L^p ESTIMATES FOR EXTENSIONS OF HOLOMORPHIC FUNCTIONS

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Introduction. The purpose of this paper is to obtain certain norm estimates for extensions and restrictions of holomorphic functions. More precisely, let D be a bounded, strictly pseudoconvex domain in $\mathbf{C}^n$ and let M be a complex submanifold which intersects ∂D transversally. Our purpose is to identify precisely the restriction to M of the Hardy spaces $H^p(D)$, $0<p\leq\infty$, as well as certain weighted Bergman spaces where the weight is a power of the distance to the boundary. Our main result is that the restriction to M is again a weighted Bergman space where the nature of the weight function depends on the codimension of M. A precise formulation is given in Section 1. The case $p=\infty$ was obtained by Henkin [8]. The case $1\leq p\leq\infty$ has been obtained by different methods by Cumenge [5]. Related results in the Hilbert space case can be found in Beatrous and Burbea [3]. Our proof is based on an integral representation due to Henkin and Leiterer [9] for holomorphic functions in a strictly pseudoconvex domain in a Stein manifold.

The paper is organized as follows. In Section 1 we give some technical definitions and a precise formulation of our main result. In Section 2 we develop some integral representations and construct an extension operator for holomorphic functions. In Section 3 we obtain some growth estimates for integral operators of the type considered in Section 2. In Sections 4, 5, and 6 we obtain L^p estimates for these operators in the cases $1<p<\infty$, $0<p\leq 1$, and $p=\infty$ respectively. Finally, in Section 7 we give a counterexample to show that strict pseudoconvexity is essential for L^p estimates with $0<p<\infty$.

1. Statement of results. Let M be a relatively compact open set with smooth boundary in a complex manifold $\tilde{M}$. For $z\in M$ let $\delta(z)$ denote the distance from z to the boundary of M with respect to some Riemannian metric on $\tilde{M}$. For any real number $s>-1$ and any positive number p, we denote by $L^p_s(M)$ the L^p space with respect to the measure $\delta^s\,dV$, where dV is the volume element on $\tilde{M}$, that is, $L^p_s(M)$ consists of all measurable functions f on M satisfying

$$\|f\|_{p,s}=\left(\int_M |f|^p\delta^s\,dV\right)^{1/p}<\infty.$$

It follows from compactness of M that $L^p_s(M)\subset L^p_t(M)$ for $-1<s\leq t$ and that the inclusion map is continuous. For $s=-1$, we define $L^p_{-1}(M)$ to be the space of all measurable functions on ∂M satisfying

$$\|f\|_{p,-1}=\left(\int_{\partial M} |f|^p\,d\sigma\right)^{1/p}<\infty.$$

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Here $d\sigma$ denotes the volume element on the hypersurface ∂M. The norm on $L_s^p(M)$ clearly depends on the choice of a Riemannian metric on $\tilde{M}$, but the vector space $L_s^p(M)$ does not, and norms arising from different metrics are equivalent. Note also that if ρ is a characterizing function for M (i.e., a smooth function on $\tilde{M}$ with $\rho<0$ on M, $\rho>0$ on $\tilde{M}\setminus\bar{M}$, and $d\rho\neq 0$ on ∂M) then an equivalent norm would be obtained by replacing δ by $-\rho$ and dV (or $d\sigma$) by any volume element on $\tilde{M}$ (or ∂M).

For $s>-1$ we denote by $A_s^p(M)$ the space of all holomorphic functions in $L_s^p(M)$, and we define $A_{-1}^p(M)$ to be the usual Hardy class consisting of holomorphic functions on M with boundary values in $L_{-1}^p(M)$. (See Stein [13].) We define $A(M)$ to be the space of holomorphic functions on M with polynomial growth: $A(M)=\{f\in \mathcal{O}(M): \sup|f(z)|\,\delta(z)^s<\infty \text{ for some } s\in\mathbf{R}\}$. It follows from plurisubharmonicity of $|f|^p$ that $A_s^p(M)\subset A(M)$ for all $s\geq -1$ and $0<p<\infty$. Thus we have

$$A(M)=\bigcup_{\substack{s\geq -1\\ 0<p\leq\infty}} A_s^p(M).$$

Now let D be a bounded, strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary, and let $\tilde{M}$ be an m-dimensional complex submanifold of a neighborhood of $\bar{D}$ which intersects ∂D transversally. Let $M=\tilde{M}\cap D$, and let $R:\mathcal{O}(D)\to\mathcal{O}(M)$ be the restriction mapping.

1.1. THEOREM. (a) *For* $s\geq -1$ *and* $0<p\leq\infty$, *R maps* $A_s^p(D)$ *continuously into* $A_{n-m+s}^p(M)$.

(b) *There is a linear operator* $E: A(M)\to A(D)$ *such that RE is the identity operator on* $A(M)$. *Moreover, for* $s\geq -1$ *and* $0<p<\infty$, *E maps* $A_{n-m+s}^p(M)$ *continuously into* $A_s^p(D)$.

1.2. COROLLARY. *For* $0<p\leq\infty$ *and* $s\geq -1$ *we have*

$$A_s^p(D)\,|_M=A_{n-m+s}^p(M).$$

In the case $p=\infty$, Corollary (1.2) is due to Henkin [8]. For $1\leq p\leq\infty$, it has been obtained by Cumenge [5] from estimates for the $\bar\partial$ problem.

2. Integral representations in strictly pseudoconvex domains. Throughout this section and the next, D will denote a bounded, strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. We fix a plurisubharmonic characterizing function ρ for D, and we let $g(z,\zeta)$ be the associated *Levi polynomial*

$$g(z,\zeta)=2\sum \rho_j(\zeta)(\zeta^j-z^j)-\sum \rho_{jk}(\zeta)(\zeta^j-z^j)(\zeta^k-z^k). \tag{1}$$

It follows from Taylor's formula and the strict plurisubharmonicity of ρ that there are positive constants C_1 and r and a neighborhood D' of $\bar{D}$ such that

$$\operatorname{Re} g(z,\zeta)\geq \rho(\zeta)-\rho(z)+C_1|z-\zeta|^2$$

for $z,\zeta\in D'$ and $|z-\zeta|\leq r$. Setting $\tilde{g}(z,\zeta)=g(z,\zeta)-2\rho(\zeta)$, it follows that

$$\operatorname{Re}\tilde{g}(z,\zeta)\geq -\rho(\zeta)-\rho(z)+C_1|z-\zeta|^2 \tag{2}$$

for $z,\zeta\in D'$ and $|z-\zeta|\leq r$.

The following result is a slight variation of a theorem of Henkin [7] and Ramirez [12].

2.1. LEMMA. *Let $\tilde{g}$, D', r, and C_1 be as above. There is a neighborhood $\tilde{D}$ of $\bar{D}$ with $\tilde{D} \supset D'$, a C^∞ function $\tilde{\Phi}$ on $\tilde{D} \times \tilde{D}$, and a positive constant C_2 such that*

(i) *for any $\zeta \in \tilde{D}$ the function $\tilde{\Phi}(\cdot, \zeta)$ is holomorphic on $\tilde{D}$;*

(ii) *$\tilde{\Phi}(\zeta, \zeta) = -2\rho(\zeta)$ for $\zeta \in \tilde{D}$, and $|\tilde{\Phi}(z, \zeta)| \geq C_2$ for $z, \zeta \in \tilde{D}$ with $|z - \zeta| \geq r/2$;*

(iii) *there is a non-vanishing C^∞ function $Q(z, \zeta)$ on*

$$\Delta_{r/2} = \{(z, \zeta) \in \tilde{D} \times \tilde{D} : |z - \zeta| \leq r/2\}$$

such that $\tilde{\Phi}(z, \zeta) = \tilde{g}(z, \zeta) Q(z, \zeta)$ on $\Delta_{r/2}$.

Note in particular that for any fixed $z \in D$ the function $\tilde{\Phi}(z, \cdot)$ is smooth and non-vanishing on $\bar{D}$. The proof of Lemma (2.1) proceeds along the same lines as that of Lemma 2.4 of [7] with the Levi polynomial replaced by $\tilde{g}(z, \zeta)$. We will omit the details.

Let $\tilde{M}$ be an m-dimensional complex submanifold of a neighborhood of $\bar{D}$ which intersects ∂D transversally, and let $M = \tilde{M} \cap D$. We do not exclude the case $m = n$, in which case $M = D$. By shrinking $\tilde{D}$ we may assume that $\tilde{D}$ is strictly pseudoconvex and that $\tilde{M}$ is a submanifold of $\tilde{D}$. In particular it follows that M is a relatively compact, strictly pseudoconvex open set in the Stein manifold $\tilde{M}$.

For $\epsilon > 0$, let $D_\epsilon = \{\rho < -\epsilon\}$ and let $M_\epsilon = M \cap D_\epsilon$. For ϵ sufficiently small, D_ϵ is a strictly pseudoconvex domain and M intersects ∂D transversally. Let $\Phi(z, \zeta) = \tilde{\Phi}(z, \zeta) + 2\rho(\zeta)$. By (ii) of Lemma (2.1), the function Φ vanishes on the diagonal in $\tilde{D} \times \tilde{D}$. Note also that for any compact set K in D there is an $\epsilon_K > 0$ such that $\Phi(z, \zeta) \neq 0$ for $z \in K$ and $-\epsilon_K \leq \rho(\zeta) \leq 0$. Applying Lemma 3.1.2 and (the proof of) Theorem (2.2.1) of Henkin and Leiterer [9] we obtain, after possibly shrinking $\tilde{D}$:

2.2. THEOREM. *There is a smooth differential form $\eta_0(z, \zeta)$ on $\tilde{M} \times \tilde{M}$ of bidegree $(m, m-1)$ in ζ and $(0,0)$ in z such that*

(i) *$\eta_0(\cdot, \zeta)$ is holomorphic on $\tilde{M}$ for any fixed $\zeta \in \tilde{M}$, and*

(ii) *for any $z \in M$ there is an $\epsilon_z > 0$ such that for*

$$0 \leq \epsilon \leq \epsilon_z \quad \text{and} \quad f \in \mathcal{O}(M_\epsilon) \cap C(\bar{M}_\epsilon)$$

we have

$$f(z) = \int_{\partial M_\epsilon} f(\zeta) \eta_0(z, \zeta) \Phi(z, \zeta)^{-m}.$$

2.3. COROLLARY. *There is an $\epsilon_0 > 0$ and for each positive integer j a smooth family of smooth differential forms $\eta_j^\epsilon(z, \zeta)$, $0 \leq \epsilon \leq \epsilon_0$, on $\tilde{M} \times \tilde{M}$ of bidegree (m, m) in ζ and $(0,0)$ in z such that*

(i) *$\eta_j^\epsilon(\cdot, \zeta)$ is holomorphic on $\tilde{M}$ for any fixed $\zeta \in \tilde{M}$ and $0 \leq \epsilon \leq \epsilon_0$, and*

(ii) *for any $z \in M$ there is an $\epsilon_z \in (0, \epsilon_0]$ such that for $0 \leq \epsilon \leq \epsilon_z$ and $f \in \mathcal{O}(M_\epsilon) \cap C(\bar{M}_\epsilon)$ we have*

$$f(z) = \int_{M_\epsilon} f(\zeta) \eta_j^\epsilon(z, \zeta) (\tilde{\Phi}(z, \zeta) - 2\epsilon)^{-m-j} (\rho(\zeta) + \epsilon)^{j-1}.$$

Proof. First, note that it follows from (ii) and (iii) of Lemma (2.1) that for any fixed $z \in M$ the function $\Phi(z, \cdot) - 2\epsilon$ is non-vanishing on $\bar{M}_\epsilon$ for $\epsilon > 0$ sufficiently small, so the integral in (ii) converges. Choose $\epsilon_1 > 0$ sufficiently small that $d\rho \neq 0$ on $U = \{|\rho| < 2\epsilon_1\}$. By multiplication by a smooth function $\chi(\zeta)$ with compact support in U and with $\chi \equiv 1$ on $\{|\rho| \leq \epsilon_1\}$, we may assume that the form η_0 of Theorem (2.2) vanishes for $\rho(\zeta) \leq -2\epsilon_1$. We will construct the forms η_j^ϵ so that in addition to (i) and (ii) they satisfy

(iii)' $\eta_j^\epsilon(z, \zeta) = 0$ for $\zeta \in M \setminus U$.

By Theorem (2.2) and Stokes' theorem we have, for sufficiently small $\epsilon > 0$,

$$\begin{aligned} f(z) &= \int_{\partial M_\epsilon} f(\zeta)\eta_0(z,\zeta)\Phi(z,\zeta)^{-m} \\ &= \int_{\partial M_\epsilon} f(\zeta)\eta_0(z,\zeta)(\tilde{\Phi}(z,\zeta)-2\epsilon)^{-m} \\ &= \int_{M_\epsilon} \bar{\partial}_\zeta[f(\zeta)\eta(z,\zeta)(\tilde{\Phi}(z,\zeta)-2\epsilon)^{-m}] \\ &= \int_{M_\epsilon} f[(\tilde{\Phi}-2\epsilon)\bar{\partial}_\zeta\eta - m\bar{\partial}_\zeta\tilde{\Phi}\wedge\eta](\tilde{\Phi}-2\epsilon)^{-m-1}. \end{aligned}$$

Thus we set $\eta_1^\epsilon = (\tilde{\Phi}-2\epsilon)\bar{\partial}_\zeta\eta - m\bar{\partial}_\zeta\tilde{\Phi}\wedge\eta$.

By induction, we assume that $j \geq 2$ and that a form η_{j-1} satisfying (i)–(iii) has already been constructed. Since η_{j-1} vanishes for ζ outside of U, we may write $\eta_{j-1}^\epsilon(z,\zeta) = \bar{\partial}\rho(\zeta)\wedge\omega_{j-1}^\epsilon(z,\zeta)$ where $\omega_{j-1}^\epsilon(z,\zeta)$ is a smooth form which vanishes for $\zeta \in \tilde{M}\setminus U$. Integration by parts yields

$$\begin{aligned} f(z) &= \int_{M_\epsilon} f(\zeta)\bar{\partial}(\rho(\zeta)+\epsilon)\wedge\omega_{j-1}^\epsilon(z,\zeta)(\tilde{\Phi}(z,\zeta)-2\epsilon)^{-m-j+1}(\rho(\zeta)+\epsilon)^{j-2} \\ &= \frac{1}{j-1}\int_{M_\epsilon} f\bar{\partial}(\rho+\epsilon)^{j-1}\wedge\omega_{j-1}^\epsilon(\tilde{\Phi}-2\epsilon)^{-m-j+1} \\ &= \frac{1}{j-1}\int_{M_\epsilon} f[(m+j-1)\bar{\partial}_\zeta\tilde{\Phi}\wedge\omega_{j-1}^\epsilon - (\tilde{\Phi}-2\epsilon)\bar{\partial}_\zeta\omega_{j-1}^\epsilon](\tilde{\Phi}-2\epsilon)^{-m-j}(\rho+\epsilon)^{j-1}. \end{aligned}$$

Thus, setting

$$\eta_j^\epsilon = \frac{1}{j-1}[(m+j-1)\bar{\partial}_\zeta\tilde{\Phi}\wedge\omega_{j-1}^\epsilon - (\tilde{\Phi}-2\epsilon)\bar{\partial}_\zeta\omega_{j-1}^\epsilon]$$

yields the desired result. □

2.4. COROLLARY. *For each non-negative integer j there is a smooth form $\eta_j(z,\zeta)$ on $\tilde{M}\times\tilde{M}$ of bidegree $(m, m-\delta_0^j)$ in ζ and $(0,0)$ in z such that*

(i) *$\eta_j(\cdot,\zeta)$ is holomorphic on $\tilde{M}$ for any fixed $\zeta \in \tilde{M}$, and*

(ii) *for $f \in A_{j-1}^1(M)$ and $z \in M$ we have*

$$f(z) = \int_{\partial M} f^*(\zeta)\eta_0(z,\zeta)\Phi(z,\zeta)^{-m} \quad \textit{if } j = 0$$

and

$$f(z)=\int_M f(\zeta)\eta_j(z,\zeta)\tilde{\Phi}(z,\zeta)^{-m-j}\rho(\zeta)^{j-1} \quad \text{if } j\geq 1.$$

Here f^* denotes the boundary value function of f.

Proof. For $j\geq 1$, by Corollary (2.3) and the Lebesgue Dominated Convergence Theorem we have

$$f(z)=\lim_{\epsilon\to 0+}\int_{M_\epsilon} f(\zeta)\eta_j^\epsilon(z,\zeta)(\tilde{\Phi}(z,\zeta)-2\epsilon)^{-m-j}(\rho(\zeta)+\epsilon)^{j-1}$$

$$=\int_M f(\zeta)\eta_j^0(z,\zeta)\tilde{\Phi}(z,\zeta)^{-m-j}\rho(\zeta)^{j-1}.$$

The case $j=0$ is handled similarly. □

Next we will indicate how the integral representations of Corollary (2.4) can be used to extend holomorphic functions on M to holomorphic functions on D. Let η_0 be the form from Corollary (2.4). The mapping $z\to\eta(z,\cdot)$ is a holomorphic mapping of $\tilde{M}$ into the Frechet space $C^\infty_{(m,m-1)}(\tilde{M})$. Since $\tilde{D}$ is pseudoconvex, it follows from Corollary 12.1 of Bungart [4] that this mapping extends to a holomorphic mapping of $\tilde{D}$ into $C^\infty_{(m,m-1)}(\tilde{M})$. We will continue to denote the extension by $\eta_0(z,\zeta)$. Define a linear operator $E_0: A^1_{-1}(M)\to\mathcal{O}(D)$ by

$$E_0 f(z)=\int_{\partial M} f^*(\zeta)\eta_0(z,\zeta)\Phi(z,\zeta)^{-m}.$$

From Corollary (2.4) we obtain:

2.5. THEOREM. *If $f\in A^1_{-1}(M)$ then $E_0 f\in\mathcal{O}(\bar{D}\backslash\partial M)$ and $E_0 f|_M=f$.*

Repetition of the proof of Corollary (2.3) yields an alternate expression for the extension operator E_0.

2.6. COROLLARY. *For each positive integer j there is a form $\eta_j(z,\zeta)$ on $\tilde{D}\times\tilde{M}$ of bidegree (m,m) in ζ and $(0,0)$ in z such that*

(i) *$\eta_j(\cdot,\zeta)$ is holomorphic on $\tilde{D}$ for any fixed $\zeta\in\tilde{M}$, and*

(ii) *the operator*

$$E_j f(z)=\int_M f(\zeta)\eta_j(z,\zeta)\tilde{\Phi}(z,\zeta)^{-m-j}\rho(\zeta)^{j-1}$$

satisfies $E_j f|_M=f$ for $f\in A^1_{j-1}(M)$, and $E_j f=E_k f$ for $f\in A^1_{k-1}(M)$ with $k\leq j$.

3. Some growth estimates. In this section we will estimate the growth at the boundary of certain integrals involving kernels of the type considered in Section 2. We will use the convention of denoting any positive constant by C, and a constant depending on a parameter r by C_r. The value of C or C_r may change from one line to the next. In addition we will use the notation $f\sim g$ to indicate that the ratio $|fg^{-1}|$ is bounded above and below by positive constants.

Recall from Section 2 that $\tilde{g}(z, \zeta) = g(z, \zeta) - 2\rho(\zeta)$, where $g(z, \zeta)$ is the Levi polynomial defined by (1).

3.1. LEMMA. $\tilde{g}(z, \zeta) = \overline{\tilde{g}(\zeta, z)} + O(|z-\zeta|^3)$.

Proof (cf. Kerzman and Stein [11]). By Taylor's formula we have

$$\tilde{g}(z, \zeta) = -2\rho(\zeta) + 2\sum \rho_j(\zeta)(\zeta^j - z^j) + \sum \rho_{jk}(\zeta)(\zeta^j - z^j)(\zeta^k - z^k)$$

$$= -2\rho(\zeta) + 2\sum_j \left(\rho_j(z) + \sum_k \rho_{jk}(z)(\zeta^k - z^k) + \sum_k \rho_{j\bar{k}}(z)(\overline{\zeta^k - z^k})\right)(\zeta^j - z^j)$$

$$- \sum_{jk} \rho_{jk}(\zeta^j - z^j)(\zeta^k - z^k) + O(|z-\zeta|^3)$$

$$= -2\rho(\zeta) + 2\sum \rho_j(z)(\zeta^j - z^j) + \sum \rho_{jk}(z)(\zeta^j - z^j)(\zeta^k - z^k)$$

$$+ 2\sum \rho_{j\bar{k}}(\zeta^j - z^j)(\overline{\zeta^k - z^k}) + O(|z-\zeta|^3).$$

Also,

$$\overline{\tilde{g}(\zeta, z)} = -2\rho(z) - 2\sum \rho_{\bar{j}}(z)(\zeta^j - z^j) - \sum \rho_{\bar{j}\bar{k}}(z)(\overline{\zeta^j - z^j})(\overline{\zeta^k - z^k}).$$

Thus $\tilde{g}(z, \zeta) - \overline{\tilde{g}(\zeta, z)}$ contains all terms of order less than 3 in the Taylor expansion for $\rho(\zeta)$ about z, so $\tilde{g}(z, \zeta) - \overline{\tilde{g}(\zeta, z)} = O(|z-\zeta|^3)$.

We now fix $p \in \partial M$. By translation and rotation of the $\mathbf{C}^n$ coordinates we may assume that the x^1 axis points along the outward normal to ∂D at p (here $\zeta^j = x^j + iy^j$), and that the $\zeta_2, \ldots, \zeta_m$ axes span the complex tangent space to ∂M at p. Note that since $\tilde{M}$ intersects ∂D transversally, the functions $\zeta^1, \ldots, \zeta^m$ form a local holomorphic coordinate system for $\tilde{M}$ in a neighborhood of p.

For fixed $z \in \bar{D}$, set $\tau_z(\zeta) = \operatorname{Im} \tilde{g}(z, \zeta) = \operatorname{Im} g(z, \zeta)$. Then

$$\frac{\partial}{\partial y^1}\bigg|_p \tau_p = \frac{\partial \rho}{\partial x^1}(p) \neq 0,$$

so it follows that the functions $\rho, \tau_p, \zeta^2, \ldots, \zeta^m$ form a local $C^\infty_?$ coordinate system for $\tilde{M}$ in a neighborhood of p. By continuity we obtain:

3.2. LEMMA. *There is a neighborhood U of p in $\mathbf{C}^m$ such that for any $z \in U$ the functions $\rho, \tau_p, \zeta^2, \ldots, \zeta^m$ form a C^∞ coordinate system for $U \cap \tilde{M}$, and the Jacobian of the transformation to the coordinates $\zeta_1, \ldots, \zeta_m$ is bounded above and below by positive constants.*

The next lemma follows immediately from (2) and Lemma (3.1).

3.3. LEMMA. *There is a positive constant r such that for $z, \zeta \in \bar{D}$ with $|z-\zeta| \leq r$ we have*

$$|\tilde{g}(z, \zeta)| \sim |\tilde{g}(\zeta, z)| \sim -\rho(z) - \rho(\zeta) + |\tau_z(\zeta)| + |z-\zeta|^2.$$

For $z \in \bar{D}$ sufficiently near $\tilde{M}$ let z_* be the point on $\tilde{M}$ which is nearest z, and set $\delta_M(z) = |z - z_*|$. We also set $\delta(z) = -\rho(z)$. We will omit the proof of the following elementary result.

3.4. LEMMA. *There is a positive constant r such that for* $\zeta \in \bar{M}$ *and* $|z-\zeta|<r$ *we have*

$$|z-\zeta|^2 \sim \delta_M(z)^2 + |z_* - \zeta|^2.$$

From Lemma (3.3) and (3.4) we obtain:

3.5. COROLLARY. *For* $\zeta \in M$ *and* $z \in \bar{D}$ *we have, for* $|z-\zeta|$ *sufficiently small,*

$$|\tilde{g}(z,\zeta)| \sim \delta(z) + \delta_M(z)^2 + \tau_z(\zeta) + |z_* - \zeta|^2.$$

In what follows, we assume that the parameter r of Lemma (2.1) is chosen small enough that the estimates of Corollary (3.5) hold for $\zeta \in M$ and $z \in \bar{D}$ with $|z-\zeta|<r$.

Let $K(z,\zeta)$ be a measurable function on $\bar{D}\times\bar{D}$, and let t be a real number. We say that K is a kernel of type t if $|K(z,\zeta)| \le C|\tilde{\Phi}(z,\zeta)|^{-t}$ where $\tilde{\Phi}$ is the function from Lemma (2.1). In view of Lemma (3.1), $K(z,\zeta)$ is a kernel of type t if and only if its transpose $K'(z,\zeta)=K(\zeta,z)$ is a kernel of type t.

Let $p \in \partial M$ and let U be as in Lemma (3.2). By shrinking U if necessary, we assume that U is contained in the ball about p of radius r. Let $U' \subset\subset U$.

3.6. LEMMA. *Let K be a kernel of type* $m+t$ *with* $t>0$ *and let* $0<\epsilon<t$. *There is a positive constant* $C=C_{\epsilon,t}$ *such that for any measurable function* ϕ *on M with support in* U' *and with* $|\phi(\zeta)| \le \rho(\zeta)^{t-1-\epsilon}$ *we have*

$$\int_M |K(z,\zeta)|\,|\phi(\zeta)|\,dV_M(\zeta) \le C(\delta(z)+\delta_M(z)^2)^{-\epsilon}.$$

Proof. Note that by Lemma (2.1) the function $\tilde{\Phi}(z,\zeta)$ is bounded away from 0 on $(\bar{D}\setminus U)\times U'$. Thus for $z \in \bar{D}\setminus U$ we have

$$\int_M |K(z,\zeta)|\,|\phi(\zeta)|\,dV_M(\zeta) \le C\int_M \delta(\zeta)^{t-1-\epsilon}\,dV_M(\zeta) \le C_{t,\epsilon}$$

for $\epsilon<t$. Using Lemma (3.2), Corollary (3.5), and (iii) of Lemma (2.1), for $z \in U$ we obtain that

$$\int_M |K(z,\zeta)|\,|\phi(\zeta)|\,dV_M(\zeta)$$
$$\le C\int_{\mathbf{C}^{m-1}}\int_0^\infty\int_0^\infty (\delta(z)+\delta_M(z)^2+\delta+\tau+|\delta'|^2)^{-m-t}\delta^{t-1-\epsilon}\,d\delta\,d\tau\,d\zeta'.$$

The change of coordinates $\delta=(\delta(z)+\delta_M(z)^2)x$, $\tau=(\delta(z)+\delta_M(z)^2)y$, and $\zeta'=(\delta(z)+\delta_M(z)^2)^{1/2}w$ transforms the last integral to

$$C(\delta(z)+\delta_M(z)^2)^{-\epsilon}\int_{\mathbf{C}^{m-1}}\int_0^\infty\int_0^\infty (1+x+y+|w|^2)^{-m-t}x^{t-1-\epsilon}\,dx\,dy\,dw.$$

Changing to polar coordinates in the w variable, one easily checks that this last integral converges if and only if $0<\epsilon<t$, which completes the proof. □

3.7. COROLLARY. *Let K be a kernel of type* $m+t$ *with* $t>0$. *Then for* $0<\epsilon<t$ *we have*

$$\int_M |K(z,\zeta)|\delta(\zeta)^{t-1-\epsilon}\,dV_M(\zeta) \le C_\epsilon(\delta(z)+\delta_M(z)^2)^{-\epsilon}$$

and

$$\int_M |K(z,\zeta)|\delta(z)^{t-1-\epsilon}\,dV_M(z) \le C_\epsilon(\delta(\zeta)+\delta_M(\zeta)^2)^{-\epsilon}.$$

Proof. The first estimate is immediate from Lemma (3.6) by an elementary partition of unity argument. The second estimate follows by applying the first to the transpose of K. □

3.8. LEMMA. *Let K be a kernel of type $n+t$ with $t>m-n$. For any $p\in\partial D$ there is a neighborhood U of p such that for any $s>0$ and $s-t<\epsilon<n-m+s$, and for any measurable function Ψ on D with support in U and $|\Psi(z)|<(\delta(z)+\delta_M(z)^2)^{-\epsilon}$, we have*

$$\int_D |K(z,\zeta)|\,|\Psi(z)|\,\delta(z)^{s-1}\,dV(z) \le C_{s,\epsilon}\delta(\zeta)^{s-t-\epsilon}$$

for every $\zeta\in M$.

Proof. If $p\in\partial D\setminus\partial M$ then it is only necessary to choose U so that $\bar U\cap\tilde M=\varnothing$. In this case $|K(z,\zeta)|\,|\Psi(z)|$ is bounded, so

$$\int_D |K(z,\zeta)|\,|\Psi(z)|\,\delta(z)^{s-1}\,dV(z) \le C_\epsilon\int_D \delta(z)\,dV(z) \le C_{s,\epsilon}$$

for $s>0$.

For $p\in\partial M$, we begin as in the proof of Lemma (3.6) by rotating and translating the $\mathbf{C}^n$ coordinates so that p becomes the origin, the x' axis points along the outer normal to ∂D, and the $z^2,\dots,z^n$ axes span the complex tangent space to ∂D at p. Since $\tilde M$ intersects ∂D transversally at p, there are (by the implicit function theorem) neighborhoods V_1 and V_2 of 0 in $\mathbf{C}^m$ and $\mathbf{C}^{n-m}$, respectively, and a holomorphic mapping $\phi=(\phi^{m+1},\dots,\phi^n)$ of V_1 into V_2 such that $\tilde M\cap(V_1\times V_2)=\{(z',z'')\in V_1\times V_2 : z''=\phi(z')\}$. Moreover, since the $z^2,\dots,z^m$ axes are tangent to $\tilde M$ at p we have $(\partial\phi/\partial z^j)(0)=0$ for $j=2,\dots,m$. We define functions w_j, $j=2,\dots,m$, by $w^j=z^j$ for $2\le j\le m$ and $w^j=z^j-\phi^j(z')$ for $m+1\le j\le n$, where $z'=(z^1,\dots,z^m)$. As before, we set $\tau_z(\zeta)=\operatorname{Im} g(z,\zeta)$. It follows that $\delta,\tau_p,w^2,\dots,w^n$ form a local C^∞ coordinate system for a neighborhood of p. Thus by continuity there is a neighborhood U' of p such that for each $\zeta\in U'$ the functions $\delta,\tau_\zeta,w^2,\dots,w^n$ form a local C^∞ coordinate system, and the Jacobian of the transformation to the $\mathbf{C}^n$ coordinates is bounded above and below by positive constants. Setting $w'=(w^1,\dots,w^m)$ and $w''=(w^{m+1},\dots,w^n)$, it follows that $|w''(z)|\le C\delta_M(z)$ and $|w(z)-w(\zeta)|\le C|z-\zeta|$ for $z,\zeta\in U'$. Thus we obtain from Corollary (3.5) the estimate

$$|\tilde g(z,\zeta)| \ge C(\delta(z)+\delta(\zeta)+|\tau_\zeta(z)|+|w'(z)-w'(\zeta)|^2+|w'(z)|^2) \tag{3}$$

for $z,\zeta\in U'\cap\bar D$ with $\zeta\in\bar M$ and $|z-\zeta|$ sufficiently small. By shrinking U' if necessary, we may assume that (3) holds for all $z,\zeta\in U'\cap\bar D$ with $\zeta\in\bar M$.

Let U be a neighborhood of p with $U \subset\subset U'$. Then $K(z, \zeta)$ is bounded on $(\bar{D} \cap U) \times (\bar{D} \setminus U')$, so for $\zeta \in M \setminus U'$ we have

$$\int_D |K(z, \zeta)|\,|\Psi(z)|\,\delta(z)^{s-1}\,dV(z) \leq C \int_D (\delta(z) + \delta_M(z)^2)^{-\epsilon}\delta(z)^{s-1}\,dV(z).$$

Using the local coordinates introduced above, one can easily check that the integral on the right converges if $\epsilon < n - m + s$ and $s > 0$, so the required estimate holds for $\zeta \in M \setminus U'$. For $\zeta \in U'$, we have, using the special coordinate system defined above and the estimate (3),

$$\int_D |K(z, \zeta)|\,|\Psi(z)|\,dV(z) \leq C \int_{\mathbf{C}^{m-1}} \int_{\mathbf{C}^{n-m}} \int_0^\infty \int_0^\infty (\delta(\zeta) + \delta + \tau + |w'|^2 + |w''|^2)^{-n-t} \times (\delta + |w''|^2)^{-\epsilon}\delta^{s-1}\,d\delta\,d\tau\,dw''\,dw'.$$

The change of variables $\delta = \delta(\zeta)x$, $\tau = \delta(\zeta)y$, $w' = \delta(\zeta)^{1/2}v'$, $w'' = \delta(\zeta)^{1/2}v''$ transforms the right-hand side to

$$C\delta(\zeta)^{s-t-\epsilon} \int_{\mathbf{C}^{m-1}} \int_{\mathbf{C}^{n-m}} \int_0^\infty \int_0^\infty (1 + x + y + |v'|^2 + |v''|^2)^{-n-t} \times (x + |v''|^2)^{-\epsilon} x^{s-1}\,dx\,dy\,dv''\,dv'.$$

Changing to polar coordinates in v' and v'', one checks that the last integral converges if and only if $s - t < \epsilon < n - m + s$. This completes the proof. □

A partition of unity argument yields:

3.9. COROLLARY. *Let K be a kernel of type $n + t$ with $t > m - n$. Then for $s > 0$ and $s - t < \epsilon < n - m + s$ we have*

$$\int_D |K(z, \zeta)|(\delta(z) + \delta_M(z)^2)^{-\epsilon}\delta(z)^{s-1}\,dV(z) \leq C_{s,\epsilon}\delta(\zeta)^{s-t-\epsilon}$$

and

$$\int_D |K(z, \zeta)|(\delta(\zeta) + \delta_M(\zeta)^2)^{-\epsilon}\delta(\zeta)^{s-1}\,dV(\zeta) \leq C_{s,\epsilon}\delta(z)^{s-t-\epsilon}.$$

Note that the functions $\tau_\zeta, w^2, \ldots, w^n$ used in the proof of Lemma (3.8) provide local C^∞ coordinates for ∂D. Thus a slight modification of the proof of Lemma (3.8) yields:

3.10. LEMMA. *Let K be a kernel of type $n + t$ with $t > m - n$. Then for $-t < \epsilon < n - m$ we have*

$$\int_{\partial D} |K(z, \zeta)|\,\delta_M(z)^{-2\epsilon}\,d\sigma(z) \leq C_\epsilon \delta(\zeta)^{-t-\epsilon}$$

and

$$\int_{\partial D} |K(z, \zeta)|\,\delta_M(\zeta)^{-2\epsilon}\,d\sigma(\zeta) \leq C_\epsilon \delta(z)^{-t-\epsilon}.$$

4. ***L^p* *estimates,*** **$1<p<\infty$.** For any kernel $K(z,\zeta)$ on $\bar{D}\times\bar{D}$, we define integral operators K_M^t $(t>m-n)$ and K_D^t $(t\geq 0)$ as follows:

$$K_M^t f(z)=\int_M f(\zeta)K(z,\zeta)\delta(\zeta)^{n-m+t-1}\,dV_M(\zeta),\quad t>m-n,$$

$$K_D^t f(z)=\int_D f(\zeta)K(z,\zeta)\delta(\zeta)^{t-1}\,dV(\zeta),\quad t>0,$$

$$K_D^0 f(z)=\int_{\partial D} f(\zeta)K(z,\zeta)\,d\sigma(\zeta).$$

4.1. THEOREM. *Let $K(z,\zeta)$ be a kernel of type $n+t$ and let $1<p<\infty$.*
(a) *If $n>m$, $t\geq 0$, and $0\leq s\leq t$ then the operator K_M^t is continuous from $L^p_{n-m+s-1}(M)$ into $L^p_{s-1}(D)$.*
(b) *If $0<s\leq t$ or if $t=s=0$, then K_D^t is continuous from $L^p_{s-1}(D)$ into $L^p_{n-m+s-1}(M)$.*

Proof. Let q be the conjugate exponent to p. By Hölder's inequality and Corollary (3.7) we have, for $z\in D$ and $0<\epsilon q<n-m+t$,

$$\begin{aligned}|K_M f(z)|&\leq\int_M |f(\zeta)|\,\delta(\zeta)^{\epsilon}\delta(\zeta)^{-\epsilon}|K(z,\zeta)|\,\delta(\zeta)^{n-m+t-1}\,dV_M(\zeta)\\
&\leq\left(\int_M |f(\zeta)|^p\delta(\zeta)^{\epsilon p+n-m+t-\epsilon}|K(z,\zeta)|\,dV_M(\zeta)\right)^{1/p}\\
&\quad\times\left(\int_M \delta(\zeta)^{-\epsilon q+n-m+t-1}|K(z,\zeta)|\,dV_M(\zeta)\right)^{1/q}\\
&\leq C_\epsilon(\delta(z)+\delta_M(z)^2)^{-\epsilon}\left(\int_M |f(\zeta)|^p\delta(\zeta)^{\epsilon p+n-m+t-1}|K(z,\zeta)|\,dV_M(\zeta)\right)^{1/p}.\end{aligned}$$

Assume that $0<s\leq t$. Then by Fubini's Theorem and Corollary (3.9) we have

$$\begin{aligned}&\int_D |K_M f(z)|^p\delta(z)^{s-1}\,dV(z)\\
&\quad\leq C_\epsilon\int_M |f(\zeta)|^p\delta(\zeta)^{n-m+t-1+\epsilon p}\\
&\qquad\times\int_D |K(z,\zeta)|\,(\delta(z)+\delta_M(z)^2)^{-\epsilon p}\delta(z)^{s-1}\,dV(z)\,dV_M(\zeta)\\
&\quad\leq C_{\epsilon,s}\int_M |f(\zeta)|^p\delta(\zeta)^{n-m+s-1}\,dV_M(\zeta),\end{aligned}$$

provided that $0<\epsilon<(n-m+s)\min\{p^{-1},q^{-1}\}$. This proves part (a) in the case $0<s\leq t$. For the case $0=s\leq t$, we use Lemma (3.10) to obtain

$$\begin{aligned}&\int_{\partial D} |K_M f(z)|^p \, d\sigma(z)\\ &\le C_\epsilon \int_M |f(\zeta)|^p \delta(\zeta)^{n-m+t-1+\epsilon p} \int_{\partial D} |K(z,\zeta)| \delta_M(z)^{-2\epsilon p} \, d\sigma(z) \, dV_M(\zeta)\\ &\le C_\epsilon \int_M |f(\zeta)|^p \delta(\zeta)^{n-m-1} \, dV_M(\zeta),\end{aligned}$$

provided that $0 < \epsilon < (n-m)\min\{p^{-1}, q^{-1}\}$.

The proof of (b) is similar. We will omit the details. □

We now turn to the proof of Theorem (1.1) in the case $1 < p < \infty$. Letting E_j denote the extension operator of Corollary (2.6), it follows from part (a) of Theorem (4.1) that $E_j P: A^p_{n-m+s}(M) \to A^p_s(D)$ for $-1 \le s \le j-1+m-n$ and $1 < p < \infty$, and part (b) of Theorem (1.1) follows immediately. To prove part (a), we apply Corollary (2.4) with $M = D$ to represent the restriction operator, and the desired estimate follows from part (b) of Theorem (4.1).

5. ***L^p estimates,*** **$0 < p \le 1$.** We begin with some technical lemmas that do not depend on pseudoconvexity. Let D be a bounded domain with smooth boundary in $\mathbf{C}^n$ or in a complex manifold. In the manifold case we assume that $\bar{D}$ has been covered by finitely many coordinate neighborhoods, and we denote the coordinates in any of these neighborhoods by $z = (z^1, \ldots, z^n)$.

For $p_0 \in D$ sufficiently near ∂D, we translate and rotate the coordinate system so that $z(p_0) = 0$ and the $\operatorname{Im} z^n$ axis is perpendicular to ∂D. Let $\mathcal{B}_\epsilon(p_0)$ denote the "ball"

$$\mathcal{B}_\epsilon(p_0) = \left\{ \sum_1^{n-1} |z^j|^2 < \epsilon\delta(p_0),\ |z^n| < \epsilon\delta(p_0) \right\}.$$

Since ∂D is smooth, it follows that there is an $\epsilon_0 > 0$ such that, for $p_0 \in D$ sufficiently near ∂D and $p \in \mathcal{B}_{\epsilon_0}(p_0)$, we have

$$\tfrac{1}{2}\delta(p_0) < \delta(p) < 2\delta(p_0). \tag{4}$$

5.1. LEMMA. *For each $0 < \epsilon < \epsilon_0$ there is a compact subset K of D, a sequence $\{p_j\}$ in $D \setminus K$, and a positive integer M such that*

(a) *the family $\{\mathcal{B}_\epsilon(p_j)\}$ covers $D \setminus K$, and*

(b) *each point of D lies in at most M of the sets $\mathcal{B}_{\epsilon_0}(p_j)$.*

Proof. By compactness, it suffices to establish the result in a small neighborhood of a fixed boundary point p_0. Let $e_1, \ldots, e_{n-1}$ be an orthonormal basis for the complex tangent space to ∂D at p_0, and let e_n denote the outward unit normal to ∂D at p_0. For $R > 0$, let L_R denote the lattice in $T_{p_0}(\partial D)$ generated by the vectors $R^{1/2}e_j$, $R^{1/2}Je_j$ $(j = 1, \ldots, n-1)$ and RJe_n. For $\eta > 0$ sufficiently small, let Π_η denote the projection of $T_{p_0}(\partial D)$ on the hypersurface $\{p : \delta(p) = \eta\}$ along the direction e_n, and let $L(R;\eta) = \Pi_\eta(L_R)$. For any ϵ satisfying $0 < \epsilon < \epsilon_0$ there are, by a straightforward continuity argument, a neighborhood U of p_0 and positive constants δ_0 and C (depending on ϵ) such that for $0<\eta<\delta_0$ the balls $\mathcal{B}_\epsilon$ centered at the points of $L(C\eta;\eta)$ cover the tube $\{p\in U: |\delta(p)-\eta|<C\eta\}$. Moreover, if U and δ_0 are sufficiently small, then the number of times a point in U is covered by the balls $\mathcal{B}_{\epsilon_0}$ is bounded by a constant independent of η.

For each non-negative integer j set $T_j=\{p\in U: \delta_0 2^{-j-1}\le\delta(p)\le\delta_0 2^{-j}\}$. Let N be the least integer greater than $1/C$ and let $2^{-j-1}\delta_0=\delta_j^0<\delta_j^1<\cdots<\delta_j^N=2^{-j}\delta_0$ be a uniform partition of the interval $[2^{-j-1}\delta_0, 2^{-j}\delta_0]$. Then the family of balls $\mathcal{B}_\epsilon$ centered at the points $L(C\delta_j^i,\delta_j^i)$, $i=1,\dots,N$, covers the tube T_j but the balls $\mathcal{B}_{\epsilon_0}$ do not intersect the tubes T_i for $i\notin\{j-1,j,j+1\}$. It follows that the points of $\bigcup_{i,j} L(C\delta_j^i,\delta_j^i)$ have the required properties. □

The next result is closely related to a classical estimate of Hardy and Littlewood on the growth of the means of holomorphic functions in the unit disc. (See [6, p. 87].)

5.2. THEOREM. *Let D be a bounded domain with smooth boundary.*
(a) *For $0<p_1\le p_2<\infty$, $t>-1$, and $p_1(s+n+1)>p_2 n$ we have*

$$\|f\|_{p_2,s}\le C\|f\|_{p_1,t}$$

for any holomorphic function f on D.
(b) *For $0<p_1\le p_2<\infty$ and $p_1(s+n+1)>p_2 n$*

$$\|f\|_{p_2,s}\le C\|f\|_{p_1,-1}$$

for any holomorphic function f on D.

In the present setting we do not know whether the estimate in (b) remains valid when $p_1(s+n+1)=p_2 n$.

Proof. It follows from the subharmonicity of $|f|^p$ that for any compact set K in D the metric in $A_s^p(D)$ is equivalent to that defined by integration over $D\backslash K$. In particular, to prove (a) it suffices to show that

$$\left(\int_{D\backslash K}|f|^{p_1}\delta^s\,DV\right)^{1/p_1}\le C\left(\int_D |f|^{p_2}\delta^t\,dV\right)^{1/p_2}$$

for some compact subset K of D.

Let K and ϵ_0 be as in Lemma (5.1) and for $\epsilon=\frac{1}{4}\epsilon_0$ let p_j be as in the lemma. For brevity we denote by $\mathcal{B}_j$ the "ball" $\mathcal{B}_\epsilon(p_j)$ and by $\tilde{\mathcal{B}}_j$ the ball $\mathcal{B}_{\epsilon_0}(p_j)$. For any $p\in\mathcal{B}_j$ let $\Delta(p)$ denote a polydisc centered at p with radius $\epsilon\delta(p_j)$ in the normal direction and radii $(\epsilon\delta(p_j))^{1/2}(n-1)^{-1/2}$ in complex tangential directions. Note that for any $p\in\mathcal{B}_j$ we have $\Delta(p)\subset\tilde{\mathcal{B}}_j$, so for any non-negative pluri-subharmonic function Ψ on $\tilde{\mathcal{B}}_j$ we have, for $p\in\mathcal{B}_j$,

$$\begin{aligned}\Psi(p)&\le C(\epsilon\delta(p_j))^{-n-1}\int_{\Delta(p)}\Psi\,dV\\&\le C(\epsilon\delta(p_j))^{-n-1}\int_{\tilde{\mathcal{B}}_j}\Psi\,dV\le C\int_{\tilde{\mathcal{B}}_j}\Psi\delta^{-n-1}\,dV.\end{aligned}$$

In particular, if f is holomorphic in D we have

$$|f(p)|^{p_1} \le C \int_{\tilde{\mathcal{B}}_j} |f|^{p_1} \delta^{-n-1}\, dV.$$

It thus follows that

$$\begin{aligned} |f(p)|^{p_2}\delta(p)^s &\le C\delta(p)^s \left(\int_{\tilde{\mathcal{B}}_j} |f|^{p_1}\delta^{-n-1}\, dV \right)^{p_2/p_1} \\ &\le C\left(\int_{\tilde{\mathcal{B}}_j} |f|^{p_1}\delta^{-n-1+s(p_1/p_2)}\, dV \right)^{p_2/p_1}. \end{aligned}$$

Integrating over $\mathcal{B}_j$ gives

$$\begin{aligned} \int_{\mathcal{B}_j} |f|^{p_2}\delta^s\, dV &\le C\delta(p_j)^{n+1}\left(\int_{\tilde{\mathcal{B}}_j} |f|^{p_1}\delta^{-n-1+s(p_1/p_2)}\, dV \right)^{p_2/p_1} \\ &\le C\left(\int_{\tilde{\mathcal{B}}_j} |f|^{p_1}\delta^{(n+1)(p_1/p_2-1)+s(p_1/p_2)}\, dV \right)^{p_2/p_1}. \end{aligned}$$

Thus we have

$$\begin{aligned} \int_{D\setminus K} |f|^{p_2}\delta^s\, dV &\le \sum_j \int_{\mathcal{B}_j} |f|^{p_2}\delta^s\, dV \\ &\le C\sum_j \left(\int_{\tilde{\mathcal{B}}_j} |f|^{p_1}\delta^t\, dV \right)^{p_2/p_1} \\ &\le C\left(\sum_j \int_{\tilde{\mathcal{B}}_j} |f|^{p_1}\delta^t \right)^{p_2/p_1} \\ &\le CM^{p_2/p_1}\left(\int_D |f|^{p_1}\delta^t \right)^{p_2/p_1}. \end{aligned}$$

Here M is the bound on the covering multiplicity from Lemma (5.1). The third inequality comes from the condition $p_1 \le p_2$. This completes the proof of (a).

The proof of (b) is similar. Fix a small positive number δ_0 and for each nonnegative integer j let $T_j = \{p \in D : \delta_0 2^{-j-1} < \delta(p) < \delta_0 2^{-j}\}$. The argument used in the proof of (a) shows that

$$\int_{T_j} |f|^{p_2}\delta^s\, dV \le C\left[(\delta_0 2^{-j})^{(n+1)(p_1/p_2-1)+s(p_1/p_2)} \int_{\tilde{T}_j} |f|^{p_1} \right]^{p_2/p_1}$$

where $\tilde{T}_j = T_{j-1} \cup T_j \cup T_{j+1}$. But the integral on the right is dominated by a constant multiple of $\delta_0 2^{-j}\|f\|_{p_1,-1}^{p_1}$, so

$$\int_{T_j} |f|^{p_2}\delta^s\, dV \le C 2^{-tj}\|f\|_{p_1,-1}^{p_1}$$

with $t = (n+1)(1-p_2/p_1)+s+p_2/p_1$, which by hypothesis is greater than 0. Thus it follows that

$$\int_{\{0<\delta<\delta_0/2\}} |f|^{p_2}\delta^s\,dV = \sum_1^\infty \int_{T_j} |f|^{p_2}\delta^s\,dV$$
$$\leq C\|f\|_{p_1,-1}^{p_1}$$

which concludes the proof of (b). □

5.3. COROLLARY. *Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ (or in an n-dimensional Stein manifold) and let $\tilde{\Phi}(z,\zeta)$ be as in Section 2. Let f be holomorphic on D and let r be an arbitrary real number.*

(a) *For $0<p\leq 1$ and $p(s+n+1)=n+1+t$ we have*

$$\left(\int_D |f(\zeta)|\,|\tilde{\Phi}(z,\zeta)|^r\delta(\zeta)^s\,dV(\zeta)\right)^p \leq C\int_D |f(\zeta)|^p|\tilde{\Phi}(z,\zeta)|^{rp}\delta(\zeta)^t\,dV(\zeta).$$

(b) *For $0<p\leq 1$ and $p(s+n+1)>n$ we have*

$$\left(\int_D |f(\zeta)|\,|\tilde{\Phi}(z,\zeta)|^r\delta(\zeta)^s\,dV(\zeta)\right)^p \leq C\sup_{\epsilon>0}\int_{\partial D_\epsilon} |f(\zeta)|^p|\tilde{\Phi}(z,\zeta)|^{rp}\,d\sigma(\zeta).$$

Proof. By Lemma (3.1), $|\tilde{\Phi}(z,\zeta)|\,|\tilde{\Phi}(\zeta,z)|^{-1}$ is bounded above and below by positive constants, so it suffices to verify the estimates with $\tilde{\Phi}(z,\zeta)$ replaced by $\tilde{\Phi}(\zeta,z)$. If z is restricted to some compact subset of D, then $|\tilde{\Phi}(\zeta,z)|^r$ is bounded, so the required estimates are immediate from Theorem (4.2). Thus, it suffices to prove the result when z is near a boundary point p_0. Let U_1, U_2 be neighborhoods of p_0 with $U_1\subset\subset U_2$. Assume that U_2 is chosen sufficiently small that there is a continuous branch of $\tilde{\Phi}(\zeta,z)^r$ on $(D\cap U_2)\times(D\cap U_2)$, and that $U_2\cap D$ has smooth boundary. Since $|\tilde{\Phi}(\zeta,z)|^r$ is bounded on $(D\setminus U_2)\times(D\cap U_1)$, the desired estimates are immediate from Theorem (4.2) if the left-hand integral is taken over $D\setminus U_2$ and z is constrained to lie in $D\cap U_1$. Thus, to complete the proof, it suffices to estimate the integral over $D\cap U_2$. But this integral can be estimated by applying Theorem (4.2) to the holomorphic function $f\tilde{\Phi}(\cdot,z)^r$ in $D\cap U_2$. □

We are now ready to prove Theorem (1.1) in the case $0<p\leq 1$. Let D and M be as in the theorem and let $f\in A_s^p(D)$ with $p\leq 1$. By Theorem (5.2) we have $f\in A_t^1(D)$ for sufficiently large t. By Corollary (2.4) (with $M=D$) we have, for any $z\in M$ and any sufficiently large positive integer t,

$$|f(z)|\leq C_t\int_D \frac{|f(\zeta)|\,\delta(\zeta)^t}{|\tilde{\Phi}(z,\zeta)|^{n+1+t}}\,dV(\zeta).$$

By Corollary (5.3),

$$|f(z)|^p\leq C_{t,p}\int_D \frac{|f(\zeta)|^p\delta(\zeta)^r}{|\tilde{\Phi}(z,\zeta)|^{p(n+1+t)}}\,dV(\zeta),$$

with $r=p(n+1+t)-(n+1)$. Thus by Fubini's Theorem and Corollary (3.9) we have, for $s>-1$ and t sufficiently large,

$$\begin{aligned}\int_M |f(z)|^p\delta(z)^{s+n-m}\,dV_M(z) &\le C_p\int_D |f(\zeta)|^p\delta(\zeta)^r\int_M \frac{\delta(z)^{s+n-m}\,dV_M(z)}{|\tilde{\Phi}(z,\zeta)|^{p(n+1+t)}}\,dV(\zeta)\\ &\le C_p\int_D |f(\zeta)|^p\delta(\zeta)^r\delta(\zeta)^{s-r}\,dV(\zeta)\\ &= \|f\|_{p,s}^p,\end{aligned}$$

which proves part (a) of Theorem (1.1) when $s>-1$. To establish part (a) in the Hardy space case, $s=-1$, note that it follows from the case $1<p<\infty$ considered in Section 4 that the measure $\delta^{n-m-1}\,dV_M$ is a Carleson measure on D. (See Hormander [10].) Thus it follows that for $0<p<\infty$,

$$\int_M |f|^p\delta^{n-m-1}\,dV_M \le C\|f\|_{p,-1}^p$$

for any holomorphic function f on D, which completes the proof of part (a).

Part (b) is similar. By Corollary (2.6), Theorem (5.1), and Corollary (5.2) we have, for any sufficiently large positive integer t,

$$|Ef(z)|^p \le C_{p,t}\int_M \frac{|f(\zeta)|^p\delta(\zeta)^r}{|\tilde{\Phi}(z,\zeta)|^{p(m+1+t)}}\,dV_M(\zeta)$$

with $r=p(m+1+t)-(m+1)$. It follows from Fubini's theorem and Corollary (3.9) (or Lemma (3.10) in the case $s=-1$) that

$$\int_D |Ef(z)|^p\delta(z)^s\,dV(z) \le C_p\int_M |f(\zeta)|^p\delta(\zeta)^{s+n-m}\,dV_M(\zeta)$$

which is the estimate of part (b) of Theorem (1.1). □

We will end this section with an approximation result which extends a theorem of Stout [14].

5.4. THEOREM. *Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$. For $0<p<\infty$ and $s\ge -1$, $\mathcal{O}(\bar{D})$ is dense in $A_s^p(D)$.*

The result of Stout is the case $s=-1$, $1\le p<\infty$.

Proof. We will use a separation of singularities argument. Cover ∂D by balls $B_1,\dots,B_N$ centered at boundary points $p_1,\dots,p_N$ and with radii sufficiently small that the outward normal ν_j to ∂D at p_j is transverse to ∂D at each point of $\bar{B}_j\cap\partial D$. Let $B_0=D$, and choose $\chi_j\in C_0^\infty(B_j)$ such that $\sum\chi_j\equiv 1$ in a neighborhood of $\bar{D}$. Let $f\in A_s^p(D)$. By Theorem (5.2), $f\in A_t^1(D)$ for sufficiently large t, so by Corollary (2.4) we have, for any sufficiently large positive integer t,

$$f(z)=\int_D f(\zeta)K(z,\zeta)\delta(\zeta)^t\,dV(\zeta),$$

where $K(z,\zeta)$ is a kernel of type $n+t+1$. Thus it follows that

$$
\begin{aligned}
f(z) &= \sum \int_D \chi_j(\zeta) f(\zeta) K(z,\zeta)\delta(\zeta)^t \, dV(\zeta) \\
&= \sum f_j(z).
\end{aligned}
$$

Clearly $f_0 \in \mathcal{O}(\bar{D})$ and f_j has boundary singularities only in B_j for $j=1,\dots,N$. We will show that $f_j \in A_s^p(D)$. In the case $1<p<\infty$, $s>-1$, this is an immediate consequence of Theorem (4.1). For the remaining cases, we use the reproducing property of the kernel to write

$$
\begin{aligned}
f_j(z) &= \chi_j(z) f(z) + \int_D (\chi_j(\zeta)-\chi_j(z)) f(\zeta) K(z,\zeta)\delta(\zeta)^t \, dV(\zeta) \\
&= \chi_j(z) f(z) + \mathcal{E}_j f(z).
\end{aligned}
$$

Thus it is enough to show that $\mathcal{E}_j f \in L_s^p(D)$. But

$$
\begin{aligned}
|\mathcal{E}_j f(z)| &\le C \int_D |f(\zeta)| \frac{|z-\zeta|\,\delta(\zeta)^t}{|\tilde{\Phi}(z,\zeta)|^{n+1+t}} \, dV(\zeta) \\
&\le C \int_D |f(\zeta)| \frac{\delta(\zeta)^t}{|\tilde{\Phi}(z,\zeta)|^{n+t+1/2}} \, dV(\zeta).
\end{aligned} \tag{5}
$$

We will first consider the case $s=-1$, $1<p<\infty$. It is easy to check, using the estimate (2) from Section 2, that

$$
\int_D \frac{\delta(\zeta)^t \, dV(\zeta)}{|\tilde{\Phi}(z,\zeta)|^{n+t+1/2}} \le C,
$$

so it follows from Jensen's inequality that

$$
|\mathcal{E}_j f(z)|^p \le C_p \int_D |f(\zeta)|^p \frac{\delta(\zeta)^t}{|\tilde{\Phi}(z,\zeta)|^{n+t+1/2}} \, dV(\zeta).
$$

By Lemma (3.10) and Fubini's theorem,

$$
\begin{aligned}
\int_{\partial D_\epsilon} |\mathcal{E}_j f(z)|^p \, d\sigma(z) &\le C_p \int_D |f(\zeta)|^p \delta(\zeta)^t \int_{\partial D_\epsilon} \frac{d\sigma(z)}{|\tilde{\Phi}(z,\zeta)|^{n+t+1/2}} \, dV(\zeta) \\
&\le C_p \int_D |f(\zeta)|^p \delta(\zeta)^{-1/2} \, dV(\zeta) \\
&= C_p \|f\|_{p,-1/2}^p \\
&\le C_p \|f\|_{p,-1}^p.
\end{aligned}
$$

We now turn to the case $0<p\le 1$. By (5) and Corollary (5.3),

$$
|\mathcal{E}_j f(z)|^p \le C \int_D \frac{|f(\zeta)|^p \delta(\zeta)^r}{|\tilde{\Phi}(z,\zeta)|^{p(n+t+1/2)}} \, dV(\zeta),
$$

with $r=p(n+t+1)-(n+1)$. For $s>-1$, it follows from Fubini's theorem that

$$
\|\mathcal{E}_j f\|_{p,s}^p \le C_{p,s} \int_D |f(\zeta)|^p \delta(\zeta)^r \int_D \frac{\delta(z)^s \, dV(z)}{|\tilde{\Phi}(z,\zeta)|^{p(n+t+1/2)}} \, dV(\zeta).
$$

Choosing t sufficiently large that $p(n+t+1/2)>s+n+1$, it follows from Corollary (3.9) that the inner integral is dominated by $\delta(\zeta)^{s-p(n+t+1/2)+n+1}=\delta(\zeta)^{s-r+p/2}$, so

$$\|\mathcal{E}_j f\|_{p,s}^p \le C_{p,s}\|f\|_{p,s+p/2}^p \le C_{p,s}\|f\|_{p,s}^p.$$

For the case $s=-1$ a similar argument, using Lemma (3.10) instead of Corollary (3.9), shows that

$$\int_{\partial D_\epsilon} |\mathcal{E}_j f|\, d\sigma \le C_p\|f\|_{p,p/2-1}^p \le C_p\|f\|_{p,-1}^p.$$

Thus, in every case, $f_j\in A_s^p$ whenever $f\in A_s^p$.

To complete the proof, it is only necessary to show that each f_j can be approximated by functions in $\mathcal{O}(\bar{D})$. For $\epsilon>0$ sufficiently small, the function $f_j^\epsilon(z)=f_j(z-\epsilon\nu_j)$ is in $\mathcal{O}(\bar{D})$. Moreover, maximal function estimates (see Stein [13]) show that $\|f_j^\epsilon\|_{p,s}\le C\|f_j\|_{p,s}$, so it follows from the Lebesgue dominated convergence theorem that for $f\in A_s^p$ we have $\|f_j-f_j^\epsilon\|_{p,s}\to 0$ as $\epsilon\to 0^+$, and the proof is complete. □

6. The L^∞ estimate. In this section we use a slightly modified version of an argument of Henkin [8]. Since L^∞ estimates for the extension operator E are elementary outside of a small neighborhood of ∂M, it suffices to show that every $p\in\partial M$ has a neighborhood U such that $|f(z)|\le\|f\|_\infty$ for $z\in U$ and $f\in H^\infty(M)$.

For $p\in\partial M$ fixed, choose local coordinates near p such that $M=D\cap\{z_{m+1}=\cdots=z_n=0\}$ in a neighborhood of p and such that ∂D is strictly convex near p. For $z_*\in M$ near p, let H_{z_*} denote the hyperplane $\{z\colon \sum\rho_j(z_*)(z^j-z_*^j)=0\}$. Since the characterizing function ρ for D can be chosen to be strictly convex near p, the following result follows immediately from Taylor's Formula.

6.1. LEMMA. *There is a neighborhood U of p such that for $z_*\in M\cap U$ and $z\in\bar{D}\cap H_{z_*}$ we have*

$$\delta(z)+\delta_M(z)^2\sim\delta(z_*).$$

It follows from the inverse function theorem that for z sufficiently near p there is a unique z_* near p satisfying $z_*^1=\sum\rho_j(z_*)(z^j-z_*^j)$, $z_*^j=z^j$ for $j=2,\dots,m$, and $z_*^j=0$ for $j=m+1,\dots,n$ (cf. [8, Lemma 6]). Thus $z_*\in M$, $z\in H_{z_*}$, and z_* depends continuously on z. From Corollary (2.6) we have

$$Ef(z)=\int_M f(\zeta)\frac{N(z,\zeta)}{\bar{\Phi}(z,\zeta)^{m+1}}\,dV_M(\zeta)=\int_M f(\zeta)K(z,\zeta)\,dV_M(\zeta),$$

so, for z sufficiently near p,

$$\begin{aligned}Ef(z)&=\int_M f(\zeta)K(z_*,\zeta)\,dV_M(\zeta)+\int_M f(\zeta)(K(z,\zeta)-K(z_*,\zeta))\,dV_M(\zeta)\\&=f(z_*)+\mathcal{E}f(z).\end{aligned}$$

Thus it suffices to show that the error term $\mathcal{E}f$ satisfies $|\mathcal{E}f(z)|\le C\|f\|_\infty$ for z sufficiently near p. Since the kernel is well behaved away from the diagonal, it suffices to estimate the integral over a small neighborhood of p. But for z and ζ near p, the kernel has the form $K(z,\zeta)=N(z,\zeta)\tilde{g}(z,\zeta)^{-m-1}$ with N smooth. By the mean value theorem, setting $z_\lambda=z_*+(1-\lambda)(z-z_*)$ for $0\le\lambda\le1$, we have for z,ζ near p

$$(6)\qquad |K(z,\zeta)-K(z_*,\zeta)|\le C\sup_\lambda\left\{\frac{\sum\rho_j(\zeta)(z^j-z_*^j)}{|\tilde{g}(z_\lambda,\zeta)|^{m+2}}+\frac{|z-z_*|\,|z_\lambda-\zeta|}{|\tilde{g}(z_\lambda,\zeta)|^{m+2}}\right\}.$$

Since $z\in H_{z_*}$ the numerator in the first term on the right can be expressed as

$$\sum(\rho_j(\zeta)-\rho_j(z_*))(z^j-z_*^j),$$

so the first term is dominated by $|z_*-\zeta|\,|z-z_*|\,|\tilde{g}(z,\zeta)|^{-m-2}$. Since $|z_*-\zeta|\le|z_*-z_\lambda|+|z_\lambda-\zeta|$ we obtain from (6) the estimate

$$\begin{aligned}|K(z,\zeta)-K(z_*,\zeta)|&\le C|z-z_*|\left[\sup_\lambda\frac{|z_*-z_\lambda|}{|\tilde{g}(z_\lambda,\zeta)|^{m+2}}+\sup_\lambda\frac{|z_\lambda-\zeta|}{|\tilde{g}(z_\lambda,\zeta)|^{m+2}}\right]\\ (7)\qquad&\le C\left[|z-z_*|^2\sup_\lambda\frac{1}{|\tilde{g}(z_\lambda,\zeta)|^{m+2}}+|z-z_*|\sup_\lambda\frac{|z_\lambda-\zeta|}{|\tilde{g}(z_\lambda,\zeta)|^{m+2}}\right]\\ &\le C\left[\delta(z_*)\sup_\lambda\frac{1}{|\tilde{g}(z_\lambda,\zeta)|^{m+2}}+\delta(z_*)^{1/2}\sup_\lambda\frac{1}{|\tilde{g}(z_\lambda,\zeta)|^{m+3/2}}\right].\end{aligned}$$

The last inequality follows from Lemma (5.1) since $|z-z_*|\le C\delta_M(z)$.

By Corollary (3.5) and Lemma (5.1), for any $z\in H_{z_*}\cap D$ which is sufficiently near p we have

$$\begin{aligned}|\tilde{g}(z,\zeta)|&\ge C(\delta(z)+\delta_M(z)^2+\delta(\zeta)+\tau_z(\zeta)+|\zeta-z_*|^2)\\ &\ge C(\delta(z_*)+\delta(\zeta)+|\mathrm{Im}\sum\rho_j(z)(\zeta^j-z^j)|+|\zeta-z_*|^2)\\ &=C(\delta(z_*)+\delta(\zeta)+|\sum\mathrm{Im}\,\rho_j(z_*)(\zeta^j-z^j)|+|\zeta-z_*|^2)\\ &\ge C|\tilde{g}(z^*,\zeta)|.\end{aligned}$$

Thus it follows from (7) that

$$|K(z,\zeta)-K(z_*,\zeta)|\le C\left(\frac{\delta(z_*)}{|\tilde{g}(z_*,\zeta)|^{m+2}}+\frac{\delta(z_*)^{1/2}}{|\tilde{g}(z,\zeta)|^{m+3/2}}\right),$$

so

$$|\mathcal{E}f(z)|\le C\|f\|_\infty\left(\delta(z_*)\int_M\frac{dV_M(\zeta)}{|\tilde{g}(z_*,\zeta)|^{m+2}}+\delta(z_*)^{1/2}\int_M\frac{dV_M(\zeta)}{|\tilde{g}(z_*,\zeta)|^{m+3/2}}\right).$$

By Lemma (3.6) we have $|\mathcal{E}f(z)|\le C\|f\|_\infty$.

This completes the proof of Theorem (1.1) in the case $p=\infty$. □

7. Concluding remarks. It seems worthy of note that global strict pseudoconvexity of ∂D is not essential in Theorem (1.1). It is only necessary to assume that D is pseudoconvex and that ∂D is strictly pseudoconvex at each point of ∂M (cf. Adachi [1]). Part (a) can be proved in this case by constructing a strictly pseudoconvex domain D' with $D'\subset D$ and $\partial M\subset\partial D'$. One then applies the result for strictly pseudoconvex domains to D'. Part (b) can be proved by replacing Lemma (2.1) by an analogue of Theorem (2.1) of Beatrous [2]. On the other hand, strict pseudoconvexity of ∂D at each point of ∂M is essential. For example, fix a number r with $0<r<1$ and set $D=\{(z,w)\in\mathbf{C}^2\colon |z|^2+e^{-1/|w|^2}<r^2\}$. Then D is a smooth, bounded, pseudoconvex domain, and ∂D is strictly pseudoconvex at each point with $w\neq 0$. Let $M=D\cap\{w=0\}$. For $F\in\mathcal{O}(D)$, set $f(z)=F(z,0)$ for $z\in\Delta_r=\{\lambda\in\mathbf{C}\colon|\lambda|<r\}$. For any fixed $z\in\Delta_r$ the function $F(z,\cdot)$ is holomorphic in the disc of radius $\eta_z=[-\log(r^2-|z|^2)]^{-1/2}$, so it follows from subharmonicity that for $0<p<\infty$

$$|f(z)|^p\le\frac{1}{2\pi\eta_z}\int_{|w|=\eta_z}|F(z,w)|^p\,ds(w),$$

where ds denotes the arc length measure. By Fubini's theorem,

$$\begin{aligned}\int_{\partial D}|F|^p\,d\sigma&=\int_{\Delta_r}\int_{|w|=\eta_z}|F(z,w)|^p(1+|z|^2\eta_z^6e^{2/\eta_z^2})^{1/2}\,ds(w)\,dA(z)\\&\ge 2\pi\int_{\Delta_r}\eta_z(1+|z|^2)\eta_z^6e^{2/\eta_z^2})^{1/2}|f(z)|^p\,dA(z)\\&\ge 2\pi\int_{\Delta_r}|z|\eta_z^4e^{1/\eta_z^2}|f(z)|^p\,dA(z)\\&=2\pi\int_{\Delta_r}|f(z)|^p\left(\log\frac{1}{r^2-|z|^2}\right)^{-2}(r^2-|z|^2)^{-1}|z|\,dA(z)\\&\ge C_\epsilon\int_{\Delta_r}|f(z)|^p(r^2-|z|^2)^{-1+\epsilon}\,dA(z)\end{aligned}\tag{8}$$

for any $\epsilon>0$. Moreover, an arbitrary $F\in H^p(D)$ can be approximated by dilations which are holomorphic in a neighborhood of $\bar{D}$, so (8) remains valid for $f\in A^p_{-1}(D)$. For $\alpha\in\mathbf{R}$ let f_α be some branch of $(z-r)^{-\alpha}$ on Δ_r. Then for $s>-1$, $f_\alpha\in A^p_s(\Delta_r)$ if and only if $\alpha<(s+2)/p$. Thus it follows from (8) that for $0<p<\infty$ and $1/p<\alpha<(1+\epsilon)/p$ we have $f_\alpha\in A^p_{-1+\epsilon}(\Delta_r)$, but f_α has no extension in $A^p_{-1}(D)$. Thus $A^p_{-1}(D)|_M\not\supset A^p_{-1+\epsilon}(M)$ for any $\epsilon>0$. Similar considerations apply to Bergman spaces on D.

REFERENCES

1. K. Adachi, *Extending bounded holomorphic functions from certain subvarieties of a weakly pseudoconvex domain,* Pacific J. Math. 110 (1984), 9–19.
2. F. Beatrous, *Hölder estimates for the $\bar\partial$ equation with a support condition,* Pacific J. Math. 90 (1980), 249–257.
3. F. Beatrous and J. Burbea, *Positive-definiteness and its applications to interpolation problems for holomorphic functions,* Trans. Amer. Math. Soc. 284 (1984), 247–270.
4. L. Bungart, *Holomorphic functions with values in locally convex spaces and applications to integral formulas,* Trans. Amer. Math. Soc. 111 (1964), 317–344.

5. A. Cumenge, *Extension dans des classes de Hardy de fonctions holomorphes et estimations de type "mesures de Carleson" pour l'equation ∂*, Ann. Inst. Fourier Grenoble 33 (1983), 59–97.
6. P. Duren, *Theory of H^p spaces,* Academic Press, New York, 1970.
7. G. M. Henkin, *Integral representations of functions holomorphic in strictly pseudoconvex domains and some applications,* Math. USSR Sb. 7 (1969), 597–616.
8. ———, *Continuation of bounded holomorphic functions from submanifolds in general position to strictly pseudoconvex domains,* Math. USSR Izv. 6 (1972), 536–563.
9. G. M. Henkin and J. Leiterer, *Global integral formulas for solving the $\bar{\partial}$-equation on Stein manifolds,* Ann. Polon. Math. 39 (1981), 93–116.
10. L. Hörmander, *L^p estimates for (pluri) subharmonic functions,* Math. Scand. 20 (1967), 65–78.
11. N. Kerzman and E. M. Stein, *The Szegö kernel in terms of Cauchy-Fantappié kernels,* Duke Math. J. 45 (1978), 197–224.
12. E. Ramirez de A., *Ein Divisionsproblem und Randintegraldarstellungen in der komplexen Analysis,* Math. Ann. 184 (1970), 172–187.
13. E. M. Stein, *Boundary behavior of holomorphic functions of several complex variables,* Princeton Univ. Press, Princeton, N.J., 1972.
14. E. L. Stout, *H^p functions on strictly pseudoconvex domains,* Amer. J. Math. 98 (1976), 821–852.

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