dm_n(x) &= \int_{X_l} \left| \int_{X_l} g_l(x,y)\, d\mu(y)\, dm_n(x) \right| \\ &\le \int_{X_l} \left\{ \int_{X_l} g_l(x,y)\, dm_n(x) \right\} d|\mu(y)| \le c_l \|\mu\|.\end{aligned}$$

This proves our assertion. Since (6) holds for all l, $c_l \downarrow 0$ implies that $U^\mu \equiv 0$ a.e. in $\mathbf{R}^n$ and, therefore, $\mu \equiv 0$. The lemma is proved. □

LEMMA 2. *Let X be a smoothly bounded finitely connected set in $\mathbf{R}^n$ for which the Dirichlet problem is solvable. Let $g(x,y) = g_X(x,y)$ be the Green function in X. Fix $x_0 \in \mathring{X}$ and extend $g(x_0,y)$ to $\mathbf{R}^n$ by setting $g(x_0,y) \equiv 0$ on $\mathbf{R}^n \setminus X$. Then*

$$\Delta_y g(x_0, y) = -\delta_{x_0} + \frac{\partial g(x_0, y)}{\partial n_y}\, dS_y\Big|_{\partial X} \tag{8}$$

in the sense of distributions. (Here, δ_{x_0} denotes the unit point-mass at x_0, $\partial/\partial n_y$ means the derivative in the direction of the inner normal $\vec{n}_y$ to ∂X, and $dS\,|_{\partial X}$ is the area measure on ∂X.)

Proof of Lemma 2. Take an arbitrary $\phi \in C_0^\infty(\mathbf{R}^n)$. Then, applying the distribution $\Delta_y g(x_0, y)$ to ϕ, we obtain from (5)

$$\begin{aligned}\langle \phi, \Delta g\rangle &= \langle \Delta\phi, g\rangle = \langle \Delta\phi, k(x_0, y) + h_{x_0}(y)\rangle \\ &= \int_X \Delta\phi(k(x_0, y) + h_{x_0}(y))\, dm_n(y) \\ &= \int_{\mathbf{R}^n} \Delta\phi k(x_0, y)\, dm_n(y) + \int_X \Delta\phi h_{x_0}(y)\, dm_n(y) \\ &\quad - \int_{\mathbf{R}^n\setminus X} \Delta\phi k(x_0, y)\, dm_n(y).\end{aligned} \tag{9}$$

As ϕ has a compact support, Green's formula yields

$$\int_{\mathbf{R}^n} \Delta\phi k(x_0, y)\, dm_n(y) = -\phi(x_0). \tag{10}$$

According to the second Green's formula, recalling that $\Delta(h_{x_0}(y)) = 0$ in X, we obtain

$$\begin{aligned}\int_X h_{x_0}\Delta\phi\, dm_n(y) &= \int_X (h_{x_0}\Delta\phi - \phi\Delta h_{x_0})\, dm_n(y) \\ &= -\int_{\partial X} h_{x_0}\frac{\partial\phi}{\partial n}\, dS + \int_{\partial X} \phi\frac{\partial h_{x_0}}{\partial n}\, dS.\end{aligned} \tag{11}$$

Also, since $k(x_0, y)$ is harmonic in $\mathbf{R}^n\setminus X$ and ϕ has a compact support, using Green's formula again, we can rewrite the last integral in (9) as follows:

$$\int_{\mathbf{R}^n\setminus X} \Delta\phi k(x_0, y)\, dm_n(y) = \int_{\partial X} k(x_0, y)\frac{\partial\phi}{\partial n}\, dS - \int_{\partial X} \phi\frac{\partial k(x_0, y)}{\partial n_y}\, dS. \tag{12}$$

Thus, combining formulas (5) and (9)–(12) and recalling that $g(x_0, y) \equiv 0$ on ∂X, we finally obtain

$$\begin{aligned}\langle \phi, \Delta g\rangle &= -\phi(x_0) - \int_{\partial X} h_{x_0}\frac{\partial\phi}{\partial n}\, dS + \int_{\partial X} \phi\frac{\partial h_{x_0}}{\partial n}\, dS \\ &\quad - \int_{\partial X} k(x_0, y)\frac{\partial\phi}{\partial n}\, dS + \int_{\partial X} \phi\frac{\partial k(x_0, y)}{\partial n_y}\, dS \\ &= -\phi(x_0) + \int_{\partial X} \phi\frac{\partial g(x_0, y)}{\partial n}\, dS = \left\langle \phi, -\delta_{x_0} + \frac{\partial g(x_0, y)}{\partial n}\, dS\Big|_{\partial X}\right\rangle.\end{aligned}$$

The lemma is proved. □

Now, according to Lemma 1, the theorem follows immediately from the following assertion.

LEMMA 3. *Let* $\{X_l\}$ *and* c_l *be the same as in Lemma* 1. *Then* $f_0 \in H(X)$ *implies that* $c_l \downarrow 0$ *as* $l \uparrow \infty$.

Proof of Lemma 3. At first, we observe that since $X_1 \supset X_2 \supset \cdots$ then, according to the maximal principal, $g_l(x, y) \le g_{l-1}(x, y)$ for all $y \in X_{l-1}$ as $x \in X_l$ is fixed. Then ($g_l(x, y) \ge 0$ for all l, x, y)

$$c_l = \max_{x \in X_l} \int_{X_l} g_l(x, y)\, dm_n(y) \le \max_{x \in X_l} \int_{X_l} g_{l-1}(x, y)\, dm_n(y)$$

$$\le \max_{x \in X_{l-1}} \int_{X_{l-1}} g_{l-1}(x, y)\, dm_n(y) = c_{l-1},$$

that is, $\{c_l\}$ is a decreasing sequence. Hence, to prove the lemma it suffices to show that $\{c_l\}$ contains a subsequence converging to zero. Since $f_0 \in H(X)$, there exists a sequence $\{h_l\}$ of functions harmonic in a neighborhood of X and such that

$$\|f_0 - h_l\|_{C(X)} < 1/2l.$$

Taking a subsequence, if necessary, we can assume that $h_l \in H(X_l)$ for all l. Moreover, since for each l_0, $f_0 - h_{l_0}$ is uniformly continuous in X_{l_0}, there exists a neighborhood U_{l_0} of X such that

$$\|f_0 - h_{l_0}\|_{C(\overline{U_{l_0}})} < 1/2l_0.$$

Also, for $l \ge l_1 = l_1(l_0) \ge l_0$ all $X_l \subset U_{l_0}$. Therefore, for each l_0 we can choose $l_0' > l_0$, $X_{l_0'} \in \{X_l\}$, $h_{l_0'} \in H(X_{l_0'})$ such that

$$\|f_0 - h_{l_0'}\|_{C(X_{l_0'})} \le 1/2l_0. \tag{13}$$

Let f_l denote the harmonic extension of $f_0|_{\partial X_l}$ into X_l. Then, by the maximum principle, from (13) it follows that

$$\|f_{l'} - h_{l'}\|_{C(X_{l'})} \le 1/2l \tag{14}$$

for all l, where $l' = l'(l) \ge l$ is as above.

Recall that $f_0 \in C_0^\infty(\mathbf{R}^n)$, and without loss of generality we can assume that $\Delta f_0 \equiv 1$ on X_1. Then for each l ($l' = l'(l) > l$) and each $x \in X_{l'}$, according to (8), (13), and (14) we obtain

$$\int_{X_{l'}} g_{l'}(x, y)\, dm_n(y) = \langle \Delta_y f_0, g_{l'}(x, y)\rangle = \langle f_0, \Delta_y g_{l'}(x, y)\rangle$$

$$= -f_0(x) + \int_{\partial X_{l'}} f_0(y) \frac{\partial g_{l'}(x, y)}{\partial n_y}\, dS_y = -f_0(x) + f_{l'}(x)$$

$$\le |h_{l'}(x) - f_0(x)| + |f_{l'}(x) - h_{l'}(x)| \le \frac{1}{2l} + \frac{1}{2l} \le \frac{1}{l}.$$

Hence, $c_{l'} \le 1/l$. Since $l' \uparrow \infty$ whereby $l \uparrow \infty$, the subsequence $c_{l'} \downarrow 0$. The lemma is proved and, therefore, the proof of the theorem is complete. □

COROLLARY 1. *Let us keep the same notation as above. Then the following are equivalent:*

(i) $H(X) = C(X)$;

(ii) $c_l \downarrow 0$ *for any sequence* $\{X_l\} : X_l \downarrow X$; *and*

(iii) $|x|^2 \in H(X)$.

Proof. (i) ⇒ (iii) is obvious; (iii) ⇒ (ii) follows from Lemma 3; and (ii) ⇒ (i) follows from Lemma 1. □

REMARK. (i)⇔(ii) has been observed by Huber [6]. However, our proof of (i)⇔(ii) is simpler than the one in [6], since it does not involve a direct construction of an annihilating measure, but rather points out a very simple function which is not approximable by harmonic functions.

3. Further remarks. The following proposition can be viewed as the quantitative version of the analog of the Hartogs–Rosenthal theorem for harmonic approximation (cf. [4, Ch. II, Cor. 8.4] and [11, Ch. V, Thm. 5.19]).

For a given compact set $X \subset \mathbf{R}^n$ let R_X denote the radius of the ball B_X whose volume is equal to the volume of X.

PROPOSITION 1. $\Lambda(X) \le R_X^2$ *if* $n = 2$ *and* $\Lambda(X) \le (n/(n-2))R_X^2$ *for* $n \ge 3$.

Proof. For the sake of brevity we will conduct the argument for $n \ge 3$. By the standard Hahn–Banach duality argument,

$$\Lambda(X) = \sup_{\substack{\mu \perp H(X) \\ \operatorname{supp}\mu \in X \\ \|\mu\| \le 1}} \left| \int_X |x|^2 \, d\mu \right|. \tag{15}$$

Since we can regard $|x|^2$ on X as the restriction of a function $\phi_0 \in C_0^\infty(\mathbf{R}^n)$, using Fubini's theorem we can rewrite (15) as follows (cf. (10)):

$$\begin{aligned} \Lambda(X) &= \sup_{\substack{\mu \perp H(X) \\ \|\mu\| \le 1}} \left| -\int_X \left\{ \int_{\mathbf{R}^n} \Delta\phi_0 k_n(x, y) \, dm_n(y) \right\} d\mu(x) \right| \\ &= \sup_{\substack{\mu \perp H(X) \\ \|\mu\| \le 1}} \left| 2n \int_X U^\mu(y) \, dm_n(y) \right| \\ &= 2n \sup_{\substack{\mu \perp H(X) \\ \|\mu\| \le 1}} \left| \int_X \left\{ \int_X k_n(x, y) \, dm_n(y) \right\} d\mu(x) \right| \\ &\le 2n \max_{x \in X} U^X(x). \end{aligned} \tag{16}$$

(Here, U^X denotes the potential of the measure $m_n|_X$.) In the second equality in (16), we used the fact that since $\mu \perp H(X)$, $U^\mu \equiv 0$ on $\mathbf{R}^n \setminus X$. As a convolution of bounded and locally integrable functions, U^X is continuous in $\mathbf{R}^n$. Therefore, it attains its maximum at a certain point $x_0 \in X$. Without loss of generality, we can assume that $x_0 = (0, \dots, 0)$. Let $B_X = \{x \colon |x| \le R_X\}$. Since

$$\mathrm{Vol}(B_X) = \mathrm{Vol}(B_X \cap X) + \mathrm{Vol}(B_X \setminus X) = \mathrm{Vol}(X \cap B_X) + \mathrm{Vol}(X \setminus B_X) = \mathrm{Vol}(X),$$

$\mathrm{Vol}(B_X \setminus X) = \mathrm{Vol}(X \setminus B_X)$. Also, for each $y_1 \in X \setminus B_X$ and each $y_2 \in B_X \setminus X$, $k_n(0, y_1) \le k_n(0, y_2)$. So, using polar coordinates we obtain

$$U^X(0) \le \int_{B_x} k_n(0, y)\, dm_n(y) = \frac{1}{\omega_n(n-2)} \int_{B_x} \frac{1}{|y|^{n-2}}\, dm_n(y) = \frac{1}{2(n-2)} R_X^2.$$

Thus, finally it follows from (16) that

$$\Lambda(X) \le \frac{n}{n-2} R_X^2 \tag{17}$$

and the proof is complete. □

REMARK. The estimate (17) is not sharp. In fact, all measures $\mu \perp H(X)$ have nontrivial positive and negative parts. Therefore, the inequality in (16) is actually strict. Below, by employing a different method, we will obtain a sharp estimate for $\Lambda(X)$.

COROLLARY 2 (the "Hartogs-Rosenthal" Theorem). *If* $\mathrm{Vol}(X) = 0$, *then* $H(X) = C(X)$.

The following proposition relates the extremal problems (3) and (15) to the best harmonic majorant of $|x|^2$ (cf. [16]).

PROPOSITION 2. *Let* ∂X *be smooth and, therefore, the Dirichlet problem is solvable on* X. *Let* u_0 *be the best harmonic majorant of* $|x|^2$ *in* X, *that is,* $u_0 \in H(X)$ *and* $u_0|_{\partial X} \equiv |x|^2$. *Then,* $\Lambda(X) = \frac{1}{2}\|u_0 - |x|^2\|_{C(X)}$. *Moreover, the harmonic function*

$$u^* = u_0 - \tfrac{1}{2}\|u_0 - |x|^2\|_{C(X)}$$

is the extremal function in (3), *that is,* $\| |x|^2 - u^* \|_\infty = \Lambda(X)$.

Proof. Since $u_0 - |x|^2$ is continuous and superharmonic on X and $\equiv 0$ on ∂X, it is positive on X and therefore attains its maximum at a certain point $x_0 \in \mathring{X}$. Consider measure

$$\mu^* = \frac{1}{2}\delta_{x_0} - \frac{1}{2}\frac{\partial g_X(x_0, y)}{\partial n_y}\, dS_y \bigg|_{\partial X}.$$

(We keep the same notation as in Section 2.) It is clear (cf. Lemma 2) that $\mu^* \perp H(X)$ and $\|\mu^*\| = 1$. Therefore, from (15), we obtain that

$$\Lambda(X) = \sup_{\substack{\mu \perp H(X) \\ \|\mu\| \le 1}} \left| \int_X (u_0 - |x|^2)\, d\mu \right| \ge \int_X (u_0 - |x|^2)\, d\mu^*$$

$$= \tfrac{1}{2}(u_0(x_0) - |x_0|^2) = \tfrac{1}{2}\|u_0 - |x|^2\|_\infty.$$

On the other hand, since $u^* \in H(X)$ and

$$\|u^* - |x|^2\|_\infty = \max\begin{cases} \max\limits_{x\in X_1} u_0 - |x|^2 - \frac{1}{2}\|u_0 - |x|^2\|, & \text{where } X_1 = \{x \in X : u^*(x) \geq |x|^2\} \\ \max\limits_{x\in X_2} \frac{1}{2}\|u_0 - |x|^2\| - (u_0 - |x|^2), & \text{where } X_2 = X \setminus X_1 \end{cases}$$

$$= \tfrac{1}{2}\|u_0 - |x|^2\|_\infty,$$

$\Lambda(X) \leq \frac{1}{2}\|u_0 - |x|^2\|_\infty$. Thus, $\Lambda(X) = \frac{1}{2}\|u_0 - |x|^2\|_\infty$ and u^* is the best harmonic approximation to $|x|^2$ in X.

Without loss of generality, we can assume that the point x_0, where $u_0 - |x|^2$ attains its maximum, is the origin. Then the above proposition can be stated as

$$\Lambda(X) = \tfrac{1}{2}u_0(0),$$

provided that $\|u_0 - |x|^2\|_\infty = u_0(0)$. A beautiful result of Payne [14], obtained by making use of deep properties of the Schwarz symmetrization, states that

$$u_0(0) \leq R_X^2, \tag{18}$$

and equality occurs if and only if up to a set of capacity zero X is a ball (see also [2, p. 70], [15], and [16]). Therefore, from Proposition 2 and (18), applying a standard approximation argument we obtain the following improvement of Proposition 1.

COROLLARY 3. *For any compact set $X \subset \mathbf{R}^n$*

$$\Lambda(X) \leq \tfrac{1}{2}R_X^2 \tag{19}$$

Moreover, if X is essential for $H(X)$ (i.e., X is equal to the closure of the set of all nonpeak points of $H(X)$), then equality in (19) *holds if and only if X is a ball.*

We finish with two more comments.

I. It is an interesting and nontrivial problem to find an appropriate lower bound for $\Lambda(X)$ in terms of simple geometric quantities (cf. [14] and [2, Ch. II, Thm. 2.9]). However, a simple estimate similar to the one in (2), and depending only on the volume and perimeter, does not hold for $\Lambda(X)$ for the following reason. Let us consider a nowhere dense set $X \subset \mathbf{R}^2$, so-called Swiss cheese, obtained by deleting from the unit disk Δ_0 a sequence of pairwise disjoint open disks Δ_j whose radii r_j have a finite sum and whose union is dense in Δ_0. Then, as is well known, X has a positive area and a finite perimeter $2\pi\sum_0^\infty r_j$ (see [5, Ch. II], [13], and [8]). However, it is known that one can still choose r_j in such a way that numbers c_l corresponding to $X_l = \Delta_0 \setminus \bigcup_{j=1}^{l}$ converge to zero. So, by Lemma 1, $H(X) = C(X)$ and therefore $\Lambda(X) = 0$ (see [6] and [12]).

II. After this paper had been submitted, the author showed that the analogues of Theorem 1, Corollaries 1 and 2, and Proposition 2 also hold for general uniformly elliptic operators of the second order with sufficiently smooth coefficients. The corresponding statements of those results with the proofs will appear elsewhere.

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