HARDY SPACES AND BMO-FUNCTIONS INDUCED BY ERGODIC FLOWS

Jun-Ichi Tanaka

1. Introduction. Let X be a measure space with probability measure m, and let $\{T_t\}_{t\in\mathbf{R}}$ be an ergodic measurable action of the real line $\mathbf{R}$ on X preserving m. The *ergodic Hilbert transform* on X, $H_X\phi$, of a function ϕ in $L^1(X)$ is defined by the formula:

$$(1.1)\qquad (H_X\phi)(x)=\lim_{\epsilon\to+0}\frac{1}{\pi}\int_{\epsilon<|t|<1/\epsilon}\phi(T_{-t}x)\frac{dt}{t}$$

for a.e. x in X. The existence of this limit is shown in [2]. Let $H^\infty(X)$ be the subalgebra of $L^\infty(X)$ consisting of functions of the form $\phi+iH_X\phi$, and let $H_0^\infty(X)$ be the subspace of all functions in $H^\infty(X)$ with mean value zero. The space $H^p(X)$ (resp. $H_0^p(X)$), $0<p<\infty$, is defined to be the closure of $H^\infty(X)$ (resp. $H_0^\infty(X)$) in $L^p(X)$. The measure m is multiplicative on $H^\infty(X)$, and $H^\infty(X)$ becomes a weak*-Dirichlet algebra in $L^\infty(X)$ (cf. [10], [16], and Proposition 5.1 in Section 5).

Let Y be a measure space with probability measure m_1, and let T be an ergodic measure preserving transformation on Y. Suppose that F is a bounded measurable function on Y, bounded away from zero, and normalized to have integral one. Throughout this paper, we shall always assume that the ergodic flow $(X,\{T_t\}_{t\in\mathbf{R}},m)$ is the "special flow under the function F" generated by ergodic dynamical system (Y,T,m_1). More precisely, we define τ to be the function by the formula

$$(1.2)\qquad \tau(y,n)=\begin{cases}\sum_{j=0}^{n-1}F(T^jy) & \text{if } n>0,\\ 0 & \text{if } n=0,\\ -\tau(T^ny,-j) & \text{if } n<0,\end{cases}$$

for each integer n and each y in Y. Let X be the region of $Y\times\mathbf{R}$ under the graph of F, that is,

$$X=\{(y,s): y\in Y \text{ and } 0\le s<F(y)\},$$

and let m be the restriction of $dm_1\times dt$ to X. Then it is easy to see that m is a probability measure on X by the hypotheses of F. By using (1.2), a measure preserving transformation group $\{T_t\}_{t\in\mathbf{R}}$ on X is defined by the formula

$$(1.3)\qquad T_t(y,s)=(T^ny,s+t-\tau(y,n))$$

Received March 23, 1984. Final revision received August 11, 1984.

This research was partially supported by a Grant-in-Aid for Scientific Research (No. 57740097) from the Japanese Ministry of Education.

Michigan Math. J. 32 (1985).

if $\tau(y,n) \leq s+t < \tau(y,n+1)$, for each (y,s) in X (see [11, Section 2]). Then $(X, \{T_t\}_{t\in \mathbf{R}}, m)$ is an ergodic flow. The theorem of Ambrose [1] asserts that every ergodic flow is isomorphic, in a measure preserving fashion, to a special one just constructed in above. Thus there is no loss of generality from assuming it throughout.

By the way, our setting may be regarded as a direct extension of local product decompositions, which is very useful for understanding compact abelian groups with ordered duals (cf. [8, Chapter II] and [15]).

Our principal objective is to establish an isometric isomorphism between $H_0^p(X)$, $1 \leq p < \infty$, and a certain Banach space associated with analytic functions on $Y \times \mathbf{R}$, and particular attention is given to ergodic theoretic generalizations of BMO-functions. Our approach should make clearer the relationship between classical Hardy spaces and analyticity on ergodic settings.

In the next section, we present some preliminary material which we shall need. In Section 3, our representation of $H_0^p(X)$, Theorem 3.1, is obtained and we provide another proof of the basic theorem concerning BMO-functions. We also study VMO-functions on strictly ergodic systems and a construction of unbounded VMO-functions is given in Section 4. We close with some remarks in Section 5.

This work was completed while the author was visiting the University of California, Berkeley, and he would like to acknowledge the hospitality of the Department of Mathematics. In particular, he would like to express his sincere gratitude to Professor Henry Helson for stimulating discussions. The author is also grateful to Professor Yuji Ito for his valuable advice about ergodic theory, and to the referee for his useful suggestions which improved the first version of this paper.

2. Preliminaries. We begin with some basic properties about the classical Hardy spaces, $H^p(dt)$, $0 < p \leq \infty$, on $\mathbf{R}$. It is known that $H^1(dt)$ is the space of all integrable functions of which Fourier transforms vanish on negative real line. We define $H^\infty(dt/(1+t^2)) = H^\infty(dt)$, that is, the space of all boundary functions of bounded analytic functions in the upper half-plane. Let $H^p(dt/(1+t^2))$, $0 < p < \infty$, be the closure of $H^\infty(dt/(1+t^2))$ in $L^p(dt/(1+t^2))$. Then f lies in $H^p(dt/(1+t^2))$ if and only if $f(t)/(t+i)^{2/p}$ lies in $H^p(dt)$. Recall that a locally integrable function f is said to be a BMO-function on $\mathbf{R}$ if the norm

$$\|f\|_{\mathrm{BMO}(\mathbf{R})} = \sup_I \frac{1}{|I|} \int_I \left| f(t) - \frac{1}{|I|} \int_I f(s)\, ds \right| dt$$

is bounded, where the supremum is taken over all bounded intervals I. Then the space $\mathrm{BMO}(\mathbf{R})$ of all BMO-functions on $\mathbf{R}$ is isomorphic to the dual space of $H^1(dt)$.

When $1 \leq p \leq \infty$, $H^p(X)$ equals the space of all functions ϕ in $L^p(X)$ such that the function of t, $\phi(T_t x)$, lies in $H^p(dt/(1+t^2))$ for a.e. x in X. In an ergodic setting, an analogous definition of BMO-functions is given by Coifman and Weiss [3]. A function ϕ in $L^1(X)$ belongs to $\mathrm{BMO}(X)$ if and only if the norm

(2.1) $$\|\phi\|_{BMO(X)} = \operatorname{ess\,sup}_{x} \sup_{\epsilon>0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |\phi(T_t x) - M_\epsilon \phi(x)|\, dt$$

is bounded, where $M_\epsilon \phi(x) = (1/2\epsilon) \int_{-\epsilon}^{\epsilon} \phi(T_t x)\, dt$. Similarly we denote by BMO$(Y\times \mathbf{R})$ the space of all functions $f(y,t)$ in $L^1(dm_1 \times dt/(1+t^2))$ of which the norm

$$\|f\|_{BMO(Y\times \mathbf{R})} = \operatorname{ess\,sup}_{y} \|f(y,\cdot)\|_{BMO(\mathbf{R})}$$

is bounded.

Let Y, T, and F be as in Section 1, and define

(2.2) $$\sigma(y,t) = (Ty, t - F(y))$$

for each (y,t) in $Y\times\mathbf{R}$. Then we may easily see that the hypotheses of F imply that $Y\times\mathbf{R}$ is the disjoint union $\bigcup_{j=-\infty}^{\infty} \sigma^j(X)$. Let π be the mapping of $Y\times\mathbf{R}$ onto X defined by

(2.3) $$\pi(y,t) = (T^n(y), t - \tau(y,n))$$

if $\tau(y,n) \le t < \tau(y,n+1)$, where $\tau(y,n)$ denotes the function defined in (1.2). Every function ϕ on X has the *automorphic extension* $\phi^{\#}$ to $Y\times\mathbf{R}$ by

(2.4) $$\phi^{\#}(y,t) = \phi\circ\pi(y,t)$$

for each (y,t) in $Y\times\mathbf{R}$. This automorphic extension will play an important role in what follows. We note here the space of all automorphic extensions, BMO$(X)^{\#}$, of functions in BMO(X) becomes a closed subspace in BMO$(Y\times\mathbf{R})$.

Let $\mathbf{B}$ be a Banach space, and let $\mathbf{F}$ be a closed subspace of $\mathbf{B}$. The dual space of $\mathbf{B}$ is denoted by $\mathbf{B}^*$. The annihilator $\mathbf{F}^\perp$ of $\mathbf{F}$ is the subspace of all L in $\mathbf{B}^*$ which is orthogonal to $\mathbf{F}$. Recall that by Hahn–Banach theorem $\mathbf{F}^*$ and $(\mathbf{B}/\mathbf{F})^*$ are isometrically isomorphic to $\mathbf{B}^*/\mathbf{F}^\perp$ and $\mathbf{F}^\perp$, respectively.

We refer to [6] for results about classical Hardy spaces and BMO-functions on $\mathbf{R}$. Analyticity in our setting can be found in [8] and [5, Chapter VII], and our reference for the basic facts about Banach spaces is [4].

Let us introduce certain Banach spaces consisting of measurable functions on $Y\times\mathbf{R}$. For $1\le p<\infty$, we define $\mathcal{L}^p$ is the space of all f in $L^1(dm_1\times dt)$ such that

(2.5) $$\begin{aligned} N_p(f) &= \sum_{j=-\infty}^{\infty} \left[\int_Y \int_0^{F(T^j y)} |f(y, s+\tau(y,j))|^p\, dm_1(y)\, ds\right]^{1/p} \\ &= \sum_{j=-\infty}^{\infty} \left[\int_X |f\circ\sigma^{-j}(x)|^p\, dm(x)\right]^{1/p} \end{aligned}$$

is bounded. Then $\mathcal{L}^p$ becomes a Banach space with the norm N_p. The dual space $(\mathcal{L}^p)^*$ of $\mathcal{L}^p$ is easily determined by the properties of the space of sequences (cf. [4, Chapter IV]). Suppose that $1/p+1/q=1$. We denote by $\mathfrak{N}^q$ the Banach space of all measurable functions g on $Y\times\mathbf{R}$ of which the norm

$$
(2.6)\qquad \begin{aligned} N_q^*(g) &= \sup_{-\infty<j<\infty}\left[\int_Y\int_0^{F(T^jy)} |g(y,s+\tau(y,j))|^q\,dm_1(y)\,ds\right]^{1/q} \\ &= \sup_{-\infty<j<\infty}\left[\int_X |g\circ\sigma^{-j}(x)|^q\,dm(x)\right]^{1/q} \end{aligned}
$$

is bounded. Then $(\mathcal{L}^p)^*$ is isometrically isomorphic to $\mathcal{N}^q$. More precisely, if L belongs to $(\mathcal{L}^p)^*$, then we may choose a unique g in $\mathcal{N}^q$ with $\|L\| = N_q^*(g)$ such that

$$
(2.7)\qquad \begin{aligned} L(f) &= \int_Y\int_{-\infty}^{\infty} f(y,t)g(y,t)\,dm_1(y)\,dt \\ &= \sum_{j=-\infty}^{\infty}\int_X (fg)\circ\sigma^{-j}(x)\,dm(x) \end{aligned}
$$

for each f in $\mathcal{L}^p$. Conversely, for a given g in $\mathcal{N}^q$, (2.7) defines a linear functional on $\mathcal{L}^p$. We notice that $\mathcal{L}^1$ and $\mathcal{N}^\infty$ are equal to $L^1(dm_1\times dt)$ and $L^\infty(dm_1\times dt)$, respectively.

By identifying $(\mathcal{L}^p)^*$ with $\mathcal{N}^q$, we obtain the following.

LEMMA 2.1. *Let $1<q\le\infty$. For a given g in $\mathcal{N}^q$ and for a positive integer k, we put*

$$
(2.8)\qquad h_k(y,t) = \frac{1}{2k+1}\sum_{j=-k}^{k} g\circ\sigma^j(y,t).
$$

Then there exists a weak- limit point h of $\{h_k\}$, which satisfies that $h=\theta^{\#}$ for some θ in $L^q(X)$ with $\|\theta\|_q\le N_q^*(g)$.*

Proof. Since $N_q^*(g\circ\sigma^j) = N_q^*(g)$ for all j, we have easily $N_q^*(h_k)\le N_q^*(g)$ for all k. Then Banach–Alaoglu theorem can be used to have a weak-* limit point h of $\{h_k\}$. Let $\{h_{k_\alpha}\}$ be a net consisting of points in $\{h_k\}$ which converges to h in weak-* topology. Then it follows from (2.8) that

$$
h_{k_\alpha}\circ\sigma(y,t) - h_{k_\alpha}(y,t)\to 0 \ \text{(in norm-topology), and}
$$

$$
h_{k_\alpha}\circ\sigma(y,t)\to h\circ\sigma(y,t) \ \text{(in weak-* topology)}
$$

They yield that $h\circ\sigma(y,t) = h(y,t)$ for a.e. (y,t) in $Y\times\mathbf{R}$. Since $N_q^*(h)\le N_q^*(g)$, it follows from (2.6) that the restriction θ of h to X has the desired properties. □

It is sometimes useful to define $\mathcal{L}^\infty$, though its dual space is difficult. The space $\mathcal{L}^\infty$ is similarly defined to be the space of all f such that

$$
(2.9)\qquad N_\infty(f) = \sum_{j=-\infty}^{\infty} \operatorname{ess\,sup}\{|f(y,t)|;\,(y,t)\in\sigma^j(X)\}
$$

is bounded. For $1\le p\le\infty$, a closed subspace $\mathcal{H}^p$ of $\mathcal{L}^p$ is defined to be the space of all f such that the function of t, $f(y,t)$, belongs to $H^1(dt)$ for a.e. y in Y. It can be easily seen that $\mathcal{H}^p$, $1\le p<\infty$, is the closure $\mathcal{H}^\infty$ in $\mathcal{L}^p$. We notice that the dual $(\mathcal{H}^p)^*$, $1\le p<\infty$, is isometrically isomorphic to $\mathcal{N}^q/(\mathcal{H}^p)^\perp$.

We finally define a linear mapping Φ of $\mathcal{H}^p$, $1 \le p \le \infty$, into $L^p(X)$ by the formula

$$\Phi(f)(y,s) = \sum_{j=-\infty}^{\infty} f \circ \sigma^j(y,s) \tag{2.10}$$

for each (y,s) in X. It follows from (2.5) and (2.9) that Φ is a bounded linear mapping of which the norm is at most one.

3. Representation of Hardy spaces and duality. The following theorem is derived from several well-known results in functional analysis. Let $\ker \Phi$ be the closed subspace of all f in $\mathcal{H}^p$ such that $\Phi(f)=0$.

THEOREM 3.1. *Let $1 \le p < \infty$, and let $\mathcal{H}^p$, $H_0^p(X)$, Φ, and* $\ker \Phi$ *be as before. Then $\mathcal{H}^p/\ker \Phi$ is isometrically isomorphic to $H_0^p(X)$ via the mapping Φ. In particular, Φ maps $\mathcal{H}^p$ onto $H_0^p(X)$.*

Proof. We first show that the image $\Phi(\mathcal{H}^p)$ of $\mathcal{H}^p$ is a dense subspace of $H_0^p(X)$. Let $1/p+1/q=1$. If ϕ lies in $H^q(X)$, then we see that

$$\iint_X \phi(y,t) f\circ\sigma^j(y,t)\, dm_1(y)\, dt = \iint_{\sigma^j(X)} \phi\circ\pi(y,t) f(y,t)\, dm_1(y)\, dt$$

for each f in $\mathcal{H}^p$, where σ and π are defined in (2.2) and (2.3), respectively. So it follows from (2.4), (2.10), and Fubini's theorem that

$$\begin{aligned}\int_X \phi(x)\Phi(f)(x)\, dm(x) &= \int_Y \int_{-\infty}^{\infty} \phi^{\#}(y,t) f(y,t)\, dm_1(y)\, dt \\ &= \int_Y \left[\int_{-\infty}^{\infty} \phi^{\#}(y,t) f(y,t)\, dt\right] dm_1(y) \\ &= 0.\end{aligned}$$

Since $H_0^p(X)^\perp = H^q(X)$, $\Phi(f)$ belongs to $H_0^p(X)$. Similarly if a function ϕ in $L^q(X)$ is orthogonal to $\Phi(\mathcal{H}^p)$, then we see easily that the function of t, $\phi^{\#}(y,t)$, lies in $H^q(dt/(1+t^2))$ for m_1-a.e. y in Y. This implies that ϕ is orthogonal to $H_0^p(X)$. Thus we obtain that $\Phi(\mathcal{H}^p)$ is dense in $H_0^p(X)$.

Since $\|\Phi(f)\|_p \le N_p(f)$ for each f in $\mathcal{H}^p$, we may regard Φ as a bounded linear mapping of $\mathcal{H}^p/\ker \Phi$ into $H_0^p(X)$. Then the adjoint Φ^* of Φ maps $H_0^p(X)^*$ into $(\mathcal{H}^p/\ker \Phi)^*$ by the formula $\Phi^*(L) = L\circ\Phi$ for each L in $H_0^p(X)^*$. We consider $\mathcal{H}^p/\ker \Phi$ as a subspace of $\mathcal{L}^p/\ker \Phi$. Notice that the dual spaces $H_0^p(X)^*$ and $(\mathcal{L}^p/\ker \Phi)^*$ are isometrically isomorphic to $L^q(X)/H^q(X)$ and $(\ker \Phi)^\perp$, respectively.

Next we show that Φ^* is an isometry of $H_0^p(X)^*$ onto $(\mathcal{H}^p/\ker \Phi)^*$. Indeed, for each U in $(\mathcal{H}^p/\ker \Phi)^*$, we can extend U to a linear functional on $\mathcal{L}^p$ with the same norm $\|U\|$ by Hahn–Banach theorem. Hence there exists a function g in $\mathfrak{N}^q$ such that g is orthogonal to $\ker \Phi$, $N_q^*(g) = \|U\|$, and (2.7) represents U. On the other hand, it is easy to see that $f(y,t) - f\circ\sigma^{-j}(y,t)$ belongs to $\ker \Phi$ for each f in $\mathcal{H}^p$ and for each integer j. Therefore we have

$$U(f+\ker\Phi)=\int_Y\int_{-\infty}^{\infty} g(y,t)f(y,t)\,dm_1(y)\,dt$$

$$=\int_Y\int_{-\infty}^{\infty} g(y,t)f\circ\sigma^{-j}(y,t)\,dm_1(y)\,dt$$

$$=\int_Y\int_{-\infty}^{\infty} g\circ\sigma^{j}(y,t)f(y,t)\,dm_1(y)\,dt$$

for all f in $\mathcal{H}^p$. Since $f_0\circ\sigma^{-j}$ lies in $\ker\Phi$ for each f_0 in $\ker\Phi$, $g\circ\sigma^j$ is also orthogonal to $\ker\Phi$. Hence every $g\circ\sigma^j$ has the same properties as g. Let h_k be the function defined by (2.8) with this g, and let h and θ be as in Lemma 2.1. Then it follows from the property of weak*-topology that (2.7) replaced g with h also represents U. Recall that $h=\theta^{\#}$ and θ lies in $L^q(X)$. So if we define $L_\theta(\phi)=\int_X\theta(x)\phi(x)\,dm(x)$ for all ϕ in $H_0^p(X)$, then L_θ is a linear functional on $H_0^p(X)$ with $\|L_\theta\|=\|\theta+H^q(X)\|$, where $\|\theta+H^q(X)\|$ denotes the quotient norm of $\theta+H^q(X)$ in $L^q(X)/H^q(X)$. Then since

$$\Phi^*(L_\theta)(f+\ker\Phi)=\int_X\theta(x)\Phi(f)(x)\,dm(x)$$

$$=\iint_X\theta(y,s)\sum_{j=-\infty}^{\infty} f\circ\sigma^j(y,s)\,dm_1(y)\,ds$$

$$=\int_Y\int_{-\infty}^{\infty}\theta^{\#}(y,t)f(y,t)\,dm_1(y)\,dt$$

$$=\int_Y\int_{-\infty}^{\infty}h(y,t)f(y,t)\,dm_1(y)\,dt$$

$$=U(f+\ker\Phi)$$

for each f in $\mathcal{H}^p$, we have that $\Phi^*(L_\theta)=U$. Furthermore, since $\|U\|=N_q^*(g)$ and $N_q^*(h)=\|\theta\|_q$, it follows from Lemma 2.1 that

$$\|\Phi^*(L_\theta)\|\geq\|\theta\|_q\geq\|\theta+H^q(X)\|=\|L_\theta\|.$$

On the other hand, since $\|\Phi^*\|=\|\Phi\|$ holds generally, we also have $\|\Phi^*(L_\theta)\|\leq\|\Phi\|\,\|L_\theta\|\leq\|L_\theta\|$. This shows that Φ^* is an isometry of $H_0^p(X)^*$ onto $(\mathcal{H}^p/\ker\Phi)^*$.

Therefore, by the closed range theorem [4, Chapter VI, 6.4], we obtain that Φ is also a bounded linear mapping of $\mathcal{H}^p/\ker\Phi$ onto $H_0^p(X)$. It is easy to assert that Φ is also isometric. This follows from the relation:

$$\|\Phi(f+\ker\Phi)\|_p=\sup\{|L(\Phi(f+\ker\Phi))|;L\in H_0^p(X)^*,\text{ and }\|L\|=1\}$$
$$=\sup\{|\Phi^*(L)(f+\ker\Phi)|;L\in H_0^p(X)^*,\text{ and }\|L\|=1\}$$
$$=\sup\{|U(f+\ker\Phi)|;U\in(\mathcal{H}^p/\ker\Phi)^*,\ \|U\|=1\}$$
$$=\tilde{N}_p(f+\ker\Phi),$$

where $\tilde{N}_p(f+\ker\Phi)$ denotes the quotient norm of $f+\ker\Phi$ in $\mathcal{H}^p/\ker\Phi$. Thus Φ is an isometry of $\mathcal{H}^p/\ker\Phi$ onto $H_0^p(X)$, and this completes the proof. □

In view of Theorem 3.1, it is interesting to provide a characterization of ker Φ.

PROPOSITION 3.2. *Let* $1 \le p < \infty$, *and let* $\mathcal{G}^p$ *be the space of all* $f(y,t) - f \circ \sigma(y,t)$ *with* f *in* $\mathcal{H}^p$. *Then* $\mathcal{G}^p$ *is a dense subspace in* ker Φ.

Proof. It is obvious that $f(y,t) - f \circ \sigma(y,t)$ lies in ker Φ for each f in $\mathcal{H}^p$, so it suffices that $(\mathcal{G}^p)^\perp$ is included in $(\ker \Phi)^\perp$. For each L in $(\mathcal{G}^p)^\perp$, there is a function g in $\mathfrak{N}^q$, $1/p + 1/q = 1$, such that (2.7) represents L. Suppose that h_k is the function defined by (2.8). Then since

$$\int_Y \int_{-\infty}^{\infty} (f(y,t) - f \circ \sigma(y,t)) g(y,t)\, dm_1(y)\, dt = 0$$

for each f in $\mathcal{H}^p$, every h_k represents L on $\mathcal{H}^p$. Therefore, by Lemma 2.1, we may find a function θ in $L^q(X)$ such that

$$L(f) = \int_Y \int_{-\infty}^{\infty} f(y,t) \theta^{\#}(y,t)\, dm_1(y)\, dt$$

$$= \int_X \Phi(f)(x) \theta(x)\, dm(x)$$

for each f in $\mathcal{H}^p$. Hence $L(f_0) = 0$ for each f_0 in ker Φ; this proves the proposition. □

We can extend the above mapping Φ to the case where $0 < p < 1$. The space $\mathcal{H}^p$ is defined to be the closure of $\mathcal{H}^\infty \cap L^p(dm_1 \times dt)$ in $L^p(dm_1 \times dt)$. Since we use the ordinary metric when $0 < p \le 1$, $\mathcal{H}^p$ is more natural than any other case. It is easy to see that Φ maps $\mathcal{H}^p$ into $H_0^p(X)$. In [11, p. 322], Muhly asked whether $\Phi(\mathcal{H}^p)$, $0 < p \le 1$, actually equals $H_0^p(X)$. Theorem 3.1 is motivated by this question and shows that at least it is true for $p = 1$.

Coifman and Weiss showed that the dual space of $H^1(X)$ is isomorphic to BMO(X) in [3, Theorem 2]. We provide another proof of this fundamental result. Their techniques are different from ours and based on deep results in harmonic analysis and uniform algebras.

THEOREM 3.3. *The dual space of* $H_0^1(X)$ *is isomorphic to* BMO(X). *More precisely, if L is a bounded real linear functional on* $H_0^1(X)$, *then there is a real function* ϕ *in* BMO(X) *satisfying that*

$$L(\psi) = \int_X \operatorname{Re} \psi(x) \phi(x)\, dm(x) \tag{3.1}$$

for each ψ *in* $H_0^1(X)$, *where* ϕ *is unique up to additive constants. Moreover there are two positive constants* C_1 *and* C_2 *such that*

$$C_1 \|L\| \le \|\phi\|_{\mathrm{BMO}(X)} \le C_2 \|L\|. \tag{3.2}$$

In order to prove Theorem 3.3, we need a definition and two lemmas.

For a given f in $L^1(dm_1 \times dt/(1+t^2))$, we define

$$H_{Y \times \mathbf{R}} f(y,t) = \lim_{\epsilon \to +0} \frac{1}{\pi} \int_{\epsilon < |t-u| < 1/\epsilon} f(y,u) \left[\frac{1}{t-u} + \frac{u}{1+u^2} \right] du$$

for a.e. (y,t) in $Y\times\mathbf{R}$ just like the ergodic Hilbert transform H_X defined by (1.1). The existence of this limit follows easily from Fubini's theorem. Since Riesz's projection theorem holds for weak*-Dirichlet algebras, if θ lies in $L^\infty(X)$, then $H_X\theta$ lies in $L^1(X)$. Then we may see that $H_{Y\times\mathbf{R}}\theta^\#(y,t)-(H_X\theta)^\#(y,t)$ is only an integrable function of y. In other words, there is a function p in $L^1(Y)$ such that

$$(H_X\theta)^\#(y,t)=H_{Y\times\mathbf{R}}\theta^\#(y,t)+p(y) \tag{3.3}$$

for a.e. (y,t) in $Y\times\mathbf{R}$. This also implies that

$$\|(H_X\theta)^\#-H_{Y\times\mathbf{R}}\theta^\#\|_{\mathrm{BMO}(Y\times\mathbf{R})}=0.$$

The next lemma is an immediate consequence of the case concerning $H^1(dt)$, so the proof is omitted (cf. [6, Chapter VI, Section 4]).

LEMMA 3.4. *The dual space* $\mathcal{H}^1$ *is isomorphic to* $\mathrm{BMO}(Y\times\mathbf{R})$. *Consequently, if f lies in* $\mathrm{BMO}(Y\times\mathbf{R})$, *then there are functions g_1 and g_2 in $L^\infty(dm_1\times dt)$ and a function q in $L^1(Y)$ such that*

$$f(y,t)=g_1(y,t)+H_{Y\times\mathbf{R}}g_2(y,t)+q(y)$$

for a.e. (y,t) *in* $Y\times\mathbf{R}$.

From this fact, Lemma 2.1 enables us to characterize functions in $\mathrm{BMO}(Y\times\mathbf{R})$ which belong to $\mathrm{BMO}(X)$.

LEMMA 3.5. *If f belongs to* $\mathrm{BMO}(Y\times\mathbf{R})$, *then the following properties are equivalent;*

(i) *$f(y,t)-f\circ\sigma(y,t)$ is only a function of y, and*

(ii) *we may choose functions θ_1 and θ_2 in $L^\infty(X)$ and a function r in $L^1(Y)$ such that*

$$(\theta_1+H_X\theta_2)^\#(y,t)=f(y,t)+r(y) \tag{3.4}$$

for a.e. (y,t) *in* $Y\times\mathbf{R}$.

Proof. (ii) implies (i), so it suffices to show that (i) implies (ii). Suppose f satisfies (i), and let g_1, g_2, and q be as in Lemma 3.4. For $j=1,2$, we denote by $(h_j)_k$ the function defined by (2.8) replaced g with g_j. Let h_j and θ_j be as in Lemma 2.1. Notice that h_j and θ_j belong to $L^\infty(dm_1\times dt)$ and $L^\infty(X)$, respectively. It follows from (i) that

$$f(y,t)=(h_1)_k(y,t)+H_{Y\times\mathbf{R}}(h_2)_k(y,t)+q_k(y)$$

for some q_k in $L^1(Y)$. We may also see that

$$\int_Y\int_{-\infty}^{\infty}\mathrm{Im}\,g(y,t)(h_2)_k(y,t)\,dm_1(y)\,dt$$

$$=-\int_Y\int_{-\infty}^{\infty}\mathrm{Re}\,g(y,t)H_{Y\times\mathbf{R}}(h_2)_k(y,t)\,dm_1(y)\,dt$$

for each g in $\mathcal{H}^\infty$. Therefore, by Lemma 3.4, we obtain that

$$f(y,t)=h_1(y,t)+H_{Y\times\mathbf{R}}h_2(y,t)+q_0(y)$$

for some q_0 in $L^1(Y)$. Since $h_j=\theta_j^\#$, it follows from (3.3) that f satisfies (3.4). □

Proof of Theorem 3.3. Let L be a bounded real linear functional on $H_0^1(X)$. By the Hahn–Banach theorem, there exist a constant C and real functions θ_1 and θ_2 in $L^\infty(X)$ such that $\|\theta_1\|_\infty+\|\theta_2\|_\infty\le C\|L\|$, and

$$L(\psi)=\int_X \operatorname{Re}\psi(x)(\theta_1(x)+H_X\theta_2(x))\,dm(x)$$

for each ψ in $H_0^1(X)$. On the other hand, it follows from (2.1) that there is a constant C' such that

$$\|H_X\theta\|_{\mathrm{BMO}(X)}\le C'\|\theta\|_\infty \quad \text{and} \quad \|\theta\|_{\mathrm{BMO}(X)}\le\|\theta\|_\infty$$

hold for each θ in $L^\infty(X)$ as in the case of BMO($\mathbf{R}$) (cf. [6, Chapter VI]). Thus $\|\theta_1+H_X\theta_2\|_{\mathrm{BMO}(X)}\le C_2\|L\|$ for some absolute constant C_2. If ϕ is a BMO-function on X, then $\phi^\#$ belongs to BMO($Y\times\mathbf{R}$). Since $\phi^\#$ satisfies the property of (i) in Lemma 3.5, we may find real functions θ_1 and θ_2 in $L^\infty(X)$ such that $\phi=\theta_1+H_X\theta_2$. If L is the functional defined by (3.1) with this ϕ, then L is bounded. Notice that, for ψ_1 and ψ_2 in $L^\infty(X)$, $\psi_1+H_X\psi_2$ induces zero functional on $H_0^1(X)$ if and only if $\psi_1+H_X\psi_2$ is constant. Therefore by the above argument, we have that $\|\phi\|_{\mathrm{BMO}(X)}\le C_2\|L\|$. Thus the closed graph theorem assures that there exists an absolute constant C_1 satisfying the inequality (3.2). This completes the proof. □

4. Unbounded VMO-functions on strictly ergodic systems. For the case of ergodic dynamical systems (Y,T,m_1), analogous definitions are also given in [3]. A function p in $L^1(Y)$ is said to belong to BMO(Y) if the norm

$$\|p\|_{\mathrm{BMO}(Y)}=\sup_n \operatorname{ess\,sup}_y \frac{1}{2n+1}\sum_{j=-n}^{n}|p(T^jy)-M_np(y)|$$

is bounded, where

$$M_np(y)=\frac{1}{2n+1}\sum_{j=-n}^{n}p(T^jy).$$

The *ergodic Hilbert transform* H_Yp of a function p in $L^1(Y)$ is defined by the formula

$$H_Yp(y)=\frac{1}{\pi}\sum_{j\ne0}p(T^{-j}y)\frac{1}{j}.$$

In this section, we always assume that T denotes a homeomorphism on compact Hausdorff space Y for which the topological dynamical system (Y,T) is strictly ergodic. The unique invariant probability measure on Y is also denoted by m_1. Thus, because of a theorem of Sarason (see [6, Chapter VI, Theorem 5.1] and [13]), the class VMO(Y) of all VMO-functions is defined to be the closure of $C(Y)$ in BMO(Y), where $C(Y)$ is the space of all continuous functions on Y.

In [12] Petersen showed, among other things, that there exist unbounded VMO-functions on Y under the condition that (Y,T) is a strictly ergodic subshift. We now show that this holds on every strictly ergodic system (Y,T).

To apply the property of Φ defined in (2.10), we consider a continuous flow built under the constant one. Let X be the quotient topological space $Y\times[0,1]$ by identifying $(y,1)$ and $(Ty,0)$ for all y in Y. We then define T_t by $T_t(y,s)=(T^{[s+t]}y, s+t-[s+t])$ for each (y,s) in $X=Y\times[0,1)$, where $[t]$ denotes the largest integer not exceeding t (compare this with the definition (1.3)). Then $(X,\{T_t\}_{t\in\mathbf{R}})$ becomes a strictly ergodic flow and the unique invariant probability measure m is the restriction of $dm_1\times dt$ to X (cf. [14]). Notice that if E is an open set in Y, then $m_1(E)=m(E\times[0,1))$ is positive.

THEOREM 4.1. *We may choose ϕ in $H_0^1(X)$ such that*

(4.1) Re ϕ *is continuous on X, and*

(4.2) *if we set $p(y)=\int_0^1 \operatorname{Im}\phi(y,s)\,ds$, then $p(y)$ is an unbounded* VMO-*function on Y, and so it is an unbounded* BMO-*function on Y.*

In order to prove Theorem 4.1, we need the following.

LEMMA 4.2. *There are a sequence $\{\phi_n\}$ of continuous functions in $H_0^\infty(X)$ and sequences $\{E_n\}$ and $\{F_n\}$ of open sets in Y which satisfy*

(4.3) $|\operatorname{Re}\phi_n(x)|<2^{-n}$ *for each x in X,*

(4.4) *the sequence $\{E_n\}$ is decreasing and* $\operatorname{Im}\phi_n(y,s)>2^n$ *for each (y,s) in $E_n\times[0,1)$, and*

(4.5) $|\operatorname{Im}\phi_n(y,s)|<2^{-n}$ *for each (y,s) in $(Y\backslash F_n)\times[0,1)$, and $m_1(F_n)<\frac{1}{3}m_1(E_{n-1})$ for $n\geq 2$.*

Proof. Assume by induction, we have found $\{\phi_1,\phi_2,\dots,\phi_{n-1}\}$, $\{E_1,E_2,\dots,E_{n-1}\}$ and $\{F_1,F_2,\dots,F_{n-1}\}$ for which the properties (4.3), (4.4), and (4.5) hold. By the definition of Hilbert transform on $\mathbf{R}$, we may choose a continuous analytic function f on $\mathbf{R}$ such that $\lim_{t\to\infty}|f(t)|t^2=0$, $\operatorname{Im}f(t)>2^n+1$ on $[0,1)$, and $|\operatorname{Re}f(t)|<2^{-(n+1)}$ on $\mathbf{R}$. Let k be an integer such that $\sum_{|j|>k}|f(s+j)|<2^{-(n+1)}$ for each s in $[0,1)$. We fix a point y_n in E_{n-1}. Then there is an open neighborhood W of y_n in E_{n-1} such that $\{T^jW; j=0,\pm1,\dots,\pm k\}$ are disjoint from each other, and $\sum_{j=-k}^{k}m_1(T^jW)<\frac{1}{3}m_1(E_{n-1})$. We set $F_n=\bigcup_{j=-k}^{k}T^jW$. Let q be a function in $C(Y)$ satisfying that $0\leq q\leq 1$, q is supported on W, and $q=1$ on some open neighborhood E_n of y_n in E_{n-1}. We define $g(y,t)=q(y)f(t)$ for each (y,t) in $Y\times\mathbf{R}$. Then since g lies in $\mathcal{H}^\infty$, $\phi_n=\Phi(g)$ lies in $H_0^\infty(X)$. It is easy to see that above ϕ_n, E_n, and F_n have the desired properties (4.3), (4.4), and (4.5). □

Proof of Theorem 4.1. Let $\{\phi_n\}$, $\{E_n\}$, and $\{F_n\}$ be as in Lemma 4.2. Since F_j contains E_j for all j, (4.5) implies that $m_1(F_{j+n})<3^{-j}m_1(E_n)$. So if we set $V_n=E_n\backslash(\bigcup_{j=1}^{\infty}F_{j+n})$, then $m_1(V_n)$ is positive. Notice that $\operatorname{Im}\phi_j(y,s)>2^j$ for each (y,s) in $V_n\times[0,1)$ and for $j=1,2,\dots,n$. We define that $\phi=\sum_{n=1}^{\infty}\phi_n$. Then it follows from (4.3) that ϕ belongs to $H_0^1(X)$, and Re ϕ satisfies (4.1). On the other hand, $|\operatorname{Im}\phi_{j+n}(y,s)|<2^{-(j+n)}$ for each (y,s) in $V_n\times[0,1)$ and for each j. So we have that $\operatorname{Im}\phi(y,s)>1+2^2+2^3+\cdots+2^n$ for each (y,s) in $V_n\times[0,1)$. This shows that $p(y)$ defined in (4.2) is unbounded. Let us show that $p(y)$ is a VMO-function on Y. Recall that there is a constant C such that $\|H_X\psi\|_{BMO(X)} \le C\|\psi\|_\infty$ for each ψ in $L^\infty(X)$. Furthermore, it follows easily from the definition (2.1) that

$$\left\|\int_0^1 \psi(y,s)\,ds\right\|_{BMO(Y)} \le \|\psi\|_{BMO(X)}$$

for each ψ in $BMO(X)$. Thus we have

$$\begin{aligned}\left\|p(y)-\int_0^1 \sum_{n=1}^k \operatorname{Im}\phi_n(y,s)\,ds\right\|_{BMO(Y)} &\le \left\|\operatorname{Im}\phi-\sum_{n=1}^k \operatorname{Im}\phi_n\right\|_{BMO(X)}\\ &\le C\left\|\operatorname{Re}\phi-\sum_{n=1}^k \operatorname{Re}\phi_n\right\|_\infty \to 0,\end{aligned}$$

as $k\to\infty$. Since the function of y, $\int_0^1 \sum_{n=1}^k \operatorname{Im}\phi_n(y,s)\,ds$, lies in $C(Y)$, this implies that the unbounded function $p(y)$ belongs to $VMO(Y)$. This completes the proof. □

Theorem 4.1 can be used to give an additional information about the difference between ergodic settings and classical ones.

COROLLARY 4.3. *There exists a continuous function θ on X such that*
(4.6) *the function of t, $\theta(T_tx)$, is infinitely differentiable for each x in X, and*
(4.7) *the ergodic Hilbert transform $H_X\theta$ of θ is unbounded.*

Proof. Let ϕ be the function in Theorem 4.1, and let g be an infinitely differentiable function on $\mathbf{R}$ such that g is supported in $(\frac{1}{4},\frac{3}{4})$ and has integral one. Then we put

$$\theta(y,t)=g(t)\cdot\int_0^1 \operatorname{Re}\phi(y,s)\,ds$$

for each (y,t) in $X=Y\times[0,1)$. It is easy to see that θ has the property (4.6). We extend $\theta-\operatorname{Re}\phi$ on X to $Y\times\mathbf{R}$ as follows:

$$h(y,t)=\begin{cases}\theta(y,t)-\operatorname{Re}\phi(y,t), & \text{for } (y,t) \text{ in } Y\times[0,1),\\ 0, & \text{otherwise.}\end{cases}$$

Observe that h lies in $L^1\cap L^\infty(dm_1\times dt)$ and that $\int_{-\infty}^{\infty} h(y,t)\,dt=0$. Hence we may easily find a function f in $\mathcal{H}^1$ such that $\operatorname{Re} f=h$ by the definition of Hilbert transform on R. Then the function $\Phi(f)$ in $H_0^1(X)$ satisfies $\operatorname{Re}\Phi(f)=\theta-\operatorname{Re}\phi$. Since $\operatorname{Re} f$ is supported in $Y\times[0,1)$ and bounded, we have that

$$\sup_y|\operatorname{Im} f(y,t)|=O(t^{-2}),\quad \text{as } t\to\infty.$$

From this fact, it is easy to see that the function of y, $\int_0^1 \operatorname{Im}\Phi(f)(y,s)\,ds$, is bounded. So the function of y, $\int_0^1 H_X\theta(y,s)\,ds$, cannot be bounded by (4.2). Thus $H_X\theta$ satisfies the desired property (4.7). □

5. Remarks. (a) We do not know whether Theorem 3.1 holds when $p=\infty$. However, just as in the proof of Theorem 3.1, we see that $\Phi(\mathcal{H}^\infty)$ is weak*-dense in $H_0^\infty(X)$. With the notion of automorphic extension defined by (2.4), this fact enables us to provide another proof to one of the most fundamental results in our direction (cf. [10, Theorem I] and [16]).

PROPOSITION 5.1. *$H^\infty(X)$ is a weak*-Dirichlet algebra under pointwise multiplication.*

Proof. If ϕ lies in $H^\infty(X)$, then $\phi^\# f$ lies in $\mathcal{H}^\infty$ for all f in $\mathcal{H}^\infty$. So we have $\int_X \phi\Phi(f)\,dm=0$. Since $\Phi(\mathcal{H}^\infty)$ is weak*-dense in $H_0^\infty(X)$, this implies that m is multiplicative on $H^\infty(X)$. On the other hand, if ϕ in $L^1(X)$ is orthogonal to $H^\infty(X)+\bar{H}^\infty(X)$, then $\phi^\#$ is orthogonal to $\mathcal{H}^\infty+\bar{\mathcal{H}}^\infty$. Therefore it follows from the ergodicity that ϕ must be a constant, so it is null. Thus we obtain that $H^\infty(X)+\bar{H}^\infty(X)$ is weak*-dense in $L^\infty(X)$. □

(b) We denote by $H^p(d\theta/2\pi)$, $1\le p\le\infty$, the classical Hardy space on the unit circle, and let $H_0^p(d\theta/2\pi)$ be as before. By considering the case where Y is one point and F is equal to 2π, Theorem 3.1 implies a relation between a certain subspace of $H^1(dt)$ and $H_0^p(d\theta/2\pi)$, $1\le p<\infty$. Furthermore, by a normal family argument, we obtain the following.

PROPOSITION 5.2. *If ϕ belongs to $H_0^1(d\theta/2\pi)$, then there exists a function f in $H^1(dt)$ with $\|\phi\|_1=\|f\|_1$ satisfying that*

$$\phi(s)=\sum_{j=-\infty}^{\infty} f(s+2\pi j)$$

for a.e. s in $[0,2\pi)$.

(c) It is not known whether there exists a function ϕ in $H_0^1(X)$ for which $H^\infty(X)\phi$ is dense in $H_0^1(X)$. This problem is old and very attractive (cf. [8, Chapter 5, Section 4]). Several equivalent forms were given by Gamelin [5, Chapter VII, Theorem 7.8]. If, however, a single generator ϕ existed, it would be represented as $\Phi(f)$ for some f in $\mathcal{H}^1$. This fact may be useful to attack the problem.

(d) Let p be a positive function in $L^1(Y)$ satisfying that $p(y)\log^+ p(y)$ does not lie in $L^1(Y)$, where $\log^+ p(y)=\log\max(p(y),1)$. Choose a positive constant c such that $F>2c$ on Y, and define a function $h(y,t)$ on $Y\times\mathbf{R}$ as follows

$$h(y,t)=\begin{cases} -p(y), & \text{for } (y,t) \text{ in } Y\times[c,2c), \\ p(y), & \text{for } (y,t) \text{ in } Y\times[0,c), \text{ and} \\ 0\ , & \text{otherwise.}\end{cases} \tag{5.1}$$

It follows easily from the definition of Hilbert transforms on $\mathbf{R}$ that there is a function f in $\mathcal{H}^1$ such that $\operatorname{Re} f=h$, so $\Phi(f)$ belongs to $H_0^1(X)$. Notice that $\operatorname{Re}\Phi(f)=h$. Let $L(X)\log^+L(X)$ be the space of all real functions ϕ for which $|\phi|\log^+|\phi|$ lies in $L^1(X)$. By (5.1), it is easy to see that $\operatorname{Re}\Phi(f)$ does not belong to $L(X)\log^+L(X)$. On the other hand, we know that if θ lies in $L(X)\log^+L(X)$, then $H_X\theta$ lies in $L^1(X)$ (cf. [7]). Thus we have the following.

PROPOSITION 5.3. Re $H^1(X)$ *is strictly larger than* $L(X)\log^+L(X)$.

By considering the case where $F=1$, a similar argument enables us to show that Proposition 5.3 also holds in the discrete setting (cf. [9, Section 3]).

PROPOSITION 5.3′. *There is a function* p *in* $L^1(Y)$ *which does not lie in* $L(Y)\log^+L(Y)$ *but* H_Yp *lies in* $L^1(Y)$.

(e) Let Y and X be as in Section 4. We remark how to relate BMO-functions on X to those on Y. Since the proof is not difficult, we omit it (cf. [3, Proof of Theorem 3]).

PROPOSITION 5.4. *If* p *belongs to* BMO(Y), *then there exists a function* ϕ *in* BMO(X) *for which*

$$p(y)=\int_0^1 \phi(y,s)\,ds \tag{5.2}$$

for a.e. y *in* Y. *Conversely,* (5.2) *defines a function in* BMO(Y) *for each* ϕ *in* BMO(X).

(f) Let Y and T be as in Section 4. Suppose that F is a positive continuous function on Y. By identifying $(y,F(y))$ and $(Ty,0)$ topologically for each y in Y, the region X of $Y\times\mathbf{R}$ under the graph F becomes a compact Hausdorff space, and the one parameter group $\{T_t\}_{t\in\mathbf{R}}$ in (1.3) defines a continuous flow $(X,\{T_t\}_{t\in\mathbf{R}})$. We may extend Corollary 4.3 to this case. It is slightly more complicated but involves no new ideas. It would be interesting to determine what kind of continuous flows can be represented by the ones defined above, because we do not know whether there exists a minimal flow on S^3, where S^3 denotes the three dimensional sphere. If there were a minimal flow on S^3, it could not be represented by a continuous flow built under a continuous function. On the other hand, every minimal flow has an analogous representation (cf. [14]).

REFERENCES

1. W. Ambrose, *Representation of ergodic flows,* Ann. of Math. (2) 42 (1941), 723–739.
2. A. Calderón, *Ergodic theory and translation invariant operators,* Proc. Nat. Acad. Sci. U.S.A. 59 (1968), 349–353.
3. R. Coifman and G. Weiss, *Maximal functions and H^p spaces defined by ergodic transformations,* Proc. Nat. Acad. Sci. U.S.A. 70 (1973), 1761–1763.
4. N. Dunford and J. Schwartz, *Linear operators, Part I,* Interscience, New York, 1958.
5. T. Gamelin, *Uniform algebras,* Prentice-Hall, Englewood Cliffs, N.J., 1969.
6. J. Garnett, *Bounded analytic functions,* Academic Press, New York, 1981.
7. H. Helson, *Conjugate series in several variables,* Pacific J. Math. 9 (1959), 513–523.
8. ———, *Analyticity on compact abelian groups.* Algebras in analysis (Birmingham, 1973), 1–62, Academic Press, London, 1975.

9. R. Jones, *The ergodic maximal function with cancellation,* Ann. Probab. 4 (1976), 91–97.
10. P. Muhly, *Function algebras and flows,* Acta Sci. Math. (Szeged) 35 (1973), 111–121.
11. ———, *Ergodic Hardy spaces and duality,* Michigan Math. J. 25 (1978), 317–323.
12. K. Petersen, *A construction of ergodic BMO functions,* Proc. Amer. Math. Soc. 79 (1980), 549–555.
13. D. Sarason, *Functions of vanishing mean oscillation,* Trans. Amer. Math. Soc. 207 (1975), 391–405.
14. J.-I. Tanaka, *Quasi-invariant measures and maximal algebras on minimal flows,* Michigan Math. J. 29 (1982), 199–211.
15. ———, *Cocycles and local product decomposition,* Trans. Amer. Math. Soc. 278 (1983), 565–572.
16. G. Weiss, *Weak*-Dirichlet algebras induced by the ergodic Hilbert transform.* L'analyse harmonique dans le domaine complexe (Montpellier, 1972), 20–27, Lecture Notes in Math., 336, 20–27, Springer, Berlin, 1973.

Department of Mathematics
University of California, Berkeley
California 94720

and

Department of Mathematics
Tsuru University
Tsuru City, Yamanashi-ken 402
Japan