PETTIS' LEMMA AND TOPOLOGICAL PROPERTIES OF DUAL ALGEBRAS

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1. Introduction and results. Let $\mathcal{H}$ be a separable, infinite-dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of (bounded, linear) operators on $\mathcal{H}$. As is well-known, $\mathcal{L}(\mathcal{H})$ is the dual space of $\mathcal{C}^1(\mathcal{H})$, the Banach space of trace-class operators on $\mathcal{H}$. A dual algebra $\mathcal{A}$ on $\mathcal{H}$ is a unital subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed in the weak* topology arising from the above duality. The study of (nonselfadjoint) dual algebras has received considerable attention since S. Brown used them to solve (positively) the invariant subspace problem for subnormal operators (cf. [5]). We recall the basic definitions and properties and refer to [4] for a detailed treatment and bibliography.

A dual algebra $\mathcal{A}$ is said to have property $(\mathbf{A}_1)$ if, for every $[L]$ in $Q_{\mathcal{A}}$ the predual of $\mathcal{A}$ $(Q_{\mathcal{A}}=\mathcal{C}_1/\perp_{\mathcal{A}})$, there exist vectors x and y in $\mathcal{H}$ such that $[L]=[x\otimes y]$ (recall that $x\otimes y$ is the rank one operator $x\otimes y(u)=(u,y)x$). More generally, given an integer n, we say that $\mathcal{A}$ has property $(\mathbf{A}_{1/n})$ if any $[L]$ in $Q_{\mathcal{A}}$ can be written as

$$(*)\qquad [L]=\sum_{i=1}^{n}[x_i\otimes y_i]\quad \text{for some } x_i, y_i \text{ in } \mathcal{H}.$$

Furthermore, $\mathcal{A}$ is said to have property $(\mathbf{A}_{1/n}(r))$ if it has property $(\mathbf{A}_{1/n})$ and if for any $C>r$ the decomposition $(*)$ can be realized with

$$\sum_{n=1}^{n}\|x_i\|\|y_i\|\le C\|[L]\|.$$

It is clear that properties $(\mathbf{A}_{1/n})$ and $(\mathbf{A}_{1/n}(r))$ make sense for arbitrary weak*-closed subspaces of $\mathcal{L}(\mathcal{H})$, and we will use them freely in that context.

Recall that the weak operator topology (WOT) on $\mathcal{L}(\mathcal{H})$ is the one defined by the seminorms $T\to|(Tx,y)|$, $x,y\in\mathcal{H}$. As is well known, the weak operator topology is weaker than the weak* topology, and it is easy to see that if $\mathcal{A}$ has property $(\mathbf{A}_{1/n})$ then both topologies coincide on $\mathcal{A}$. This latter condition has taken on increasing importance in recent years in the theory of dual algebras, especially in applications to the invariant subspace problem. For instance, in [2] it was shown that a C_{00}-contraction T—with spectrum containing the unit circle $\mathbf{T}$ and such that the weak* and weak operator topologies agree on $\mathcal{A}_T$—has a nontrivial invariant subspace. (As usual we denote by $\mathcal{A}_T$ the dual algebra generated by T.) Later in [8] the hypothesis "$T\in C_{00}$" was removed. In that same paper analogous results were obtained with the condition "(1) $(\mathcal{A}_T, w^*)=(\mathcal{A}_T,\text{WOT})$" replaced by "(1′) $\mathcal{A}_T=\mathcal{W}_T$", where $\mathcal{W}_T$ denotes the WOT closed algebra generated by T. In connection with these conditions we recall that only recently it has been

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shown that they are not automatically satisfied. Indeed, the first example of an operator T such that (1) is not satisfied was given in [15] (a somewhat conceptually simpler example was produced later in [6]), while still more recently, [16] exhibits the first example of an operator T in $\mathcal{L}(\mathcal{H})$ such that $A_T \neq \mathcal{W}_T$.

In the present note we prove that a dual algebra (in fact any weak* closed subspace of $\mathcal{L}(\mathcal{H})$) on which the weak* and weak operator topologies agree is necessarily WOT closed. This is obtained as a consequence of our main result, which we now state.

THEOREM 1. *Let $\mathcal{A}$ be a weak* closed subspace of $\mathcal{L}(\mathcal{H})$ on which the weak* and weak operator topologies agree. Then $\mathcal{A}$ has property $(\mathbf{A}_{1/n}(r))$ for some n and r.*

Note that the equality of the weak* and weak operator topologies on $\mathcal{A}$ says exactly that for each $[L]$ in $Q_{\mathcal{A}}$ there exists $n = n_{[L]}$ such that

$$(*) \qquad [L] = \sum_{i=1}^{n_{[L]}} [x_i \otimes y_i] \quad \text{for some } x_i, y_i,$$

that is, in the terminology of [8], that every $[L]$ in $Q_{\mathcal{A}}$ is of finite length. In other words, Theorem 1 says that if each $[L]$ is of finite length then the lengths can be bounded by the same number with a "control" on the norms of the vectors realizing $(*)$.

In the case where we already know that the lengths of all the elements in $Q_{\mathcal{A}}$ are bounded by m, we obtain control on the norms in $(*)$ but at the expense of doubling the length.

THEOREM 2. *Let $\mathcal{A}$ be a weak* closed subspace of $\mathcal{L}(\mathcal{H})$ with property $(\mathbf{A}_{1/m})$. Then $\mathcal{A}$ has property $(\mathbf{A}_{1/2m}(r))$ for some r.*

At least for $n = 1$, Theorem 2 is the best possible in this generality since [3] provides an example of a dual algebra having property $(\mathbf{A}_1)$ but not property $(\mathbf{A}_1(r))$ (thus answering in the negative a question of [10]). In relation to the "lengths", we mention that for every integer p there exist singly generated dual algebras having property $(\mathbf{A}_{1/p})$ but not $(\mathbf{A}_{1/p-1})$ (cf. [7]).

We postpone the proofs of Theorems 1 and 2 until the next section, and now give the promised corollary.

COROLLARY 3. *Let $\mathcal{A}$ be a weak* closed subspace of $\mathcal{L}(\mathcal{H})$ on which the weak* and weak operator topologies agree. Then $\mathcal{A}$ is* WOT *closed. In particular a weak* closed subspace with property $(\mathbf{A}_{1/n})$ for some n is* WOT *closed.*

REMARK. The last assertion of our corollary (with $n = 1$) answers positively Question 2 of [11] and Conjecture 6.1a of [1].

Proof. In view of Theorem 1, we have to show that a weak* closed subspace of $\mathcal{L}(\mathcal{H})$ with property $(\mathbf{A}_{1/n}(r))$ is WOT closed. The argument is well known to anyone working in the area; we give it for completeness.

Since any $[L]$ in $Q_{\mathcal{A}}$ is WOT continuous, it extends uniquely to a WOT continuous linear functional (still denoted $[L]$) on $\mathcal{W}_{\mathcal{A}}$, the WOT closure of $\mathcal{A}$. Moreover, for any B in $\mathcal{W}_{\mathcal{A}}$ we will have $\langle B, [L]\rangle = \sum_{i=1}^{m} (Bx_i, y_i)$ whenever

$$(*) \qquad [L] = \sum_{i=1}^{m} [x_i \otimes y_i] \quad \text{in } Q_{\mathcal{A}}.$$

Fix $C > r$; for any $[L]$ we obtain $(*)$ with $m = n$ and $\sum_{i=1}^{m} \|x_i\| \|y_i\| \le C\|[L]\|$. We then have, for any B in $\mathcal{W}_{\mathcal{A}}$ and any $[L]$ in $Q_{\mathcal{A}}$, $|\langle B, [L]\rangle| \le C\|B\| \|[L]\|$. This shows that $[L] \to \langle B, [L]\rangle$ is a bounded linear functional on Q_A. Therefore there exists $A \in \mathcal{A}$ such that $\langle B, [L]\rangle = \langle A, [L]\rangle$, $[L] \in Q_{\mathcal{A}}$. In particular, for every $x, y \in \mathcal{H}$,

$$(Bx, y) = \langle B, [x \otimes y]\rangle = \langle A, [x \otimes y]\rangle = (Ax, y),$$

that is, $B = A$. Hence $\mathcal{W}_{\mathcal{A}} = \mathcal{A}$. □

2. Proofs of Theorems 1 and 2. Our proofs are based on Pettis' Lemma, and we first briefly review the material necessary to state it. A *Polish space* is a separable topological space whose topology can be defined by a complete metric. A *Souslin space* (also called analytic space) is a Hausdorff topological space which is the image of a Polish space under a continuous map. We refer to [9] and [14] for more details. Recall that a subset Y of a topological space X is said to be *meager* in X if Y is contained in a countable union of closed subsets of X each of which has empty interior. Pettis' Lemma can be stated as follows.

PETTIS' LEMMA [13]. *Let X be a (Hausdorff) topological vector space and let Y be a subset of X which is a Souslin space (when equipped with the relative topology). If Y is non-meager in X then $Y - Y$ is a neighborhood of* 0 *in X.*

We now turn to the proof of Theorem 1. Let $\mathcal{A}$ satisfy $(\mathcal{A}, w^*) = (\mathcal{A}, \text{WOT})$; for $n, p \in \mathbf{N}$ we introduce the subset $\mathcal{F}_{n,p}$ of $Q_{\mathcal{A}}$:

$$\mathcal{F}_{n,p} = \left\{ [L] \in Q_{\mathcal{A}} \,\middle|\, \exists (x_i)(y_i),\ 1 \le i \le n, \text{ satisfying } [L] = \sum_{i=1}^{n} [x_i \otimes y_i] \text{ and } \sum_{i=1}^{n} \|x_i\| \|y_i\| \le p \right\}.$$

Clearly, $\mathcal{F}_{n,p}$ is a Souslin space as the image of the complete separable metric space

$$\left\{ ((x_i)(y_i)) \in \mathcal{H}^{(2n)} \,\middle|\, \sum_{i=1}^{n} \|x_i\| \|y_i\| \le p \right\}$$

under the continuous map $((x_i), (y_i)) \to \sum [x_i \otimes y_i]$.

On the other hand, the hypothesis easily implies

$$\bigcup_{n,p} \mathcal{F}_{n,p} = Q_{\mathcal{A}}.$$

Since a countable union of meager subsets is meager and since $Q_{\mathcal{A}}$ is non-meager in itself by the Baire category theorem, there exists n_0, p_0 such that $\mathcal{F}_{n_0, p_0}$ is non-meager. By Pettis' Lemma we obtain that $\mathfrak{F}_{n_0,p_0}-\mathfrak{F}_{n_0,p_0}$ contains some closed ball of radius $\rho>0$ centered at the origin of $Q_{\mathfrak{A}}$. Since $\mathfrak{F}_{n_0,p_0}-\mathfrak{F}_{n_0,p_0}$ is contained in $\mathfrak{F}_{2n_0,2p_0}$, an easy computation shows that $\mathfrak{A}$ has property $(\mathbf{A}_{1/2n_0},(2p_0/\rho))$, thus concluding the proof of Theorem 1. We note that Theorem 1 could also be proved via an easy adaptation of the proof of [12, Theorem 1.3].

To prove Theorem 2 we observe that if $\mathfrak{A}$ has property $(\mathbf{A}_{1/m})$ then $\bigcup_p \mathfrak{F}_{m,p}=Q_{\mathfrak{A}}$. Therefore $\mathfrak{F}_{m,p}$ is non-meager for some p (in fact for all p). Thus, for instance, $\mathfrak{F}_{m,1}-\mathfrak{F}_{m,1}$ contains some close ball of radius ρ, and it follows that $\mathfrak{A}$ has property $(\mathbf{A}_{1/2m},(2/\rho))$.

We have learned from C. Pearcy that P. G. Dixon has also proved Theorems 1 and 2 using similar arguments.

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