THE UNIQUENESS OF DECOMPOSITION OF A CLASS OF MULTIVALENT FUNCTIONS

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1. Introduction and statement of main theorem. Let P be a nonconstant polynomial. A curve ℓ will be called a *curved P-ray* if there is a path γ from $[0, \infty)$ onto ℓ such that $\gamma(t) \to \infty$ as $t \to \infty$, and $P \circ \gamma$ is one-to-one. Although a curved P-ray may contain critical points of P, we will be interested here in curved P-rays that do not.

Let $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{P}$ denote (respectively) the unit disc $\{z : |z| < 1\}$, the complex plane, and the Riemann sphere. Also, for any subset A of $\mathbf{C}$ let $\mathrm{Int}(A)$, $\mathrm{Bd}(A)$, $\mathrm{Ext}(A)$, and $\mathrm{Cl}(A)$ denote (respectively) the interior, boundary, exterior, and closure of A in $\mathbf{C}$. Denote by S the familiar class of all functions f analytic and univalent in $\mathbf{B}$ satisfying $f(0) = 0$ and $f'(0) = 1$.

Let f be a function which is analytic in $\mathbf{B}$ and has $p-1$ critical points (counting multiplicity). Also, suppose that $f = P \circ \phi$, where P is a polynomial of degree p and $\phi \in S$. This decomposition may not be unique in the sense that there may be another polynomial Q of degree p and another univalent function $\sigma \in S$ such that $f = Q \circ \sigma$. At the end of this paper we give an example of a function with non-unique decomposition. This example can be read independently of the rest of the paper. For another example see Lyzzaik [5].

The purpose of this paper is to give a quite general sufficient condition that guarantees unique decomposition, as follows.

THEOREM 1. *Let f be a function which is analytic in $\mathbf{B}$ and has $p-1$ critical points (counting multiplicity). Suppose $f = P \circ \phi$, where P is a polynomial of degree p and $\phi \in S$. Also, suppose that $B = P^{-1}\{P(z) : z$ is a critical point of $P\}$, and that there is a disjoint collection W of curved P-rays ℓ in $\mathbf{C} - \phi(\mathbf{B})$ such that $B \cap (\mathbf{C} - \phi(\mathbf{B})) \subset \bigcup_{\ell \in W} \ell$. If $f = Q \circ \psi$, where Q is a polynomial of degree p and $\psi \in S$, then Q and ψ are identical to P and ϕ, respectively.*

Note that B is the finite set of critical points of P and points mapped by P to the images of critical points. Since $\phi(\mathbf{B})$ contains all critical points of P it follows that $A = B \cap (\mathbf{C} - \phi(\mathbf{B}))$ is finite and contains no critical point of P.

The example we give at the end of this paper is one of the simplest functions that does not satisfy the hypotheses of the theorem.

We will show that the class of functions described in this theory properly contains the class $K(p)$ of close-to-convex functions of order p as defined by Livingston [3].

2. Proof of theorem. The proof of Theorem 1 will be executed in a sequence of lemmas. For convenience let $\Phi = \phi(\mathbf{B})$, and let $A = B \cap (\mathbf{C} - \Phi)$.

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Recall the assumptions of Theorem 1, and let $h=\psi\circ\phi^{-1}$, so that h is a conformal map between Φ and $\psi(\mathbf{B})$. Since $P\circ\phi=Q\circ\psi$ in $\mathbf{B}$, h is a (single-valued and analytic) branch of $Q^{-1}\circ P$. For the purpose of this proof we will assume that W consists only of that finite number of curved P-rays required to cover A. Furthermore, we assume (without loss of generality) that the initial point of each curved P-ray in W belongs to A.

Since the proof of Theorem 1 is fairly long, let us look at what we know and what we are trying to show, so that it will be easier to understand why there is a difficulty. We want to show that h is the identity function. Since ψ and ϕ are in S, $h(z)=z+a_2z^2+\cdots$. If we could just show that h, with domain Φ, has an analytic continuation to a univalent function on $\mathbf{C}$, then h would have to be linear. When this is combined with the fact that $h(0)=0$ and $h'(0)=1$, we would be able to conclude that $h(z)=z$, as desired. This would finish the proof of Theorem 1. The notation we use for analytic continuation is adopted from Conway [1].

Suppose $\gamma: [0,1]\to\mathbf{C}$ is a path such that $\gamma(0)\in\Phi$. We wish to show that the function element (h,Φ) admits analytic continuation along γ. Since h is a branch of $Q^{-1}\circ P$, P is a polynomial, and Q^{-1} has only finitely many branch points, the continuation proceeds perfectly smoothly until $P(\gamma(t))$ comes to a branch point of Q^{-1}. Furthermore, since $f=P\circ\phi=Q\circ\psi$ we must have $\{P(z): P'(z)=0\}=\{Q(z): Q'(z)=0\}$. It follows that if $P(\gamma(t))$ lies at a branch point of Q^{-1} then $\gamma(t)\in B=P^{-1}\{P(z): P'(z)=0\}$ defined above. Even when we exclude the points of the set A, the other points of B, those in Φ can cause difficulty if the path γ leaves Φ and then returns to a point of B in Φ. We start with the following.

LEMMA 1. *The function h extends to a locally univalent function in* $\mathbf{C}-\bigcup_{\ell\in W}\ell$.

Note that $\mathbf{C}-\bigcup_{\ell\in W}\ell$ is a simply connected domain that contains Φ, but no point of A. Therefore, we find it entirely reasonable to expect that whenever γ returns to Φ, the continuation of $(h;\Phi)$ returns to the same germ so that (h,Φ) admits analytic continuation along all γ in $\mathbf{C}-\bigcup_{\ell\in W}\ell$. Thus, by the Monodromy Theorem, h would extend to an analytic function in $\mathbf{C}-\bigcup_{\ell\in W}\ell$.

Proof. Let Γ be a Jordan curve in Φ whose inner domain contains $B\cap\Phi$. Let Δ be the outer domain of Γ, and let $K=\Delta\cap\Phi$. Then K is a doubly connected region.

Suppose $\gamma: [0,1]\to\mathbf{C}-\bigcup_{\ell\in W}\ell$ is a path such that $\gamma(0)\in\Phi$. We shall show that the functional element (h,Φ) admits an analytic continuation along γ.

First, let us suppose that γ is contained in $\mathrm{Cl}(\Delta)-\bigcup_{\ell\in W}\ell$. Then $\gamma\cap B=\varnothing$. Since $\{P(z): P'(z)=0\}=\{Q(z): Q'(z)=0\}$ and h is a branch of $Q^{-1}\circ P$ in Φ, h admits an analytic continuation along γ, via P followed by Q^{-1} both applied locally. In this case we need to show that if $\gamma(1)\in\Phi$ then the continuation of (h,Φ) along γ yields (h,Φ) at $\gamma(1)$. Thus, suppose γ is contained in $\mathrm{Cl}(\Delta)-\bigcup_{\ell\in W}\ell$ with $\gamma(1)\in\Phi$. Let $\{(h_t,\Phi_t); 0\le t\le 1\}$, $h_0=h$, and $\Phi_0=\Phi$ be an analytic continuation of (h,Φ) along γ. We show that $[h_1]_{\gamma(1)}=[h]_{\gamma(1)}$. There exists a path σ: $[0,1]\to\mathrm{Cl}(\Delta)\cap\Phi$ such that $\sigma(0)=\gamma(0)$, $\sigma(1)=\gamma(1)$, and σ and γ are fixed endpoint homotopic arcs in $\mathrm{Cl}(\Delta)$. Note that the singleton set $\{(h,\Phi)\}$ is an analytic continuation along σ. Since $[h_0]_{\gamma(0)}=[h]_{\sigma(0)}$, the Monodromy Theory yields $[h_1]_{\gamma(1)}=[h]_{\sigma(1)}=[h]_{\gamma(1)}$.

Now suppose that γ is contained in $\mathbf{C}-\bigcup_{\ell\in W}\ell$ with $\gamma(0),\gamma(1)\in K$. Suppose further that (h,Φ) admits an analytic continuation (using the notation above) along γ. We need to show that $[h_1]_{\gamma(1)}=[h]_{\gamma(1)}$. Suppose this is false; then, from the above, γ must meet the inner domain of Γ. Similarly, γ cannot lie entirely in Φ. Let $v=\sup\{t:0<t<1$ and $\gamma(t)\in\Gamma\}$. Since $\gamma\cap\Gamma$ is closed, $\gamma(v)\in\Gamma$ and $0<v<1$. It follows from the preceding paragraph that $[h_v]_{\gamma(v)}\neq[h]_{\gamma(1)}$; otherwise, $[h_1]_{\gamma(1)}=[h]_{\gamma(1)}$. Now let $u=\inf\{t:0<t<1,\ \gamma(t)\in\Gamma,$ and $[h_t]_{\gamma(t)}\neq[h]_{\gamma(t)}\}$. Note that u exists, $0<u<1$, and $\gamma(u)\in\Gamma$. Suppose $[h_u]_{\gamma(u)}=[h]_{\gamma(u)}$. Then $h_u\equiv h$ in some open neighborhood of $\gamma(u)$. This implies by the continuity of γ that there is a $\delta>0$ such that $[h_t]_{\gamma(t)}=[h]_{\gamma(t)}$ for all $t\in(u-\delta,u+\delta)$, which contradicts the definition of u. Hence $[h_u]_{\gamma(u)}\neq[h]_{\gamma(t)}$. By the definition of u, $\gamma(t)\notin\Gamma$ for t near u, $t<u$. As a consequence of the previous paragraph, the subarc $\gamma\,|_{[0,u]}$ of γ must have an interior point in the inner domain of Γ. Let $s=\sup\{t:t\in(0,u)$ and $\gamma(t)\in\Gamma\}$. Then s exists, $\gamma(s)\in\Gamma$, and $s<u$. From the definition of u we have $[h_s]_{\gamma(s)}=[h]_{\gamma(s)}$. Observe that the restriction of γ to (s,u) lies either in Δ or in the inner domain of Γ. In either case we can conclude that $[h_u]_{\gamma(u)}=[h]_{\gamma(u)}$, a contradiction. Therefore, our assumption is false and $[h_1]_{\gamma(1)}=[h]_{\gamma(1)}$.

We complete the proof by showing that (h,Φ) admits analytic continuation along γ, where γ is now assumed to be general. Suppose that there is a τ, $0<\tau\leq 1$, such that (h,Φ) admits a continuation $\{(h_t,\Phi_t):0\leq t<\tau\}$ only along the subarc $\gamma\,|_{[0,\tau)}$ of γ. It follows directly that $\gamma(\tau)\in B\cap\Phi$ and $\gamma\,|_{[0,\tau)}\cap(\mathbf{C}-\Phi)\neq\varnothing$. Thus there exist $t_1,t_2\in(0,\tau)$ such that $\gamma(t_1),\gamma(t_2)\in K$ and the subarcs of γ corresponding to the intervals $[0,t_1]$ and $[t_2,\tau]$ are contained in Φ. From the preceding paragraph we conclude that the branches of the continuation at $\gamma(t_1)$ and $\gamma(t_2)$ are $[h]_{\gamma(t_1)}$ and $[h]_{\gamma(t_2)}$, respectively. This, however, implies that (h,Φ) continues analytically beyond $\gamma\,|_{[0,\tau)}$, and that the branch at $\gamma(\tau)$ is $[h]_{\gamma(t)}$. Therefore, (h,Φ) continues analytically along γ.

By virtue of the Monodromy Theorem, since $\mathbf{C}-\bigcup_{\ell\in W}\ell$ is simply connected, h extends to an analytic function in $\mathbf{C}-\bigcup_{\ell\in W}\ell$. This function is locally univalent since it is a branch of $Q^{-1}\circ P$ and P has no critical values in $\mathbf{C}-\Phi$. This completes the proof. □

We now assume that h has been continued analytically to all of $\mathbf{C}-\bigcup_{\ell\in W}\ell$, which is all of $\mathbf{C}$ except for a finite number of disjoint curved P-rays. Further analytic continuation of h to a larger simply connected domain can be viewed as a local matter, continuing h up each curved P-ray in W until locally univalent continuation fails or h continues the total length. This process leads us to a unique minimal collection V of curved P-subrays of W such that h extends locally univalently to $\mathbf{C}-\bigcup_{\ell\in V}\ell$. The crux of the proof of Theorem 1 is in showing that V is empty.

LEMMA 2. *Every curved P-ray in V has its initial point* $\zeta_0\in A$, *and* $\lim_{\zeta\to\zeta_0}h(\zeta)$ *is a critical value of* Q, *where the limit is taken within* $\mathbf{C}-\bigcup_{\ell\in V}\ell$.

Proof. If $\zeta_0\notin A$, then h can be continued through ζ_0 since there will be a neighborhood $\mathbf{N}$ of ζ_0 such that P will be univalent on $\mathbf{N}$ and the appropriate branch of Q^{-1} will be univalent on $P(\mathbf{N})$. This is impossible since V is minimal. □

That the limit of $h(\zeta)$ (as $\zeta \to \zeta_0$ within $\mathbf{C} - \bigcup_{\ell \in V} \ell$) exists and is a critical point of Q is clear from the local nature of branches of Q^{-1} near any point. However, we provide a proof that this limit exists. Let D be an open disc centered at $P(\zeta_0)$. We choose D small enough so that $Q^{-1}(D)$ is a disjoint union of components, G, each of which under Q covers every point of D the same number of times with the exception of $P(\zeta_0)$, which is covered exactly once. Note that although $Q|_{\bar{G}}^{-1}$ is not necessarily a (single valued) function, it is continuous at $P(\zeta_0)$. We may suppose that D is small enough so that P maps some neighborhood M of ζ_0 univalently onto D. The curve ℓ in V with initial point ζ_0 may divide M into several components. Choose the one component, $\mathbf{M}_0$, that has ζ_0 on its boundary. Since h is defined on $\mathbf{M}_0$, and P is univalent on $\mathbf{M}_0$, there is a branch of Q^{-1} on $P(\mathbf{M}_0)$ such that $h = Q^{-1} \circ P$ on $\mathbf{M}_0$. Also, as $\zeta \to \zeta_0$ in $\mathbf{M}_0$, $P(\zeta) \to P(\zeta_0)$ and $Q^{-1}(P(\zeta)) \to Q^{-1}(P(\zeta_0))$, a critical point of Q.

Of course, V is empty, but we don't know it yet. What we do to get around this is build a Riemann surface that contains the image of h on $\mathbf{C} - \bigcup_{\ell \in V} \ell$. Using the terminology of Springer [7], we show that our Riemann surface is a smooth, unlimited covering of $\mathbf{C}$. This leads directly to the fact that the Riemann surface is a one-sheeted covering surface of $\mathbf{C}$. From there it will be relatively easy to show that h is the identity function.

LEMMA 3. *Every $\ell \in V$ is contained in a simply connected domain G such that* $\mathrm{Bd}(G)$ *is a Jordan arc with infinite endpoints, and P is univalent in* $\mathrm{Cl}(G)$. *Moreover, if G and G' are two such domains then* $\mathrm{Cl}(G) \cap \mathrm{Cl}(G') = \varnothing$.

Proof. We construct a disjoint collection of neighborhoods of the curves in V such that P is univalent in each neighborhood. Let $\ell = \ell(t)$, $0 \le t < \infty$, be a curve in V.

We provide an inductive procedure to cover ℓ. Let D_1 be the largest open disc centered at $\ell(0)$ such that D_1 meets no other curve in V, and P is one-to-one on $D \cup \ell$. Let H_1 be the open disc centered at $\ell(0)$ with radius half that of D_1. That D_1 exists may be seen by assuming the contrary, and then choosing sequences of points, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, such that $a_n \in \mathbf{C} - \ell$ and $a_n \to \ell(0)$, $b_n \in \ell$, and $P(a_n) = P(b_n)$. We may assume without loss of generality that $b_n \to b \in \ell$. Either $b = \ell(0)$ so that $\ell(0)$ is a critical point of P, or $b \neq \ell(0)$ so that $P(\ell(0)) = P(b)$ and P is not one-to-one on ℓ. Either case is contradictory to the hypotheses.

Assume that D_k and H_k, $1 \le k \le n$, have been chosen with center $\ell(t_k)$. We choose D_{n+1} in the following manner. Let t_{n+1} be the first point greater than t_n such that $\ell(t_{n+1}) \in \mathrm{Bd}(H_n)$. Let D_{n+1} be the largest open disc centered at $\ell(t_{n+1})$ such that D_{n+1} meets no other curve in V, and P is one-to-one on

$$\ell \cup \left(\bigcup_{k=1}^{n} H_k \right) \cup D_{n+1}.$$

That such a D_{n+1} exists is obvious since, for small radii, it would be inside D_n. Let H_{n+1} be the disc centered at $\ell(t_{n+1})$ with radius half that of D_{n+1}.

We claim that $\ell \subset \bigcup_{k=1}^{\infty} H_k$. Let $N(\ell) = \bigcup_{k=1}^{\infty} H_k$. If $\ell \not\subset N(\ell)$, then the centers $\ell(t_k) \to a \in \ell$, $a \notin N(\ell)$. We will show that there is an open disc D centered at a

such that D meets no other curve in V, and P is one-to-one on $\ell \cup N(\ell) \cup D$. Suppose no such D exists. Choose sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $a_n \in \mathbf{C} - [\ell \cup N(\ell)]$, $b_n \in \ell \cup N(\ell)$, $P(a_n) = P(b_n)$ for all n, $a_n \to a$, and $b_n \to b \in \mathrm{Cl}[\ell \cup N(\ell)]$. If $b \in \ell$ we have the same contradiction obtained in the selection of D_1. Thus, suppose that $b \in \mathrm{Cl}(H_s)$ for some s, and $b \notin \ell$. But then P cannot be one-to-one on $\ell \cup (\bigcup_{k=1}^{s-1} H_k) \cup D_s$ since $P(b) = P(a)$. Thus (for all k sufficiently large) $\mathrm{Cl}(D_k) \subset D$, a contradiction to the maximality hypothesis for D_k. Hence $\ell \subset N(\ell)$.

If $\ell, \ell' \in V$, $\ell \neq \ell'$, then $N(\ell) \cap N(\ell') = \varnothing$. If this were not the case, then there would be a point $a \in N(\ell) \cap N(\ell')$ so that $a \in H_k \cap H_j'$, where H_k is one of the discs forming $N(\ell)$ and H_j' is one of the discs forming $N(\ell')$. Assume, without loss of generality, that the radius of H_k is as big as the radius of H_j'. Then the center of H_j' lies in D_k, a contradiction to the hypothesis that $D_k \cap \ell' = \varnothing$.

If each $N(\ell)$ is replaced with the smallest simply connected domain that contains $N(\ell)$ then P will still be univalent on the new domains, and the new collection of domains will continue to be pairwise disjoint. We designate each of the enlarged domains by the same symbol, $N(\ell)$.

Since $\lim_{t \to \infty} \ell(t) = \infty$, infinity is an accessible boundary point (on **P**) of $N(\ell) - \ell$ from both sides of ℓ. Thus one easily forms a Jordan curve in $(N(\ell) - \ell) \cup \{\infty\}$ that "bounds" a simply connected domain $G \subset N(\ell)$ with $\ell \subset G$. It is also clear that $\mathrm{Cl}(G) \cap \mathrm{Cl}(G') = \varnothing$ since $\mathrm{Cl}(G) \subset N(\ell)$ and $\mathrm{Cl}(G') \subset N(\ell')$. This completes the proof. □

As pointed out earlier, our goal is to show that $V = \varnothing$. To do this, suppose that $V \neq \varnothing$ and let X be the Riemann surface of h in $\mathbf{C} - \bigcup_{\ell \in V} \ell$ taken over the image plane. We intend to reach a contradiction through a global study of X.

The Riemann surface X is constructed in the usual manner of points, $(z, h(z))$, and inherits both the topological and conformal structures of $\mathbf{C} - \bigcup_{\ell \in V} \ell$ under h. Let π denote the projection map $\pi((z, h(z))) = h(z)$, and het $\tilde{h}$ be the bijection $\tilde{h}(z) = (z, h(z))$. Then $\tilde{h}$ is conformal, π is analytic, and $h = \pi \circ \tilde{h}$.

LEMMA 4. *X, π can be embedded in a smooth, unlimited covering Ψ, Π of* $\mathbf{C}$.

Proof. Let **G** be the collection of all domains G obtained in Lemma 3. Lemma 3 implies that h is univalent in $\mathrm{Cl}(G) - \ell$, where $\ell \in V$ is in G. Clearly $h(\mathrm{Bd}(G))$ is a Jordan arc with its two ends at infinity. Thus it divides $\mathbf{C}$ into two simply connected domains, one of which, D_G, contains $h(\mathrm{Cl}(G) - \ell)$.

For every $G \in \mathbf{G}$ let $\tilde{D}_G = \{(D_G, w) : w \in D_G\}$. Extend π to the "domain" $\tilde{D}_G$ by $\Pi(D_G, w) = w$. Put on each $\tilde{D}_G$ the topology that makes $\Pi \mid_{\tilde{D}_G}$ a homeomorphism.

Let Ψ_0 be the free union of X and the domains $\tilde{D}_G$, $G \in \mathbf{G}$. Identify points $T \in \tilde{D}_G$ with $U \in \tilde{h}(\mathrm{Cl}(G) - \ell)$ if $\Pi(T) = \Pi(U)$. Let Ψ be the quotient space of Ψ_0 by this identification. For convenience, we view X and each $\tilde{D}_G$ as subsets of Ψ. Also, we view Π as defined on Ψ. In fact, by Dugundji (see [2, pp. 120–136]), X and each $\tilde{D}_G$ are open in Ψ.

We shall show that the pair Ψ, Π is a smooth and unlimited analytic covering of $\mathbf{C}$. It is routine to verify that Ψ is connected and Hausdorff. Since each $\tilde{D}_G$ is homeomorphic to D_G under Π, and X is homeomorphic to $\mathbf{C} - \bigcup_{\ell \in V} \ell$ under

$\tilde{h}^{-1}$, Ψ is a surface. We take $\{(\tilde{D}_G, \Pi \mid \tilde{D}_G): G \in \mathbf{G}\} \cup \{(X, \tilde{h}^{-1})\}$ to be the atlas of local homeomorphisms. This atlas forms a conformal structure on Ψ; Ψ becomes a Riemann surface and Π becomes a locally univalent map. Thus Ψ, Π is a smooth covering surface of $\mathbf{C}$ in the sense of Springer [7]. We still must show that Ψ, Π is an unlimited covering surface of $\mathbf{C}$.

Let us first show that $\Pi: \Psi \to \mathbf{C}$ is surjective. Since $\mathrm{Bd}[h(\mathbf{C} - \bigcup_{\ell \in V} \ell)]$ is contained in $\bigcup_{G \in \mathbf{G}} D_G$, and $h(\mathbf{C} - \bigcup_{\ell \in V} \ell)$ contains $\mathrm{Bd}(D_G)$ for all $G \in \mathbf{G}$,

$$\mathrm{Bd}\left[h\left(\mathbf{C} - \bigcup_{\ell \in V} \ell\right) \cup \left(\bigcup_{G \in \mathbf{G}} D_G\right)\right] = \varnothing.$$

Thus $h(\mathbf{C} - \bigcup_{\ell \in V} \ell) \bigcup_{G \in \mathbf{G}} D_G = \mathbf{C}$; Π is surjective.

Let $\gamma: [0,1] \to \mathbf{C}$ be a path, and let R_0 be a point of Ψ above $\gamma(0)$. Suppose there is a lift through R_0 of $\gamma \mid_{[0,t)}$. Let $t_n \uparrow t_0$. Since $h = Q^{-1} \circ P$, Ψ is finite-sheeted over $\mathbf{C}$. Therefore we may assume that all P_n, where P_n is the lift of $\gamma(t_n)$, lie in one chart E which is X or D_G. By the nature of $\mathrm{Bd}(D_G)$ and $\mathrm{Bd}(h(\mathbf{C} - \bigcup_{\ell \in V} \ell))$ we may assume that $P_n \to P_0 \in \Psi$. Π is univalent in a neighborhood of P_0, so that the lift of γ may be continued through t_0. This completes the proof. □

The proof of Theorem 1 is now easily completed. Since Ψ, Π is a smooth unlimited covering surface of $\mathbf{C}$, it is in fact a one-sheeted covering (see Springer [7, p. 88]). Recall that $h: \mathbf{C} - \bigcup_{\ell \in V} \ell \to \mathbf{C}$ is $h = \Pi \circ \tilde{h}$, so that h is univalent. Since V has been assumed nonempty, this leads to a contradiction to Lemma 2: $h(\Phi)$ contains all the critical values of Q, so that h cannot both be univalent and satisfy $\lim_{\zeta \to \zeta_0} h(\zeta)$ equals a critical value of Q. We finally conclude that $V = \varnothing$.

By the definition of V, h is a locally univalent entire function. Since h is a branch of $Q^{-1} \circ P$, h has a pole at infinity. Hence $h(z) = az + b$. But $h(0) = 0$ and $h'(0) = 1$, so h is the identity map. Therefore ϕ and ψ are identical, and so are P and Q.

3. Applications and examples. Lyzzaik [4] has shown that if f belongs to Livingston's class $K(p)$ of close-to-convex functions of order p then $f = P \circ \phi$, where P is a polynomial of degree p and $\phi \in S$. Furthermore, Lyzzaik [5] has shown that this decomposition is unique. Theorem 1 is a generalization of this latter result. We give a corollary that provides a new proof of Lyzzaik's result.

A curved P-ray ℓ is called a *P-ray* if $P(\ell)$ is a Euclidean ray.

COROLLARY 1. *Let f be a function which is analytic in $\mathbf{B}$ and has $p-1$ critical points (counting multiplicity). Suppose $f = P \circ \phi$, where P is a polynomial of degree p and $\phi \in S$. Also, suppose that $\mathbf{C} - \phi(\mathbf{B})$ is a union of P-rays such that any two of them either have disjoint interiors or one is a subset of the other, and that each P-ray starts on* $\mathrm{Bd}(\phi(\mathbf{B}))$. *Then this decomposition of f is unique.*

Proof. A finite number of the P-rays in the ruling of $\mathbf{C} - \phi(\mathbf{B})$ contain all points of $B \cap (\mathbf{C} - \phi(\mathbf{B}))$. Assume, without loss of generality, that no one of the P-rays in this finite collection is a subset of another. All that is needed to satisfy the hypotheses of Theorem 1 is to extract a collection of P-subrays that are *disjoint* and still contain $B \cap (\mathbf{C} - \phi(\mathbf{B}))$. If the initial point of a P-ray ℓ lies on the interior of the P-ray ℓ', remove an initial segment of ℓ. If two or more P-rays have the same initial point ζ, and ζ is not an interior point of some ray in the finite collection, then remove an initial segment from all but one of the P-rays. The resulting collection of P-rays is disjoint, and may be assumed to cover $B \cap (\mathbf{C} - \phi(\mathbf{B}))$. By Theorem 1 the decomposition of f is unique. □

The class of functions described in Corollary 1 is the closure under uniform convergence on compact subsets of **B** of $K(p)$, under the restriction that no critical point is lost in the limit (see Livingston [3], and Lyzzaik and Styer [6]). It is an open question whether or not this restricted closure properly contains $K(p)$.

We finish with a couple of examples of "pretzel" functions. The first example shows that the decomposition $f = P \circ \phi$ need not be unique. The second example shows that the decomposition may be unique, even though the hypotheses of Theorem 1 are not satisfied. In other words, this second example shows that Theorem 1 gives a sufficient, but not necessary, condition for uniqueness of decomposition.

EXAMPLE 1. See Figure 1. The function f has simple critical points at a and b. Symmetry is not important in this example.

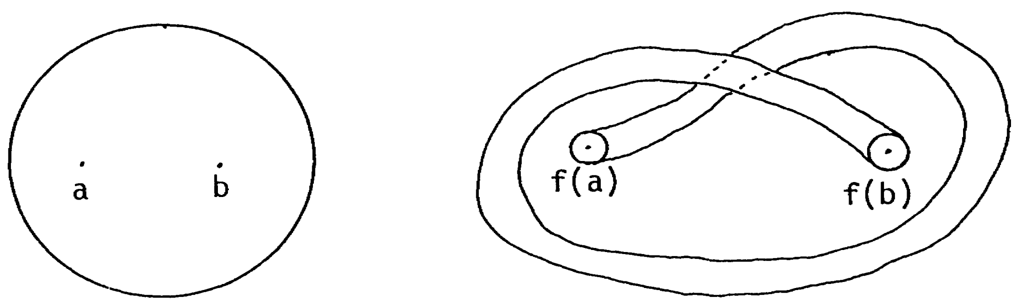

Figure 1

The three discs in Figure 2(a) or 2(b) represent the three sheets of the image of a polynomial of degree three. They are to be interpreted as superimposed upon each other; the matching dotted lines of the slits are to be identified, and the matching solid lines of the slits are to be identified. Thus, the inside ends of the slits become the two branch points of the Riemann surface of the image of the polynomial of degree 3.

Figure 2 shows the Riemann surface of the image of f embedded in the Riemann surface of the image of a polynomial of degree 3. In Figure 2(a) the embedding is made one way, and in Figure 2(b) it is made a different way.

A curve cannot be drawn from c or d to infinity, outside of the image of f, the projection of which is one-to-one into the plane. The two parts of Figure 2 correspond to two distinct decompositions of f, $f = P \circ \phi$ and $f = Q \circ \psi$, where P and Q are polynomials of degree 3 and $\phi, \psi \in S$.

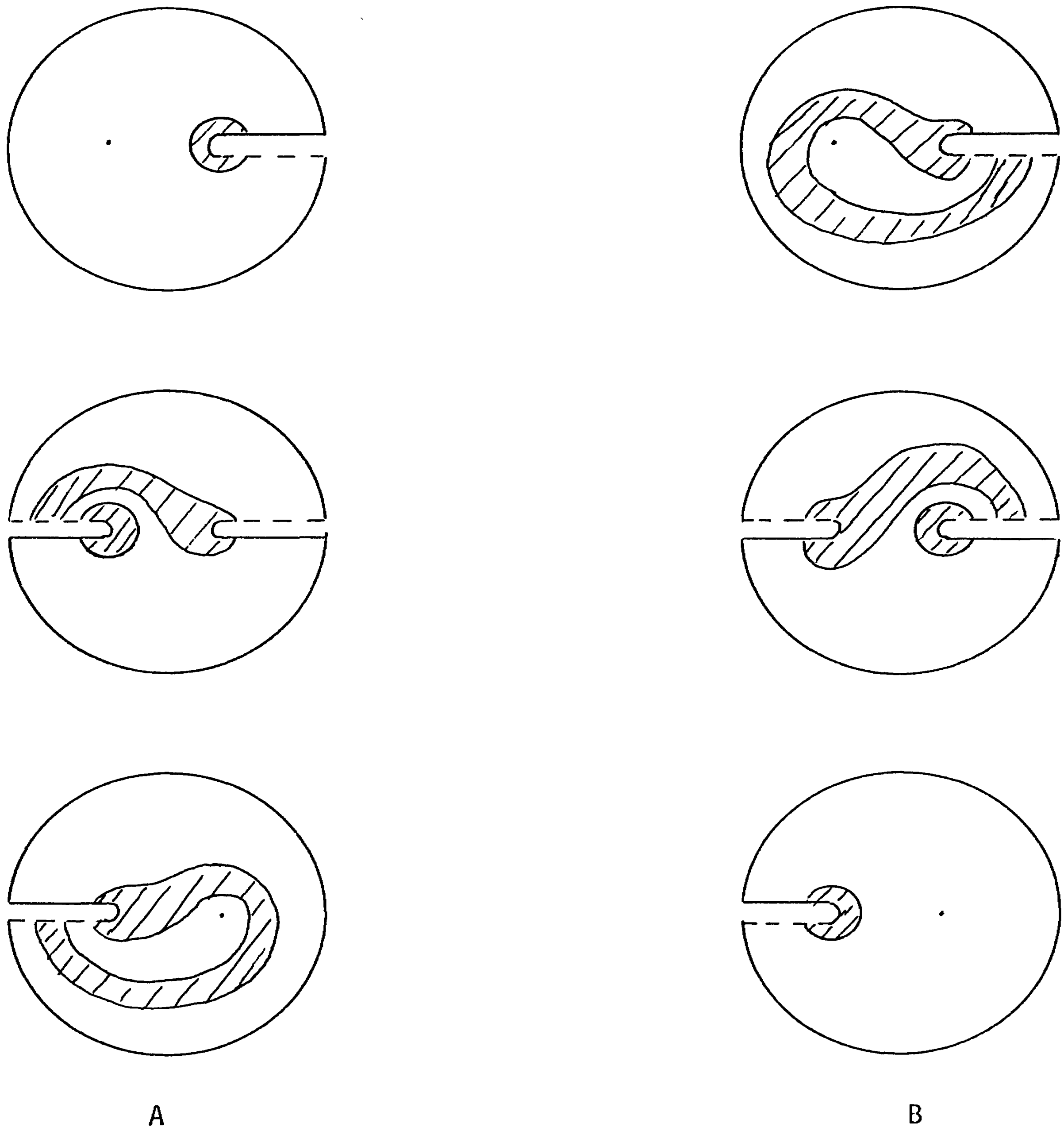

Figure 2

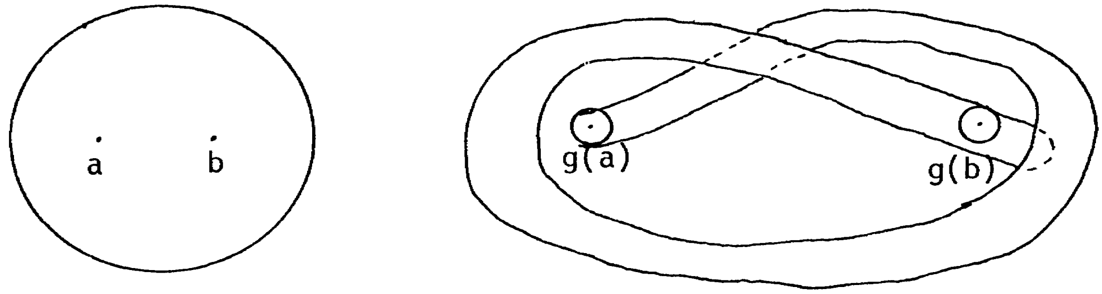

Figure 3

EXAMPLE 2. See Figure 3. We see that g is obtained by a very small modification in the image of f. However, of the two corresponding possible decompositions, the one corresponding to Figure 2(a) works while the one corresponding to Figure 2(b) forces ψ to be bivalent. See Figure 4.

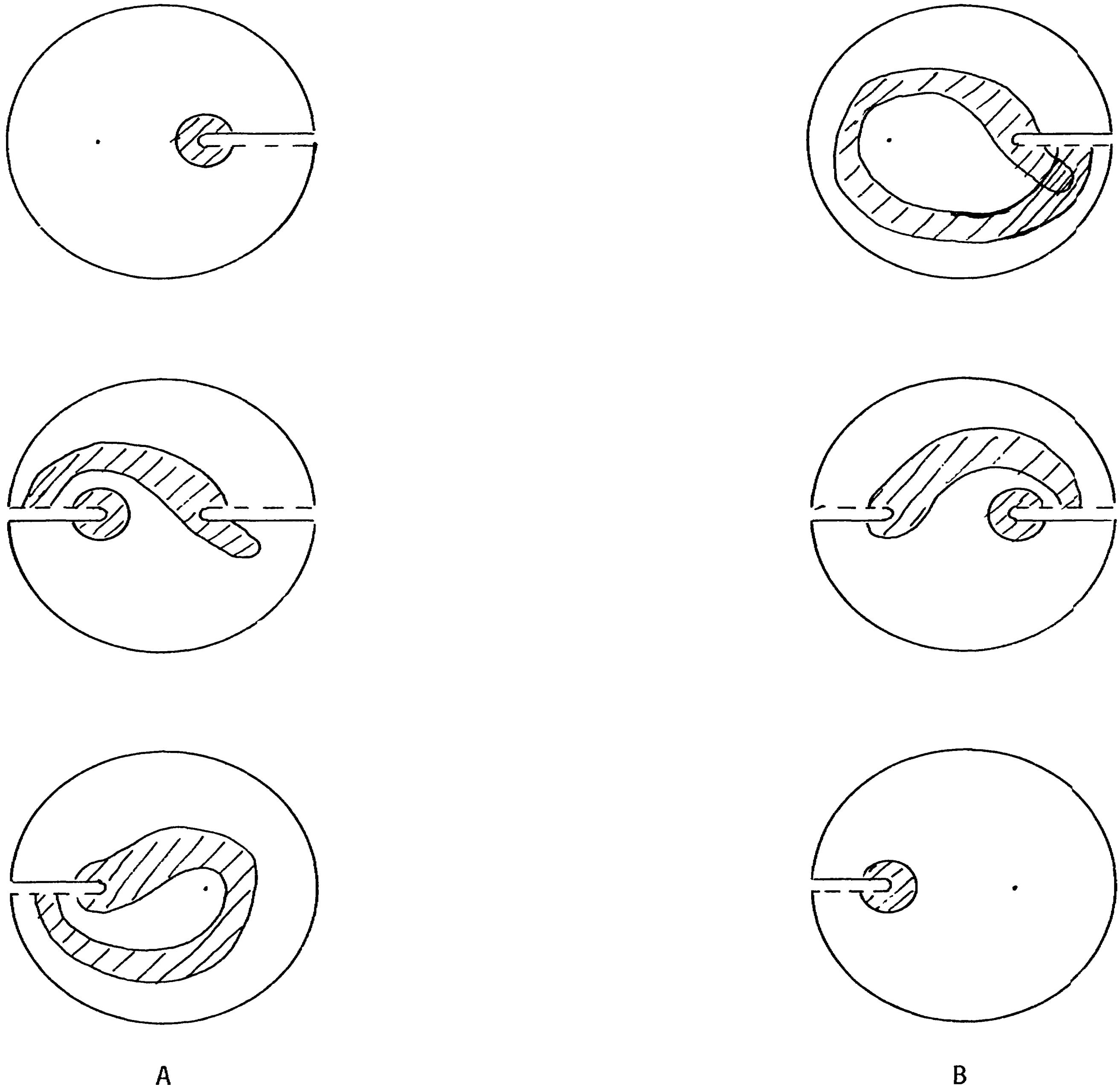

Figure 4

One sees that even though there is unique decomposition $g = P \circ \phi$, and P is a polynomial of degree 3, and $\phi \in S$; still a curve cannot be drawn from c to infinity, outside of the image of g, the projection of which is one-to-one into the plane. This shows that Theorem 1 is a sufficient, but not necessary, condition for the uniqueness of decomposition. A good necessary and sufficient condition for uniqueness of decomposition is still an open problem.

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