THE EXTREMAL PSH FOR THE COMPLEMENT OF CONVEX, SYMMETRIC SUBSETS OF $\mathbf{R}^N$

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Introduction. For a compact subset E of $\mathbf{C}^N$, we define the extremal function

$$\Phi_E(z)=\sup\{|p(z)|^{1/\deg p}\}.$$

The supremum is taken over all polynomials in N complex variables with $\|P\|_E \le 1$.

This function has been investigated and used in connection with polynomial approximation in $\mathbf{C}^N$ by Siciak [4]. Zahariuta [6] and with a different method Siciak [5] have shown that

$$\log \Phi_E(z)=\sup\{u(z)\}; \tag{0.1}$$

here the supremum is taken over all plurisubharmonic functions v with $v(z)\le \log(|z|+1)+C$ and $v(z)\le 0$ on E.

Similar functions have been used in connection with potential theory in $\mathbf{C}^N$ and also when investigating the complex Monge–Ampére equation; see for example Bedford [1, 2] and Bedford and Taylor [3] where further references can be found.

In this note we give a fairly explicit formula for Φ_E when E is convex, symmetric with respect to 0 and contained in $\mathbf{R}^N\subseteq \mathbf{C}^N$. This calculation will also give us a complex foliation of $\mathbf{C}^N$ such that Φ_E is harmonic on each leaf except at the intersections between the leaves and E.

Definitions and formulation of the main result. We will always consider $\mathbf{R}^N$ as a subset of $\mathbf{C}^N$. When we talk about the topology of a subset of $\mathbf{R}^N$ we usually refer to the $\mathbf{R}^N$-topology.

Let S_N denote the unit sphere in $\mathbf{R}^N$, and for $\xi\in S_N$, $z\in \mathbf{C}^N$ define $\xi\cdot z= \xi_1 z_1+\cdots+\xi_N z_N$.

For a convex symmetric set $E\subseteq \mathbf{R}^N$ with nonempty interior (symmetric means symmetric with respect to 0, i.e. $E=-E$), we have a representation

$$E=\{z\in \mathbf{C}^N : a(\xi)\xi\cdot z\in[-1,1] \text{ for all } \xi\in S_N\}. \tag{1.1}$$

Here $a(\xi)$ is a continuous function on S_N which can be chosen as the reciprocal of half the width of E in the direction ξ. The function $a(\xi)$ is unique if E has a tangent plane at every boundary point. We now define

$$F(\xi,z)=a(\xi)\xi\cdot z+\sqrt{(a(\xi)\xi\cdot z)^2-1}. \tag{1.2}$$

Here we always choose the sign of the root function to make $|F|\ge 1$. This choice makes, for a fixed ξ, $F(\xi,z)$ into a holomorphic mapping of $\{z: a(\xi)\xi\cdot z\notin[-1,1]\}$ onto the complement of the unit disc in $\mathbf{C}$, with $|F(\xi,z)|\le C(|z|+1)$.

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Our representation of E (1.1) shows that for every $z \notin E$ there is a ξ such that $a(\xi)\xi \cdot z \notin [-1,1]$ and therefore $|F(z,\xi)| > 1$.

We are now ready to define our candidate for Φ_E. Let

$$\Psi_E(z) = \sup_{\xi \in S_N} |F(\xi, z)|. \tag{1.3}$$

The preceding discussion shows that $\log \Psi_E(z)$ is plurisubharmonic, $\log \Psi_E(z) \le \log(|z|+1)+C$ and $\log \Psi_E(z) \le 0$ if $z \in E$. This shows that $\Psi_E(z) \le \Phi_E(z)$. We also know that $\log \Psi_E(z) > 0$ if $z \notin E$. Our main result can now be stated.

THEOREM 1. *If E is a convex symmetric subset of $\mathbf{R}^N$ with nonempty interior then*

$$\Psi_E(z) = \Phi_E(z).$$

The function $\log|F(\xi,z)|$ for fixed ξ is the projection $\mathbf{C}^N \to \mathbf{C}$ in the ξ direction followed by the largest subharmonic function in $\mathbf{C}$ which is 0 on the projection of E and grows not faster than $\log|z|$ as $|z| \to \infty$. We now obtain the following as a corollary.

THEOREM 2. *The set $E_R = \{z \in \mathbf{C}^N : |\Phi_E(z)| \le R\}$ is convex if E is a convex symmetric set in $\mathbf{R}^N$ with nonempty interior.*

Proof.

$$E_R = \{z \in \mathbf{C}^N : \Psi_E(z) \le R\} = \bigcap_{\xi \in S_N} \{z \in \mathbf{C}^N : |F(\xi,z)| \le R\}$$

But we know that $\{z \in \mathbf{C} : |z+\sqrt{z^2-1}| \le R\}$ is convex (it is an ellipse) therefore $\{z \in \mathbf{C}^N : |F(\xi,z)| \le R\}$ and also E_R must be convex.

If E has empty interior (in $\mathbf{R}^N$) then after a suitable rotation in $\mathbf{R}^N$, E is a subset of $\mathbf{R}^{N-1}$ so we can use our methods in $\mathbf{C}^{N-1}$ and in $\mathbf{C}^N \setminus \mathbf{C}^{N-1}$ we have $\Phi_E(z) = +\infty$. This is seen from the functions $P_k(z) = k \cdot z_n$. We have $|P_k(z)| = 0$ on $\mathbf{C}^{N-1}$ and $\Phi_E(z) \ge |P_k(z)| \to \infty$ on $\mathbf{C}^N \setminus \mathbf{C}^{N-1}$.

The case when $a(\xi)$ is differentiable. We first prove the theorem in the case when $a(\xi)$ is differentiable. In this case $F(\xi,z)$ is a differentiable function of ξ if $a(\xi)\xi \cdot z \notin [-1,1]$, so we can find the maximum of $|F(\xi,z)|$ for a fixed $z \notin E$ by differentiating with respect to ξ.

Let $z_0 \in E$ be fixed. Then the supremum in (1.3) is attained for some ξ_0. Now fix a basis $\{e_k\}_{k=1}^{N-1}$ for the tangent space of S_N at ξ_0. Then we have for the derivatives

$$\frac{\partial}{\partial e_k} \log|F(\xi,z_0)|_{\xi=\xi_0} = \operatorname{Re} \frac{\partial}{\partial e_k} \log|F(\xi,z_0)|_{\xi=\xi_0} = 0 \quad \text{for } k = 1, \dots, N-1.$$

If we use the definition of $F(\xi,z)$ (1.2) and note that $(\partial\xi/\partial e_k)|_{\xi=\xi_0} = e_k$, we find (for $k = 1, \dots, N-1$) that

$$\frac{\partial a}{\partial e_k}(\xi_0)\xi_0 \cdot z_0 + a(\xi_0) e_k \cdot z_0 = i\lambda_k \sqrt{(a(\xi_0)\xi_0 \cdot z_0)^2 - 1}, \tag{2.1}$$

where $\lambda = (\lambda_1, \dots, \lambda_{N-1})$ is a vector in $\mathbf{R}^{N-1}$. For ξ_0 and λ fixed and an arbitrary z these equations define a 1-complex dimensional set in $\mathbf{C}^N$. After a change of variables, we can assume that ξ_0 is the basis vector in the z_N direction and e_k is the basis vector in the z_k direction. Then we obtain the equations

$$(2.2)\quad z_k=-\frac{1}{a(\xi_0)}\left(\frac{\partial a}{\partial e_k}(\xi_0)z_N-i\lambda_k\sqrt{(a(\xi_0)z_N)^2-1}\right);\quad k=1,\dots,N-1.$$

This set we parametrize by letting

$$a(\xi_0)z_N=\frac{1}{2}\left(\zeta+\frac{1}{\zeta}\right)\quad\text{with } |\zeta|\geq 1.$$

If $|\zeta|=1$, then $a(\xi_0)z_N\in[-1,1]$ and $|F(\xi_0,z)|=1$.

We know that $|F(\xi_0,z_0)|>1$ so z_0 must correspond to a ζ_0 with $|\zeta_0|>1$. If we now remember the choice of sign for the root function in (1.2) we see that

$$\sqrt{(a(\xi_0)z_N)^2-1}=\frac{1}{2}\left(\zeta-\frac{1}{\zeta}\right).$$

Thus we have the solution set of (2.2) given as

$$(2.3)\quad \begin{cases} z_k=-\dfrac{1}{2a(\xi_0)}\left(\left(\dfrac{\partial a}{\partial e_k}(\xi_0)-i\lambda_k\right)\zeta+\left(\dfrac{\partial a}{\partial e_k}(\xi_0)+i\lambda_k\right)\dfrac{1}{\zeta}\right), & k=1,\dots,N-1\\ z_N=\dfrac{1}{2a(\xi_0)}\left(\zeta+\dfrac{1}{\zeta}\right), & \text{with } |\zeta|\geq 1. \end{cases}$$

This is a complex manifold with boundary which we call Σ_{λ,ξ_0}. We must remember that this representation is valid only after a suitable change of coordinates in $\mathbf{C}^N$.

From our construction we know that at $z=z_0$, $|F(\xi,z)|$ attains its maximum as a function of ξ for $\xi=\xi_0$. Our next step is to show that this is true for all $z\in\Sigma_{\lambda,\xi_0}$. We need the following result.

LEMMA. *If w_1 and w_2 are complex numbers then*

$$(2.4)\qquad |w_1+\sqrt{w_1^2-1}\,|=|w_2+\sqrt{w_2^2-1}\,|$$

if and only if there is a $\theta\in R$ so that

$$w_2=\cos\theta\cdot w_1+i\sin\theta\sqrt{w_1^2-1}.$$

The square root is always chosen to make $|w+\sqrt{w^2-1}\,|\geq 1$.

Proof. Let $w_1=\frac{1}{2}(\zeta_1+1/\zeta_1)$ and $w_2=\frac{1}{2}(\zeta_2+1/\zeta_2)$ with $|\zeta_1|$ and $|\zeta_2|\geq 1$. Formula (2.4) now becomes $|\zeta_1|=|\zeta_2|$ or equivalently $\zeta_2=e^{i\theta}\zeta_1$. If we now insert this in the formulas for w_1 and w_2 we get precisely $w_2=\cos\theta\cdot w_1+i\sin\theta\sqrt{w_1^2-1}$.

Now take a $\xi_1\neq\pm\xi_0$. Since ξ_0 and $\{e_k\}$, $k=1,\dots,N-1$, form a basis in $\mathbf{R}^N$ we can find real constants $c_0,\dots,c_{N-1}$ so that

$$\xi_1=c_0\xi_0+\sum_{k=1}^{N-1}c_ke_k.$$

This can be rewritten as

$$\xi_1=\left(c_0-\sum_{k=1}^{N-1}\frac{c_k}{a(\xi_0)}\cdot\frac{\partial a}{\partial e_k}(\xi_0)\right)\xi_0+\sum_{k=1}^{N-1}\frac{c_k}{a(\xi_0)}\left(\frac{\partial a}{\partial e_k}(\xi_0)\xi_0+a(\xi_0)e_k\right).$$

For every $z\in\mathbf{C}^N$ we thus have

$$a(\xi_1)\xi_1\cdot z=d_0a(\xi_0)z_N+\sum_{k=1}^{N-1}d_k\left(\frac{\partial a}{\partial e_k}(\xi_0)z_N+a(\xi_0)z_k\right).$$

(Here we have used $\xi_0\cdot z=z_N$ and $e_k\cdot z=z_k$.) $d_0,\dots,d_{N-1}$ are real constants independent of z. If $z\in\Sigma_{\lambda,\xi_0}$ then we can use (2.1) to get

$$a(\xi_1)\xi_1\cdot z=d_0a(\xi_0)z_N+i\left(\sum_{k=1}^{N-1}d_k\lambda_k\right)\sqrt{(a(\xi_0)z_N)^2-1}.$$

If for some $z\in\Sigma_{\lambda,\xi_0}$ but not in E we have $|F(\xi_0,z)|=|F(\xi_1,z)|$, then our lemma shows that there is a θ so that

$$a(\xi_1)\xi_1\cdot z=\cos\theta a(\xi_0)z_N+i\sin\theta\sqrt{(a(\xi_0)z_N)^2-1}.$$

But $a(\xi_0)z_N\notin[-1,1]$, and a simple calculation shows that $a(\xi_0)z_N$ and $i\sqrt{(a(\xi_0)z_N)^2-1}$ are $\mathbf{R}$ linearly independent, so we must have $d_0=\cos\theta$ and $\sum d_k\lambda_k=\sin\theta$. Now we can use the lemma again and conclude that

$$|F(\xi_0,z)|=|F(\xi_1,z)|\quad\text{for all } z\in\Sigma_{\lambda,\xi_0}.$$

Since Σ_{λ,ξ_0} is connected we see that $F(\xi,z)$ has its maximum as a function of ξ at $\xi=\xi_0$ for all $z\in\Sigma_{\lambda,\xi_0}$, or in other words $\Psi_E(z)=|F(\xi_0,z)|$ on Σ_{λ,ξ_0}.

We now use the parametrization (2.3) of Σ_{λ,ξ_0} and find that $\Psi_E(z)=|F(\xi_0,z)|=|\zeta|$ on Σ_{λ,ξ_0}. When $|\zeta|$ is large then we see from (2.3) that $|\zeta|\sim C|z|$, so $\log\Psi_E(z)$ is harmonic on $\Sigma_{\lambda,\xi_0}\setminus E$, $\log\Psi_E(z)=0$ on $\Sigma_{\lambda,\xi_0}\cap E$, and $\log\Psi_E(z)\sim\log|z|+C$ as $|z|$ goes to ∞ on Σ_{λ,ξ_0}. We now compare this to $\log\Phi_E(z)$. This is a plurisubharmonic function in $\mathbf{C}^N$ so its restriction to Σ_{λ,ξ_0} must be subharmonic, and from (0.1) we see that it has the same behaviour when z goes to infinity and as z approaches E as does Ψ_E. Thus on Σ_{λ,ξ_0}, $\log\Phi_E(z)-\log\Psi_E(z)$ is a subharmonic function of ζ which is 0 for $|\zeta|=1$ and is bounded as $|\zeta|$ goes to ∞. Thus $\log\Phi_E(z)\le\log\Psi_E(z)$ on Σ_{λ,ξ_0} and since z_0 was arbitrary this is true in all of $\mathbf{C}^N$ and the theorem is proved in the case of a differentiable function $a(\xi)$.

The nondifferentiable case. If $a(\xi)$ cannot be chosen as a differentiable function, then take a decreasing sequence $a_n(\xi)$ of differentiable functions converging uniformly to $a(\xi)$. Let

$$E_n=\{a:a_n(\xi)\xi\cdot z\in[-1,1]\text{ for all }\xi\in S_N\}.$$

Since $a_n\searrow a$ we see that $E_n\nearrow E$ so we must have, for all z,

$$\Psi_E(z)\le\Phi_E(z)\le\Phi_{E_n}=\Psi_{E_n}(z).$$

The function $\Psi_E(z)$ is continuous as a function of E or equivalently $a(\xi)$, so

$$\Psi_{E_n}(z)\searrow\Psi_E(z)\quad\text{as } n\to\infty.$$

Therefore

$$\Phi_E(z) = \Psi_E(z) \quad \text{for all } z \in \mathbf{C}^N$$

and the proof is complete. □

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