FORMAL CHARACTER TABLES

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1. Introduction. The character table of a finite group satisfies so many tight arithmetic restrictions that it is hard to imagine that such a table can exist without a corresponding group. Indeed, in the case of the Lyons–Sims group LyS and the Fischer–Griess group F_1, the character tables were computed before the existence of the groups was established. Of course, extensive and consistent subgroup information about these groups was also known prior to the existence proofs. In §2, a definition is given for a formal character table, as well as a method for producing examples that are not necessarily character tables of groups. It does not appear likely that necessary and sufficient conditions will be found for a square matrix in order that it be a character table of a finite group (problem 6 of [2]).

The method itself produces examples of formal character tables in which a row exists containing exactly two nonzero entries. This can happen in the character table of a group, and the structure of such groups has been determined in [4]. Theorem 2.4 may be used to obtain examples of formal character tables in which no row contains only two nonzero entries.

Theorem 3.1 may be useful to show that Definition 2.2 does lead to formal character tables (in particular examples) which are not actual character tables, while Theorem 3.2 gives an infinite family of examples.

2. "Character tables" that are not character tables. Some of the more well-known arithmetic properties of character tables are formalized in the definition below.

DEFINITION 2.1. A $k\times k$ matrix X is a formal character table if the following five conditions are satisfied:

(1) The first row of X consists entirely of 1's and the first column consists of positive integers x_{i1}. Every entry x_{ij} is a sum of exactly x_{i1} nth roots of 1 where $n=\sum_{i=1}^{k} x_{i1}^2$. (The integer n will be referred to as the order of X).

(2) The algebraic conjugate of any row (resp. column) of X is a row (resp. column) of X.

(3) Define $c_j=\sum_{i=1}^{k}|x_{ij}|^2$. Then $\sum_{i=1}^{k} x_{pi}\overline{x_{qi}}/c_i=\delta_{pq}$. (Notice that $c_1=n$ and $c_i\geq 1$ by (1), so the sum is defined.)

(4) The (pointwise) product of any two rows of X is a nonnegative integral combination of the rows of X.

(5) Define a_{pqr} for $1\leq p,q,r\leq k$ by

$$a_{pqr}=\frac{n}{c_p c_q}\sum_{s=1}^{k}\frac{x_{sp}x_{sq}\overline{x_{sr}}}{x_{s1}}.$$

Then a_{pqr} is a positive integer or zero for all p,q,r.

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Each property listed above, of course, is satisfied by the character table of any finite group (subject to reordering the rows or columns of X, or even, for some people, taking a transpose). One good reference for character theory is [5].

Of course, other well known properties of character tables may be included in this list, but many of these are already consequences of (1)–(5) above. For convenience, we list a few of these. Numbers and letters in parentheses after each assertion indicate specifically which statements imply the assertion.

A. For every p and q, $\sum_{i=1}^{k} x_{ip}\overline{x_{iq}} = \delta_{pq} c_q$ (3).
B. For every j, c_j is a positive integer (1, 2).
C. For every j, c_j divides n (1, 2, 5, A).
D. The integers a_{pqr} are structure constants for a commutative and associative algebra. That is, if $\{b_1, b_2, \dots, b_k\}$ is a basis for a $\mathbf{C}$-space, say Z, and if a multiplication is defined on Z by the equations $b_p b_q = \sum_{r=1}^{k} a_{pqr} b_r$ and linearity, then Z is a commutative and associative algebra (3, 5).
E. For each i, the linear function $\omega_i : Z \to \mathbf{C}$ given by $\omega_i(b_j) = x_{ij} n / c_j x_{i1}$ is an algebra homomorphism. In particular, $\omega_i(b_j)$ is an eigenvalue of the integer matrix $A^{(j)} = (a_{jpq})$. The corresponding eigenvector is v_i where $v_i^T = (\omega_i(b_1), \omega_i(b_2), \dots, \omega_i(b_k))$. Hence $\omega_i(b_j)$ is an algebraic integer for all i and j (3, 5, D).
F. For every i, x_{i1} divides n (1, 3, E).

In an actual character table X, the integers c_j are the centralizer orders for the various conjugacy class representatives. Notice that property (1) of Definition 2.1 may be refined so that x_{ij} is required to be a sum of exactly x_{i1} c_jth roots of unity. Other possible refinements include the incorporation of power maps, and the introduction of p-blocks, but this will not be done here.

The following definition and theorem provide a way of constructing new formal character tables from old ones.

DEFINITION 2.2. Let X be a $k \times k$ matrix which is a formal character table of order n. Assume $n = ds$ where d and s are positive integers with s dividing d. Define (X, s) to be the $(k+1) \times (k+1)$ matrix given in blocked form by

$$(X, s) = \left[\begin{array}{c|c} \vec{v} & X \\ \hline d & \vec{w} \end{array}\right],$$

where $\vec{v}$ is the first column of X and $\vec{w} = (-s, 0, 0, \dots, 0)$.

As a way of motivating this definition, it is interesting to interpret what (X, s) is when X happens to be the character table of an actual group H. In this case write $|H| = ds = s^2 t$ and let G be a group with a normal subgroup N of order $t+1$ and quotient G/N isomorphic to H. (Take $G = H \times N$, for example.) Set $\sigma = (1/st)(\rho_G - \rho_{G/N})$, where ρ denotes the regular character. The class function σ is not generally a character of G, but an easy calculation shows it has norm 1. The matrix (X, s) consists of σ appearing in the bottom row, and the irreducible characters of G whose kernels contain N, with redundant columns deleted. In other words, (X, s) is a "condensed" version of a character table of G. It is possible to prove that (X, s) is a formal character table using the ordinary characters of G. However, (X,s) is always a formal character table whenever X is, as the next result shows.

THEOREM 2.3. *Let X be a formal character table of order n where $n=ds$ and $s\mid d$. Then (X,s) is a formal character table of order $n+d^2=d(d+s)$.*

Proof. For notational convenience, index the rows of (X,s) by the integers $1,2,\ldots,k+1$ and the columns by $0,1,\ldots,k$. Let x^*_{ij} denote the ij entry of (X,s) so that $x^*_{ij}=x_{ij}$ for $i\le k$ and $j\ge 1$. Also $x^*_{i0}=x_{i1}$ for $i\le k$ and $x^*_{k+1,0}=d$, $x^*_{k+1,1}=-s$ while $x^*_{k+1,j}=0$ for $j>1$. We also define $c^*_j=\sum_{i=1}^{k+1}|x^*_{ij}|^2$.

To verify property (1) of Definition 2.1 it suffices to consider only the entries of the last row of (X,s). Any zero entry of that row (corresponding to $x^*_{k+1,j}$ for $j>1$) can be expressed as a sum of f fth roots of 1 if $f=gcd(d,c_j)>1$, and hence, as a sum of d fth roots of 1. If $gcd(d,c_j)=1$ then, as $c_j\mid ds$ and $s\mid d$, we must have $c_j=1$. This is possible by the "second orthogonality relation" (property A) only when $n=ds=1$, and in this case $(X,s)=(X,1)$ is the character table of the cyclic group of order 2.

Consider next the entry $x^*_{k+1,1}=-s$. Clearly, -1 is a sum of the d/s fth roots of 1 which are different from 1 where $f=d/s+1$. Notice that f divides $d+s$ which in turn divides $s(d+s)=c^*_1$. Hence, $-s$ is a sum of d fth roots of 1 where $f\mid c^*_1$. This proves that property (1) of Definition 2.1 holds for (X,s).

Property (2) obviously holds for (X,s). The "first orthogonality relation" (property (3)) holds for any pair of rows not including the last row by using the identity

$$\frac{1}{c^*_0}+\frac{1}{c^*_1}=\frac{1}{c_1}$$

and the "first orthogonality relation" for X. Calculating inner products with the last row is easy since there are only two nonzero entries. The result is that property (3) holds for (X,s).

Clearly (4) holds for the product of any pair of rows of (X,s) not involving the last row. If $\vec{r}_i$ denotes the ith row of (X,s) then the pointwise product of $\vec{r}_i$ with $\vec{r}_{k+1}$ is obviously $x_{i1}\vec{r}_{k+1}$ for $i\le k$, while for $i=k+1$ the product is

$$\sum_{i=1}^{k} x_{i1}\vec{r}_i+(d-s)\vec{r}_{k+1}$$

(where we have used property A applied to X). This verifies that (X,s) satisfies (4).

The last property of Definition 2.1 unfortunately is the messiest one to verify. Define

$$a^*_{pqr}=\frac{n^*}{c^*_p c^*_q}\sum_{s=1}^{k+1}\frac{x^*_{sp}x^*_{sq}\overline{x^*_{sr}}}{c^*_{s1}}$$

for $0\le p,q,r\le k$ and $n^*=n+d^2=d(d+s)$. Since this expression is symmetric in p and q, it suffices to prove a^*_{pqr} is a nonnegative integer for $p\le q$. Define $\tilde{0}=1$ and $\tilde{i}=i$ for $1\le i\le k$. Then, after some computation, we have the result that

$$a^*_{pqr}=\left(1+\frac{d}{s}\right)\frac{c_{\bar p}}{c^*_p}a_{\bar p q\bar r} \quad \text{for } q>1.$$

This is easily checked to be a nonnegative integer using $c^*_p=c_{\bar p}$ for $p>1$ while $c^*_{\bar 1}=s(d+s)$, $c^*_{\bar 0}=d(d+s)$ and $c_{\bar 0}=c_{\bar 1}=ds$. For $p\le q\le 1$ we tabulate:

$$a^*_{11r}=\begin{cases}0 & \text{if } r>1\\ (d-s)/s & \text{if } r=1\\ d/s & \text{if } r=0\end{cases}$$

$$a^*_{01r}=\begin{cases}0 & \text{if } r>1 \text{ or } r=0\\ 1 & \text{if } r=1\end{cases}$$

$$a^*_{00r}=\begin{cases}0 & \text{if } r\ge 1\\ 1 & \text{if } r=0.\end{cases}$$

In each case, a^*_{pqr} is a nonnegative integer, and Theorem 2.3 is proved. □

Notice that if X is a formal character table of order n then (X,s) is always defined for $s=1$. Of course, if n is square free then (X,s) is defined only for $s=1$. There are severe restrictions on s if (X,s) is the character table of a group (see Theorem 3.1 below).

This method of obtaining new formal character tables from old ones may be iterated. It is amusing to observe that if $X=[1]$ (the character table of the identity group), then $(X,1)$ is the character table of C_2 (a cyclic group of order 2) and $((X,1),1)$ is the character table of S_3 (symmetric group on 3 letters). Also, if X is the character table of $C_2\times C_2$ then $(X,2)$ is the character table of D_8 or Q_8 (dihedral or quaternion group of order 8) and $((X,2),1)$ is the character table of $Q_8\ltimes E_9$ where E_9 is an elementary abelian group of order 9 on which Q_8 acts Frobeniusly.

If X and Y are matrices, let $X\otimes Y$ denote the Kronecker product of X and Y. If X and Y are character tables of groups then so is $X\otimes Y$. The next result applies this to formal character tables.

THEOREM 2.4. *Let X and Y be formal character tables. Then $X\otimes Y$ is a formal character table. Moreover, $X\otimes Y$ is the character table of a group if and only if X and Y are character tables of groups.*

Proof. If X is a $k\times k$ matrix and Y is an $l\times l$ matrix then $X\otimes Y$ is a $kl\times kl$ matrix whose rows and columns are naturally indexed by the pairs $\{(i,j)\mid 1\le i\le k,\ 1\le j\le l\}$. The (i,j), (i',j') entry of $X\otimes Y$ is $x_{ii'}y_{jj'}$. The verification that $X\otimes Y$ is a formal character table is straightforward and is omitted.

If X and Y are character tables of groups, say G and H, then $X\otimes Y$ is the character table of $G\times H$.

Suppose conversely that $X\otimes Y$ is the character table of some group, say G, where X and Y are formal character tables. Then the conjugacy classes of G and the irreducible characters of G are indexed by the set $\{(i,j)\mid 1\le i\le k,\ 1\le j\le l\}$ so that if $\chi_{(i,j)}$ is the irreducible character corresponding to (i,j) and if $g\in G$ belongs to class (i',j') then $\chi_{(i,j)}(g)=x_{ii'}y_{jj'}$.

Let N be the union of those conjugacy classes of G corresponding to indices of the form $(i,1)$ when $1\leq i\leq k$ and let $\mathfrak{X}=\{\chi_{(1,j)}\mid 1\leq j\leq l\}$. An easy calculation shows that $N\subseteq\bigcap_{\chi\in\mathfrak{X}}\ker\chi$. If $g\in\bigcap_{\chi\in\mathfrak{X}}\ker\chi$ and g belongs to the class (i',j') then an application of the "second orthogonality relation" (property A) applied to columns 1 and j' of Y yields $j'=1$ so that $g\in N$. Thus, $N=\bigcap_{\chi\in\mathfrak{X}}\ker\chi$ is a normal subgroup of G.

If $\chi_{(i,j)}$ is any irreducible character of G with kernel containing N then the "first orthogonality relation" (property 3) applied to rows 1 and i of X shows $i=1$ so that $\chi_{(i,j)}\in\mathfrak{X}$. Thus, $\mathfrak{X}$ contains exactly those irreducible characters of G with kernel containing N. The character table of G/N is constructed from the character table of G by deleting all rows coresponding to characters not in $\mathfrak{X}$ and by deleting redundant columns. When this is done for $X\otimes Y$, the resulting matrix is Y so that Y is the character table of G/N. Similarly, X is the character table of a group (in fact, a factor group of G) completing the proof of Theorem 2.4. □

The last two theorems may be used to produce examples of formal character tables (that are not character tables of groups) in which every row contains more than just two nonzero entries.

We close this section with an example of a formal character table that is not readily dismissed as being the character table of a group. The matrix below is $(X,2)$ where X is the character table of A_5.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & 0 & (1+\sqrt{5})/2 & (1-\sqrt{5})/2 \\ 3 & 3 & -1 & 0 & (1-\sqrt{5})/2 & (1+\sqrt{5})/2 \\ 4 & 4 & 0 & 1 & -1 & -1 \\ 5 & 5 & 1 & -1 & 0 & 0 \\ 30 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix above may be "constructed" as follows. The alternating group A_5 is isomorphic to $\mathrm{SL}(2,4)$ and so acts on a 2-dimensional vector space V over $\mathrm{GF}(4)$. Let G denote an extension of V by A_5 relative to this action, and regard V as a normal subgroup of G. Now G acts transitively on the nonprincipal characters of V, so if λ denotes one of these, then the inertia group of λ has index 15 in G. Call this inertia group P. Thus $|P:V|=4$ and P is a Sylow 2-subgroup of G. This leads to the two cases:

(i) λ has four extensions to P.

(ii) There exists an irreducible character θ of P such that $\theta_V=2\lambda$.

Actually, the second case leads to a contradiction, but never mind that the group doesn't exist, it does lead to the "character table" given above! As if to compensate for the nonexistence of a group in case (ii), there actually occur two nonisomorphic groups corresponding to case (i), and these have the same character table.

The formal character table given above satisfies even more properties than those listed in Definition 2.1. For example, "defect" and "Brauer characters" may be defined for $p=2,3$ and 5. Characters in the "principal block" for $p=3$ and 5 form a tree, and those of defect zero vanish on appropriate p-singular elements. In fact, I am aware of no arithmetic property of character tables in general that fails for the "character table" given above (short of insisting that it come from a group, of course).

If power maps are used however, it is possible to show that the third column must correspond to an element g of order 2 or 4. In either case, the determinant of $\rho(g)$ is -1 where ρ is a representation affording the last row of the table. However, the only linear character of G is the principal one, and this is a contradiction. (This argument is due to Marty Isaacs.)

The example above is part of an infinite family of examples, none of which are actual character tables. See Theorem 3.2 below. The *ad hoc* argument above using power maps only works for the first term of this family.

3. Conclusion. Definition 2.2 and Theorem 2.3 in the previous section produced examples of formal character tables of the form (X, s) in which the last row vanishes except for the first two entries. If (X, s) is the character table of an actual group G, then the two conjugacy classes of G corresponding to the first two columns of (X, s) are contained in the kernels of the characters of G corresponding to all but the last row. This happens in general whenever an irreducible character vanishes on all but two classes [4, Theorem 2.1]. Moreover, X itself is necessarily a character table of a group in this case. The following theorem imposes extra arithmetic conditions on (X, s) for it to be an actual character table.

THEOREM 3.1. *Suppose X and (X, s) are character tables of groups. Then, there exists a prime p such that s is a power of p, and if the order of X is n, then n has the form s^2t where $t+1$ is a power of p.*

Proof. Let G be a group whose character table is (X, s) and let χ be the irreducible character of G corresponding to the last row of (X, s). Set $N = \{x \in G \mid \chi(x) \neq 0\}$. Then N may also be described as the intersection of the kernels of all the irreducible characters of G other than χ, so N is a normal subgroup of G. Since all its non-identity elements are conjugate in G, N is an elementary abelian p-group for some prime p.

Set $\bar{G} = G/N$ and adopt the bar convention for this quotient. As X is the character table of $\bar{G}$ we have $n = |\bar{G}|$. Now, by definition, n has the form $n = s^2t$ and χ has degree $d = st$, so that G has order $n + d^2 = s^2t(t+1)$ and hence N has order $t+1$. The first paragraph shows that $t+1$ is a power of p and it remains to prove that s is a power of p.

Let y be any p'-element of G. As is well-known,

$$C_{\bar{G}}(\bar{y}) = \overline{C_G(y)},$$

because y is a p'-element and N is a p-group. Assuming also that $y \neq 1$ and computing centralizer orders from X and (X, s), we have $|C_G(y)| = |C_{\bar{G}}(\bar{y})|$. Thus $|C_G(y)| = |\overline{C_G(y)}|$ and so $C_N(y) = 1$. This means that for all $x \in N^{\#}$, $C_G(x)$ is a p-group. Computing from (X, s) we get $|C_G(x)| = n + s^2 = s^2(t+1)$, so $s^2(t+1)$ is a power of p, as desired. (In fact, $C_G(x)$ is a Sylow p-subgroup of G.) □

Some special cases of Theorem 3.1 are worth pointing out. For example, if G has character table (X, s) where $t = 1$ then $|Z(G)| = t+1 = 2$ and s is a power of 2.

Hence, G is a 2-group of order $2s^2$. Moreover, G is a group of central type. Little more can be said about G since, if T is any finite 2-group, there exists a 2-group G of central type in which $|Z(G)|=2$ such that $G/Z(G)$ contains a subgroup isomorphic to T. (See Theorem 1.2 of [3] or Theorem 6.3 of [4].)

At the other extreme is $s=1$, and this is easily seen to correspond to a doubly-transitive Frobenius group (allowing $|G|=2$ as a degenerate case).

It is possible to define, for a given formal character table X, formal character subtables of X, as well as power maps. This has been done already in [1], where the resulting structure is referred to as a pseudo-group. Obviously, any addition made to Definition 2.1 will affect the proofs (or even the statements) of Theorems 2.3 and 2.4. For example, if an axiom asserting the existence of power maps is added to Definition 2.1, which coincides with the axioms (IV)–(VI) for pseudo-groups in [1], then Theorem 2.3 remains true under the additional hypotheses that both s and $(d/s)+1$ are powers of the same prime p. In view of Theorem 3.1, this is not a serious restriction.

The following result provides for the infinite family of formal character tables that was promised at the end of the last section.

THEOREM 3.2. *Let s be a power of the prime p and let $q=s^2$. If X is the character table of* $\mathrm{SL}(2,q)$, *then (X,s) is a formal character table which is not the character table of any group.*

Proof. Assume G is a group with character table (X,s), and let χ be the irreducible character of G corresponding to the last row of (X,s). Then, as in the proof of Theorem 3.1, $N=\{x\in G\,|\,\chi(x)\neq 0\}$ is a normal elementary abelian p-subgroup of G, here of order q^2. As before, set $\bar{G}=G/N$, which has character table X. Using X it is not difficult to conclude that a Sylow p-subgroup of $\bar{G}$ is elementary abelian of order q and has normalizer in G of order $q(q-1)$. (The case $p=2$ is slightly easier than $p>2$.) Thus, if P is a Sylow p-subgroup of G, its normalizer in G has order $q^3(q-1)$. Let H be a complement to P in its normalizer in G, so that $|H|=q-1$. Let $Z=Z(P)\cap N$, so $Z\neq 1$. Since any non-identity p'-element y of G satisfies $C_N(y)=1$ (as in the proof of Theorem 3.1), H acts fixed point freely on Z. Hence $|Z|\geq|H|+1=q$. If $x\in P$ and $x\notin N$, then $C_G(x)$ contains x and Z, and so has order strictly greater than q. However, the obvious calculation from (X,s) yields the contradiction $|C_G(x)|=q$. □

We remark that Theorem 3.2 also follows from Theorem 6.2 of [4] since $N=O_p(G)$ implies by that theorem the contradiction that G must be a doubly-transitive Frobenius group with Frobenius kernel N.

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