ANALYTIC FUNCTIONS OF FINITE VALENCE, WITH APPLICATIONS TO TOEPLITZ OPERATORS

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Dedicated to George Piranian on the occasion of his retirement

1. Introduction. This paper concerns the extent to which the valence of a function analytic on the unit disc determines the form of that function. If f is analytic on $\mathcal{U}=\{|z|<1\}$ and $w \in \mathbf{C}$, then the valence of f at w, denoted $v_f(w)$, is the number of solutions $z \in \mathcal{U}$ of $f(z)=w$, counting multiplicities. In [2], Baker, Deddens, and Ullman show that if f is an entire function, and if k is the smallest nonzero value of v_f, then $f(z)=h(z^k)$ for some entire function h. They ask whether the appropriate analogue of this result holds for functions which are not entire, with the role of z^k played by a k-fold Blaschke product. We construct an example showing that the answer is no.

Our main effort concerns the study of pairs of functions whose valences are related. We prove a conjecture of Lee Rubel concerning entire functions complementing the Baker, Deddens, and Ullman result and show that it, too, fails if the functions are not entire. Given functions f and g with identical, finite valence functions, we investigate two structural relationships which may hold between them: One concerns the existence of a homeomorphism of the unit circle $\mathbf{T}$ which transforms the boundary values of f to those of g. The other concerns the existence of a common function h from which both f and g are obtained by composition. We show that the second relationship holds if and only if the first holds with a distinguished type of homeomorphism.

The paper concludes with applications of our results to the study of Toeplitz operators. A new condition is added to those of Carl Cowen [5] on similarity of analytic Toeplitz operators. This does not extend to rational Toeplitz operators; indeed, it leads to an example contradicting some published results concerning their similarity. A closer look at this example suggests the possible relevance of ideas used in the analytic case to a correct formulation. We also point out other implications of our work in the study of Toeplitz operators, including an answer to a question of Thomson [8] on commutants.

Throughout the paper we mention open questions which our work suggests. A word concerning terminology: The functions we study are defined on the unit disc $\mathcal{U}$. To say that a function f is entire, for instance, means that f is the restriction to $\mathcal{U}$ of an entire function.

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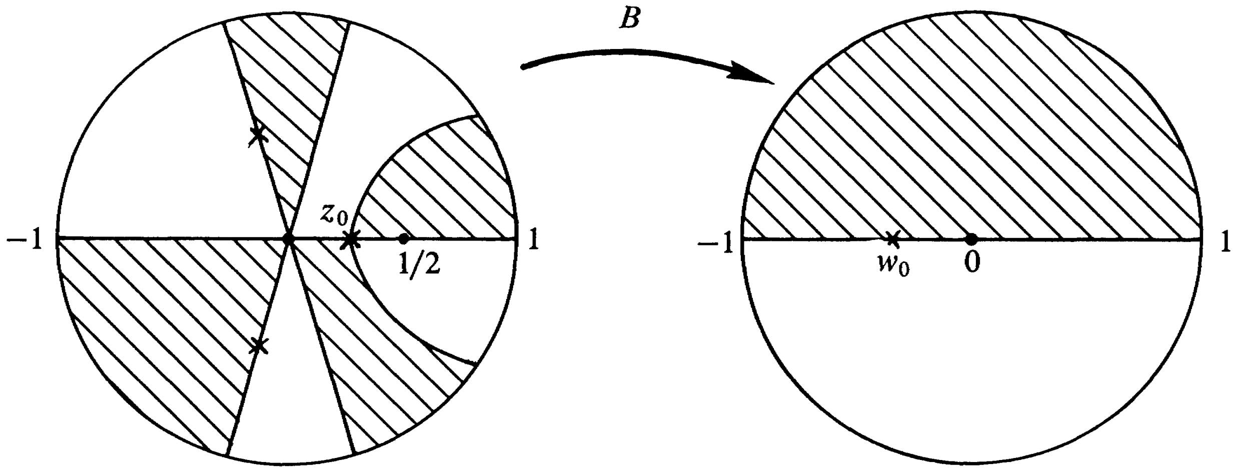


Figure 1

2. A central role in this work is played by the function F which is constructed in this section. It is a modification of the finite Blaschke product

$$B(z)=z^3(2z-1)/(2-z), \quad z\in\mathcal{U}.$$

B is a 4-sheeted branched covering map of the unit disc onto itself, which simply means it is a proper map of $\mathcal{U}$ onto $\mathcal{U}$ which assumes each value four times, counting multiplicities. It is analytic across $\mathbf{T}=\{|z|=1\}$ and wraps $\mathbf{T}$ about itself four times in the positive direction. The derivative of B vanishes twice at $z=0$ and once at $z_0=(3-\sqrt{5})/2$.

To understand the mapping properties of B, it may be helpful to refer to Figure 1. In the domain, on the left, the interior lines approximate the pre-image under B of the real axis, and the pre-images of the values 0, 1, −1, and $w_0=B(z_0)$ are indicated. The shaded (resp. unshaded) portions of the domain each map one-to-one onto the shaded (resp. unshaded) half of the image disc.

We obtain F by modifying the image surface associated with B. Denote by C the circle of radius 15/16 centered at $z=1/16$, and by γ the image of C under $z\to z^2$. The parameterized path $p(t)=[1/16+(15/16)e^{it}]^2$, $0\le t\le 4\pi$ traces twice about γ. Now use the branch of B^{-1} determined near 1 by $B^{-1}(1)=1$ to define the path

$$\Gamma(t)=B^{-1}(p(t)), \quad 0\le t\le 4\pi,$$

with $\Gamma(0)=1$. It is not difficult to see that Γ is a simple closed curve. Approximations to it and to γ are shown in the domain and range discs, respectively, of Figure 2. Finally, let ω be a conformal mapping of the unit disc onto the region interior to Γ. Our function F is obtained as the composition

$$F(z)=(B\circ\omega)(z), \quad z\in\mathcal{U}.$$

We will develop the properties of F in the sequel as they are needed.

A few words may be helpful in understanding this construction. The image surface of B, call it $\mathcal{W}$, may be visualized as four copies of the unit disc attached

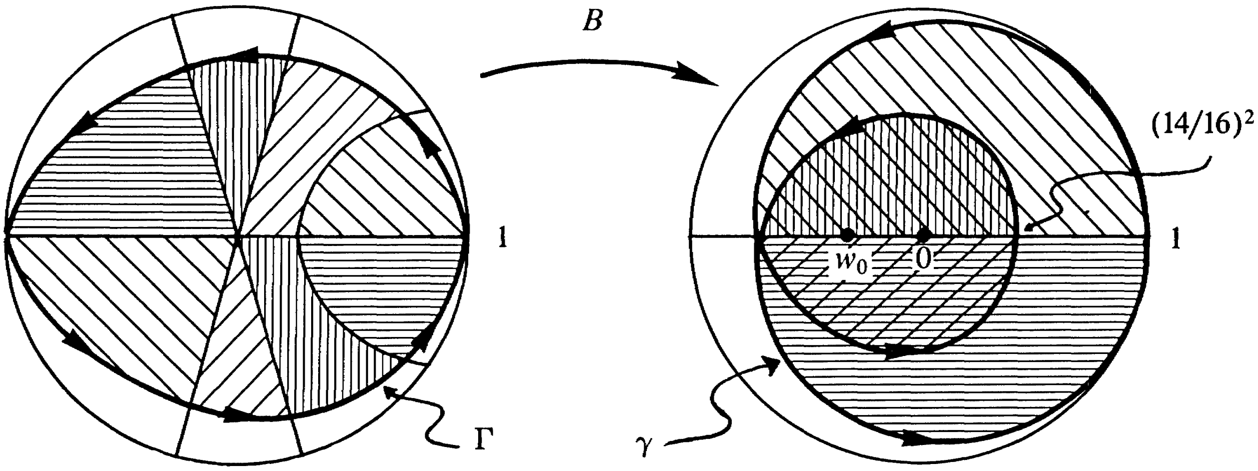


Figure 2

to one another and lying over the unit disc in the plane. We lift the curve γ to $\mathcal{W}$; it takes two transits of γ to obtain a closed curve. The image surface for F is obtained by simply removing the fringe, that portion of $\mathcal{W}$ outside of this curve.

3. For a function f analytic on an open set containing $\mathcal{U}$, we define the *valence function* of f by

$$v_f(w) = \text{card}[f^{-1}\{w\} \cap \mathcal{U}], \quad w \in \mathbf{C},$$

with due accounting for multiplicities, and we let

$$k_f = \min\{v_f(w) : w \in f(\mathcal{U})\}.$$

Using this notation, we now state the result proved by Baker, Deddens, and Ullman in [2].

THEOREM 1. *Let f be an entire function and let $k = k_f$. Then there exists an entire function h with*

$$f(z) = h(z^k), \quad z \in \mathcal{U},$$

and with $k_h = 1$.

The function $z \to z^k$ is a k-to-1 mapping of $\mathcal{U}$ onto itself, so the valence function of f is simply k times that of h. The function z^k seems to enter here because it is the only entire function which forms a k-sheeted branched covering of $\mathcal{U}$. However, there are many such coverings which are not entire: they consist precisely of the k-fold Blaschke products

$$b(z) = \lambda \prod_{j=1}^{k} (z - a_j)/(1 - \bar{a}_j z), \quad z \in \mathcal{U}.$$

Here, λ is a unimodular constant and $a_1, a_2, a_3, \ldots, a_k \in \mathcal{U}$ are the zeros of b. This leads Baker, Deddens, and Ullman to ask whether Theorem 1 remains true for functions f analytic on $\bar{\mathcal{U}}$ if z^k is replaced by some k-fold Blaschke product. The

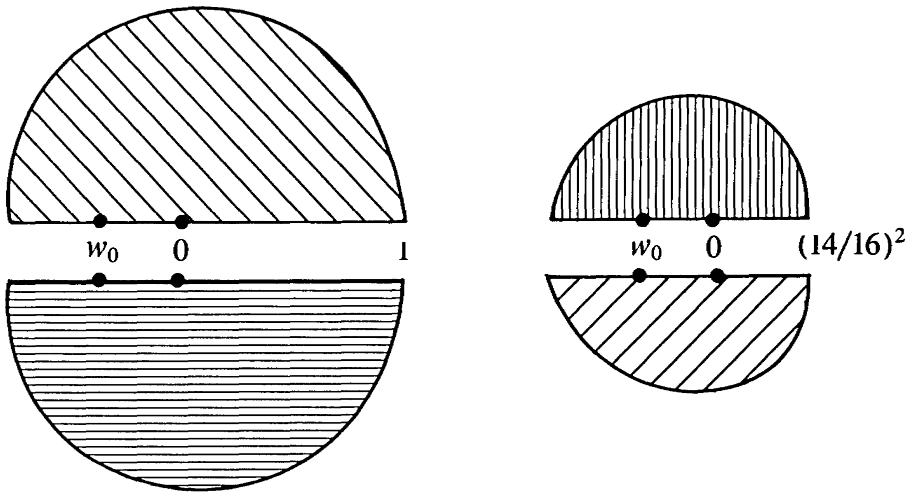


Figure 3

question is raised again by Thomson in [8] (see §6). Our function F provides a counterexample.

PROPOSITION 1. *The function F is analytic on $\bar{\mathcal{U}}$, $k_F = 2$, yet F is not of the form $F \equiv h \circ b$ for any analytic function h and 2-fold Blaschke product b.*

Proof. The Blaschke product B of Section 2 is analytic on $\bar{\mathcal{U}}$, hence on the curve Γ (see Figure 2). Moreover, B is locally univalent on Γ and maps it to the curve γ. Since this is the image of a circle under z^2, we conclude that Γ is analytic. By the Schwarz reflection principle, the conformal mapping ω of $\mathcal{U}$ onto the interior of Γ may be extended analytically to Γ. The composition $F \equiv B \circ \omega$ is therefore analytic on $\bar{\mathcal{U}}$.

Based on Figure 2, one can recognize four pieces which comprise the range of F (see Figure 3). The image surface of F can be reconstructed using two of each type of piece and pasting them together along the appropriate segments over the real axis. This is a rather loose description, but it is clear that each point in the range of F is covered at least two times, with those points interior to the smaller loop of γ covered four times.

The reason we base our construction of F on B has to do with its mapping properties. Specifically, we show that the multiple-valued analytic function $B^{-1} \circ B$ has a single nontrivial branch. In Figure 4, the domain of B has been reproduced from Figure 1, with the pre-images of the point $w = i/2$ labelled as z_1, z_2, z_3, and z_4. With any two of these, z_i and z_j, we can define a branch ψ of $B^{-1} \circ B$ in a neighborhood of z_i so that $\psi(z_i) = z_j$. Of course, if $i = j$, then ψ continues to the identity function on $\mathcal{U}$. *A priori,* one would expect three other, nontrivial branches. Using Figure 4, we will show these are all continuations of one another. A general study of such compositions was made by the author in [7]; however, as this case is very straightforward, the result may be shown directly.

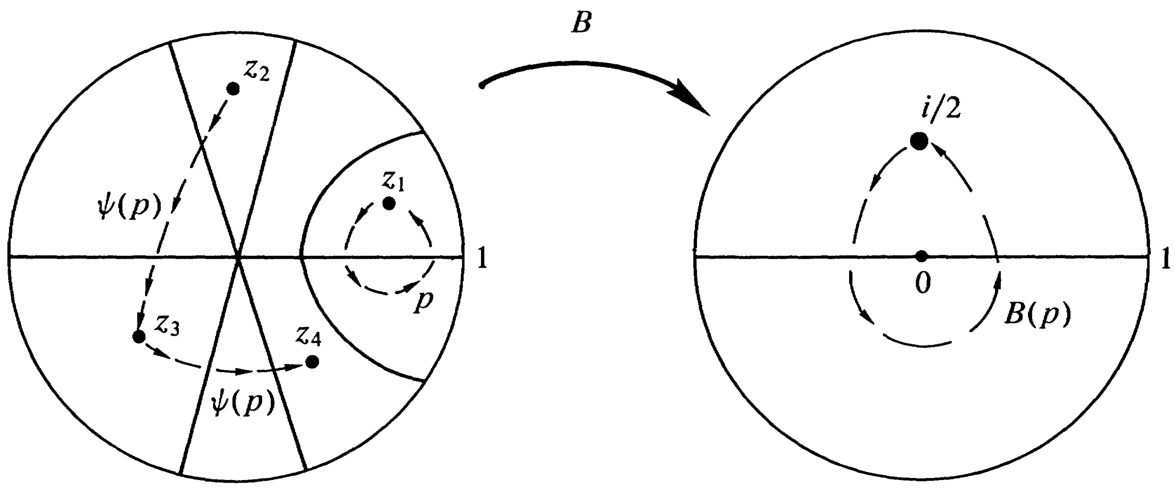


Figure 4

Begin by defining ψ near z_1 so that $\psi(z_1)=z_2$. Consider the continuation of ψ along the closed path p of Figure 4. During the first transit of p, the values of ψ go from z_2 to z_3. During the next, they go from z_3 to z_4. That is, the branches of $B^{-1}\circ B$ mapping z_1 to z_3 and to z_4 are both continuations of the branch mapping z_1 to z_2. The same behavior occurs for the four pre-images of any point in $\mathcal{U}\setminus\{0,w_0\}$.

Since F is obtained from B by suitably restricting its domain, a similar analysis shows that $F^{-1}\circ F$ also has a single nontrivial branch. Suppose that $F\equiv h\circ b$. Then

$$F^{-1}\circ F\equiv b^{-1}\circ h^{-1}\circ h\circ b;$$

so by an appropriate choice of h^{-1} and b^{-1}, we obtain a nontrivial branch of $b^{-1}\circ b$. Since the 2-fold Blaschke product b is a proper map of $\mathcal{U}$ onto itself, the modulus of the continuation of $b^{-1}\circ b$ must approach 1 along any path approaching the unit circle $\mathbf{T}$. However, the nontrivial branch of $F^{-1}\circ F$ does not have this property. Consider, for instance, the point $z=(14/16)^2$. It is both in the range of F and among its boundary values, say $z_5\in\mathcal{U}$ and $e^{it_0}\in\mathbf{T}$ with $F(z_5)=F(e^{it_0})=(14/16)^2$. One can find a path in $\mathcal{U}$ ending at e^{it_0} along which the continuation of $F^{-1}\circ F$ converges to $z_5\in\mathcal{U}$. This shows that $F^{-1}\circ F$ cannot be of the form $b^{-1}\circ b$, and completes the proof of the proposition. □

4. In this section we prove a conjecture of Lee Rubel (private communication) concerning entire functions and show that it, too, fails for functions which are assumed to be analytic only on $\bar{\mathcal{U}}$.

THEOREM 2. *Suppose that f and g are entire functions with $f(\mathcal{U})=g(\mathcal{U})$. Then there exist an entire function h, a unimodular constant λ, and positive integers m and n with*

$$f(z)=h(z^m) \quad \textit{and} \quad g(z)=h(\lambda z^n)$$

for all $z\in\mathbf{C}$.

Proof. The proof is essentially the same as that given by Baker, Deddens, and Ullman for Theorem 1, so we will not repeat all the details.

First assume that the derivatives of f and g have the same number of zeros at the origin. We wish to consider a branch of the function $\psi = f^{-1} \circ g$ or, what is the same thing, an analytic function $w = \psi(z)$ which satisfies the equation $f(w) - g(z) = 0$. The boundary of the set $f(\mathfrak{U}) = g(\mathfrak{U})$ consists of a finite number of analytic arcs. Clearly one can find arcs J_1 and J_2 of $\mathbf{T}$ on which f and g, respectively, have nonvanishing derivatives and which they map to the same analytic arc. If $w \in J_1$ and $z \in J_2$ satisfy $f(w) = g(z)$, we may define ψ locally at z so that $\psi(z) = w$ and so that ψ maps J_2 to J_1. Following Baker, Deddens, and Ullman, we may now use Schwarz reflection to show that ψ is *algebraic,* and the maximum principle to show that $f(0) = g(0)$ and that $\psi(z)$ is always finite for $z \neq 0, \infty$. With our assumption on the derivatives of f and g at the origin, we obtain the same growth estimates as they do for ψ at 0 and, by reflection, ∞. Their arguments then show that ψ has the form $\psi(z) = \eta z$, some $\eta \in \mathbf{T}$. From the definition of ψ, we obtain

$$f(\eta z) = g(z), \quad z \in \mathfrak{U}.$$

The conclusion of the theorem therefore holds with $h \equiv f$, $n = m = 1$, and $\lambda = \eta$.

For the case of general functions f and g, choose integers j and k so that $f(z^j)$ and $g(z^k)$ have derivatives which vanish to the same order at $z = 0$. These modified functions still satisfy the hypotheses of the theorem, so they satisfy the above identity for some $\eta \in \mathbf{T}$. Now it is an easy exercise using Theorem 1 to obtain the conclusion of the theorem. □

REMARKS. (a) In this proof we used the hypotheses on f and g only to obtain the arcs J_1 and J_2. Thus the conclusion follows from the seemingly weaker hypothesis that f and g map arcs of $\mathbf{T}$ to the same arc.

(b) It is natural to ask whether $\mathfrak{U}$ can be replaced by other domains in this result. As a starting point, we pose the

QUESTION 1. Let Ω be the interior of an ellipse centered at the origin. Suppose f and g are entire, $f(\Omega) = g(\Omega)$, and the derivatives of f and g have the same order of zero at $z = 0$. Is it necessarily the case that $f(z) = g(\pm z)$?

Suppose now that f and g are assumed to be analytic only on $\bar{\mathfrak{U}}$ with $f(\mathfrak{U}) = g(\mathfrak{U})$. The analogue of Theorem 2 would imply the existence of a function h analytic on $\bar{\mathfrak{U}}$ and finite Blaschke products b_1 and b_2 with

$$f \equiv h \circ b_1 \qquad g \equiv h \circ b_2. \tag{1}$$

We show that this generalization fails by using the function F along with a companion function G.

We obtain G by a construction similar to that used in Section 2 for F. The only difference is that we replace the Blaschke product B used there by $B_1(z) = z^4$. In particular, the valence and range of G are the same as those of F, and G is analytic on $\bar{\mathfrak{U}}$. The mapping properties of G may be inferred from Figure 5, which corresponds to Figure 2 for F.

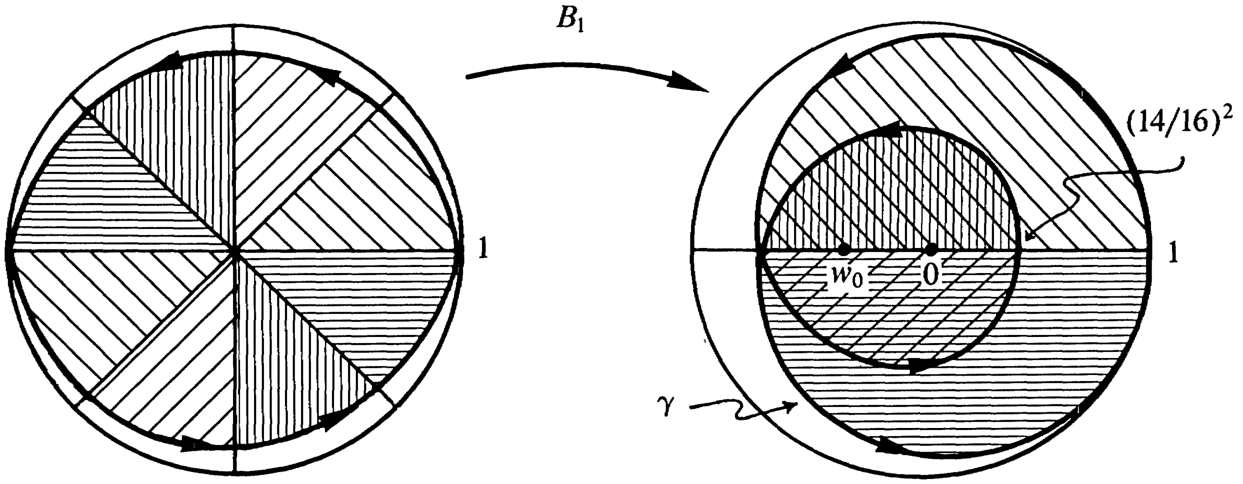


Figure 5

Suppose the equations (1) were to hold with $f \equiv F$ and $g \equiv G$. From Section 3 we know that b_1 could only be a Möbius transformation of $\mathcal{U}$, i.e., a 1-fold Blaschke product. Since F and G have the same valence, b_2 would likewise be a Möbius transformation. Consequently, $F \equiv G \circ \sigma$ for some Möbius transformation σ. This, however, is not possible since F has a point which is a simple branch point, while G has only a higher order branch point.

We end this section with a corollary to Theorem 2.

COROLLARY. *Suppose that f and g are entire, and $v_f = v_g$. Then there is a unimodular constant λ with $f(z) = g(\lambda z)$ for $z \in \mathbf{C}$.*

5. In this section we consider pairs of functions with identical, finite valence functions; and we investigate two structural relationships which may hold between them. Let f and g be bounded analytic functions on $\mathcal{U}$. We say they have *property* (P) if there are finite Blaschke products b_1 and b_2 having the same number of zeros, and a bounded analytic function h with

$$\text{(P)} \qquad f \equiv h \circ b_1, \qquad g \equiv h \circ b_2.$$

We say f and g have *property* (Q) if there is a homeomorphism ψ of $\mathbf{T}$ with

$$\text{(Q)} \qquad f \circ \psi \equiv g, \text{ a.e. on } \mathbf{T}.$$

We refer to ψ as a *distinguished* homeomorphism if ψ is of the form $b_3^{-1} \circ b_4$ where b_3 and b_4 are finite Blaschke products. Note that in order for this to define a homeomorphism of $\mathbf{T}$, it is necessary and sufficient that b_3 and b_4 have the same number of zeros. Also note that Möbius transformations of $\mathbf{T}$ are distinguished homeomorphisms.

The main result of this section is the following, which will be used when we discuss Toeplitz operators.

THEOREM 3. *Suppose f and g are bounded analytic functions on $\mathcal{U}$. Then they have property* (P) *if and only if they have property* (Q) *for a distinguished homeomorphism ψ.*

Some observations will be helpful before beginning the proof. If b is a finite Blaschke product, then $\mathcal{W}_b$ will denote its image surface, that is, the surface to which b^{-1} continues. If b has k zeros, $\mathcal{W}_b$ will be a k-sheeted ramified covering of $\mathcal{U}$. It can be pictured as k unit discs attached with appropriate branch points and lying over $\mathcal{U}$. Suppose f is bounded and analytic on $\mathcal{U}$ and suppose Ω is an open set in $\mathcal{U}$ on which a single valued branch of b^{-1} is specified. The corresponding function element $f \circ b^{-1}$ defined on Ω continues analytically to a Riemann surface $\mathcal{W}$ with natural projection $\pi : \mathcal{W} \to \mathcal{U}$. Since b is a covering map, and b^{-1} has at most algebraic singularities and k branches, $(\mathcal{W}, \pi)$ will be a ramified covering of $\mathcal{U}$ with m sheets, where $m \leq k$. If $m = k$, then $\mathcal{W}$ will be conformally equivalent to $\mathcal{W}_b$. If $m < k$, then $\mathcal{W}$ is obtained from $\mathcal{W}_b$ by identifying certain sheets on which f has the same behavior. In particular, this will occur precisely when there is an open set Ω in $\mathcal{U}$ on which distinct, single valued branches β_1 and β_2 of b^{-1} exist with

$$f \circ \beta_1 \equiv f \circ \beta_2 \text{ on } \Omega.$$

Finally, note that there is no difficulty defining expressions of the form $f \circ b^{-1}$ on $\mathbf{T}$. Since b is analytic and locally invertible in a neighborhood of $\mathbf{T}$, one can easily move into the interior of $\mathcal{U}$ slightly; $f \circ b^{-1}$ merely represents the (multiple valued) boundary function for the multiple valued function $f \circ b^{-1}$ on $\mathcal{U}$. In Theorem 3, we specify that f and g be bounded only for convenience—we simply needed a space in which each function has boundary values which determine it uniquely. An analogous result holds, for instance, if f and g are meromorphic functions of bounded characteristic on $\mathcal{U}$.

Proof of Theorem 3. If f and g have property (P), then any branch of $b_1^{-1} \circ b_2$ gives a homeomorphism ψ on $\mathbf{T}$ for which (Q) holds.

For the converse, assume f and g have property (Q) with $\psi \equiv b_3^{-1} \circ b_4$. As we remarked earlier, b_3 and b_4 will necessarily have the same number, k, of zeros. It might appear at first that (P) holds for b_1 and b_2 equal to b_3 and b_4, respectively. This, however, is not the case in general; there may be a finite Blaschke product b_5 so that (P) holds for products b_1 and b_2 which are related to b_3 and b_4 by

$$b_3 \equiv b_5 \circ b_1, \qquad b_4 \equiv b_5 \circ b_2.$$

The proof turns out to be rather subtle.

From (Q) we can readily obtain an open set Ω in $\mathcal{U}$ on which single valued branches of b_3^{-1} and b_4^{-1} exist and satisfy

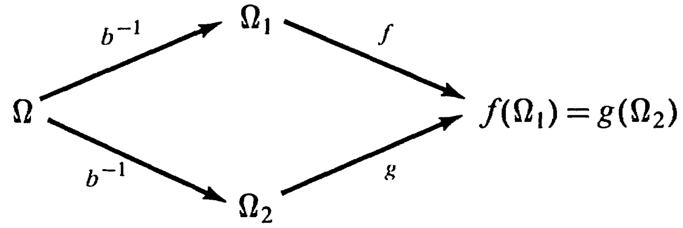
$$f \circ b_3^{-1} \equiv g \circ b_4^{-1} \text{ on } \Omega. \tag{2}$$

Let $\mathcal{W}$ denote the surface to which these function elements continue, and suppose it has m sheets. We consider first the case that $m = k$; the case $m < k$ will reduce to this.

When $m = k$, the surfaces $\mathcal{W}_{b_3}$ and $\mathcal{W}_{b_4}$ are conformally equivalent to $\mathcal{W}$, and hence to one another. This implies that there is a Möbius transformation σ of $\mathcal{U}$ with $b_3 \circ \sigma \equiv b_4$. Replacing f by $f \circ \sigma$, we may assume without loss of generality that $b_3 \equiv b_4 \equiv b$ in (2), giving

(3) $$f\circ b^{-1}\equiv g\circ b^{-1} \text{ on } \mathcal{U}.$$

This equation does not imply that $f\equiv g$, since the branches of b^{-1} on the left and right may differ. Let the branch of b^{-1} on the right (resp. left) of (3) satisfy $b^{-1}(\Omega)=\Omega_1$ (resp. $b^{-1}(\Omega)=\Omega_2$). We obtain the following commutative diagram:

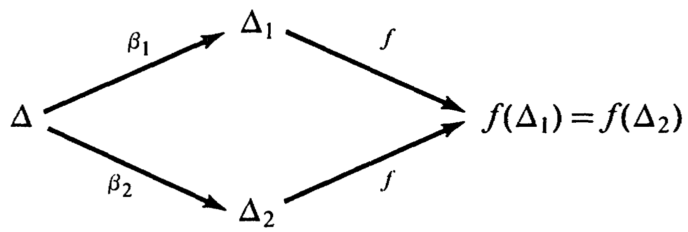


Consider the branch of $b^{-1}\circ b$ mapping Ω_2 to Ω_1 which is taken from the diagram. In general, such a branch continues to a ramified covering of $\mathcal{U}$. In our case, however, it must be single valued on $\mathcal{U}$. For, suppose otherwise: Then there is a closed path p in $\mathcal{U}$ starting at a point of Ω_2 so that along the closed path $b(p)$ the branch β_1 of b^{-1} defined by $\beta_1(\Omega)=\Omega_1$ continues to a branch β_2 of b^{-1} with $\beta_2(\Omega)=\Omega_3\neq\Omega_1$. From the diagram, $f\circ b^{-1}\circ b$ is identically equal to g, and hence is single valued along p. The branches $f\circ\beta_1$ and $f\circ\beta_2$ of $f\circ b^{-1}$ are then identical on Ω. By our earlier remarks, this would imply that the surface for $f\circ b^{-1}$ has less than k sheets, contradicting our assumption on m. Therefore, the branch of $b^{-1}\circ b$ taken from our diagram is single valued on $\mathcal{U}$ and consequently represents a Möbius transformation γ of $\mathcal{U}$. Thus $f\circ\gamma\equiv g$ and we conclude that f and g have property (P).

Suppose that $m<k$. First, consider $\mathcal{W}$ as the surface for $f\circ b_3^{-1}$. Since $m<k$, $\mathcal{W}$ is obtained from $\mathcal{W}_{b_3}$ by identifying certain sheets. In particular, we can find a small open set Δ in $\mathcal{U}$ so that two distinct branches of b_3^{-1}, call them β_1 and β_2, are single valued on Δ and have the property that

(4) $$f\circ\beta_1\equiv f\circ\beta_2 \text{ on } \Delta.$$

Choosing Δ smaller, if necessary, we may assume f is univalent on $\Delta_1=\beta_1(\Delta)$ and $\Delta_2=\beta_2(\Delta)$ and that $\Delta_1\cap\Delta_2=\varnothing$. We have the commuting diagram:



Chasing arrows, we obtain

(5) $$f^{-1}\circ f\equiv b_3^{-1}\circ b_3$$

as maps of Δ_2 onto Δ_1. We now appeal to the theory of function pairs developed in [7]. To each analytic function α on $\mathcal{U}$ is associated a collection $\mathcal{P}_\alpha$ of ordered pairs of self maps of $\mathcal{U}$. These can be used to determine, among other things, whether α results from a composition of other functions. In our situation, (5) implies that $\mathcal{P}_f\cap\mathcal{P}_{b_3}$ contains a nontrivial function pair. By [7, Theorem 7], there is an analytic function b_1 on $\mathcal{U}$ with $\mathcal{P}_f \cap \mathcal{P}_{b_3} = \mathcal{P}_{b_1}$ and ([7, Theorem 5])

$$(6) \qquad f \equiv f_1 \circ b_1 \qquad b_3 \equiv b_5 \circ b_1$$

for some functions f_1 and b_5 analytic on $b_1(\mathcal{U})$. By [7, Theorem 14], b_1 is an inner function, implying that both b_5 and b_1 must be finite Blaschke products. Observe that the number of zeros of b_5 is m for the following reason:

$$f_1 \circ b_5^{-1} \equiv f_1 \circ b_1 \circ b_1^{-1} \circ b_5^{-1} \equiv f \circ b_3^{-1}$$

has a surface with m sheets, so the surface for b_5 has at least m sheets. If it had more, we would proceed as we just have to find a nontrivial finite Blaschke product of which both f_1 and b_5 are functions. This would contradict the equality $\mathcal{P}_f \cap \mathcal{P}_{b_3} = \mathcal{P}_{b_1}$.

If we consider $\mathcal{W}$ as the surface for $g \circ b_4^{-1}$, we obtain in the same manner finite Blaschke products b_6 and b_2 and g_1 analytic on $\mathcal{U}$ with

$$(7) \qquad g \equiv g_1 \circ b_2 \qquad b_4 \equiv b_6 \circ b_2.$$

As with b_5, b_6 must have m zeros. For suitably chosen branches of the inverses we now have

$$(8) \qquad f_1 \circ b_5^{-1} \equiv f \circ b_3^{-1} \equiv g \circ b_4^{-1} \equiv g_1 \circ b_6^{-1}.$$

Consequently, we are in the situation handled before, namely, $f_1 \circ b_5^{-1}$ and $g_1 \circ b_6^{-1}$ continue to a surface with m sheets, while b_5 and b_6 have m zeros. Therefore, f_1 and g_1 satisfy $f_1 \circ \gamma \equiv g_1$ for some Möbius transformation γ of $\mathcal{U}$. Replacing b_1 by $\gamma^{-1} \circ b$ and f_1 by $f_1 \circ \gamma$ in (6), we may assume without loss of generality that $f_1 \equiv g_1$. Denoting this common function by h, we see that (6) and (7) imply property (P). This completes the proof of Theorem 3. □

The functions F and G constructed earlier show that this theorem fails if ψ is not required to be a distinguished homeomorphism. Although $v_F \equiv v_G$, property (P) fails for F and G. We can show, however, that (Q) holds by choosing (as in the proof of Theorem 2) a branch of $F^{-1} \circ G$ mapping one arc of $\mathbf{T}$ to another. The derivatives of F and G do not vanish on $\mathbf{T}$, so analytic continuation yields a mapping ψ of $\mathbf{T}$ into $\mathbf{T}$ for which $F \circ \psi \equiv G$. Since $v_F \equiv v_G$, the argument principle shows ψ is a homeomorphism. By the theorem, ψ is not distinguished. Does it have any special properties? Curiously enough, the answer is yes: ψ is the restriction to $\mathbf{T}$ of an algebraic function, as are distinguished homeomorphisms. This is a consequence of the fact that F and G are themselves algebraic. In particular, note that each has a square root mapping $\mathcal{U}$ onto the disc $\{|z - 1/16| < 15/16\}$ with at most algebraic singularities in $\mathcal{U}$. One easily verifies using a Schwarz reflection that these roots are algebraic, implying the same about the functions themselves. Indeed, G is seen to be rational, though F is not. In general, if $v_f \equiv v_g$ then, as we saw explicitly in the case of F and G, the image surfaces for f and g consist of the same pieces pasted together in different ways, resulting in different branch structures. Can this difference always be accounted for by some algebraic relationship, some algebraic "change of coordinates" ψ?

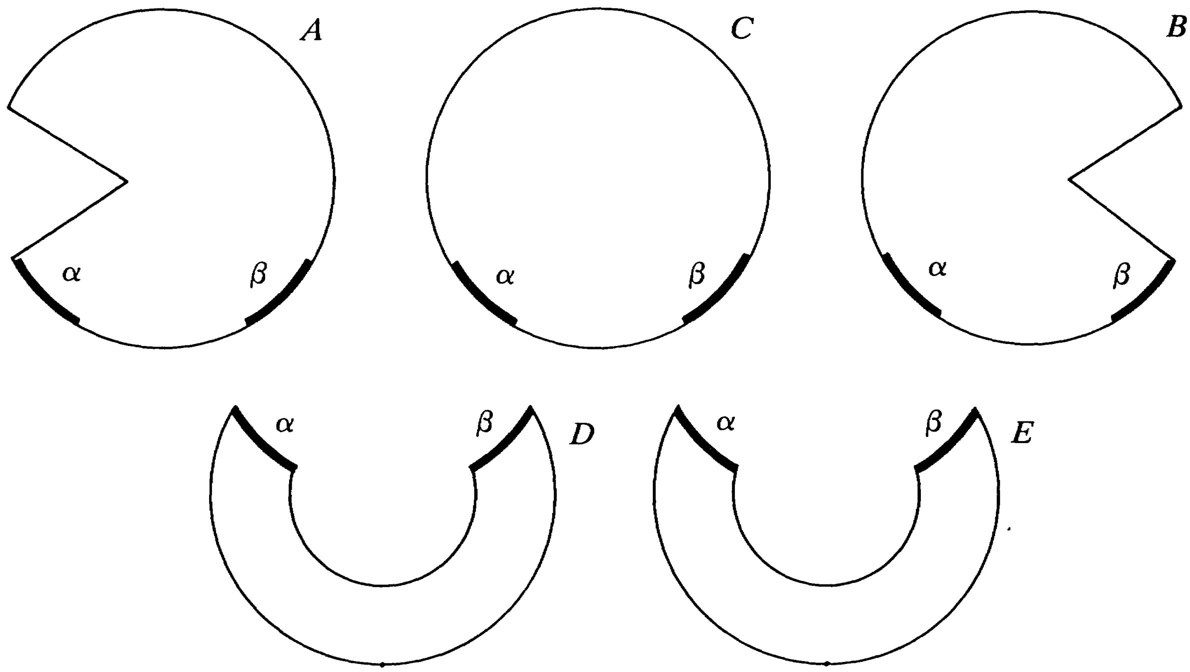


Figure 6

QUESTION 2. If f and g are analytic on $\bar{\mathcal{U}}$ with identical valence functions, does there exist an algebraic homeomorphism of $\mathbf{T}$ with $f \circ \psi \equiv g$?

An affirmative answer may not seem so unlikely, in light of the well-known fact that no matter how pathological is the simply connected region Ω, conformal maps of $\mathcal{U}$ onto it can differ at most by composition with a Möbius transformation. Also, positive results on the following structural question would tend to affirm Question 2.

QUESTION 3. Suppose f is analytic on $\bar{\mathcal{U}}$, with $|f| < 1$. Does there exist a constant k depending only on the valence function v_f, an analytic function $\omega: \mathcal{U} \to \mathcal{U}$, and a k-fold Blaschke product b such that $f \equiv b \circ \omega$?

Recent results of A. Lyzzaik (private communication) suggest an affirmative answer.

On the other hand, we conclude this section with an example showing that the requirement that f and g be analytic across $\mathbf{T}$ in Question 2 cannot be omitted. Functions I_1 and I_2 will be obtained as Riemann mappings of $\mathcal{U}$ onto simply connected Riemann surfaces $\mathcal{W}_1$ and $\mathcal{W}_2$. Both surfaces are constructed from the pieces in Figure 6. For $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively, we make these identifications:

$$A \xrightarrow{\alpha} D \xrightarrow{\beta} B \xrightarrow{\alpha} E \xrightarrow{\beta} C,$$

$$A \xrightarrow{\alpha} D \xrightarrow{\beta} C \xrightarrow{\alpha} E \xrightarrow{\beta} B.$$

That is, for $\mathcal{W}_1$ we attach A to D along the arc α; D to B along the arc β; etc., and likewise for $\mathcal{W}_2$. Each surface should be visualized as lying over the set Ω of Figure 7. Each is simply connected, and the natural projection maps to Ω, denoted

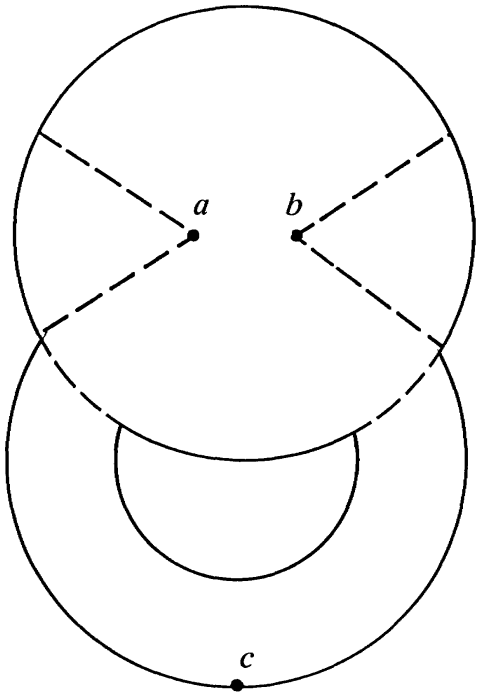


Figure 7

$p_1: \mathcal{W}_1 \to \Omega$ and $p_2: \mathcal{W}_2 \to \Omega$, endow them with conformal structures. Choose Riemann mappings $\tilde{I}_1: \mathcal{U} \to \mathcal{W}_1$ and $\tilde{I}_2: \mathcal{U} \to \mathcal{W}_2$ and define $I_1 \equiv p_1 \circ \tilde{I}_1: \mathcal{U} \to \Omega$ and $I_2 \equiv p_2 \circ \tilde{I}_2: \mathcal{U} \to \Omega$.

Since $\mathcal{W}_1$ and $\mathcal{W}_2$ cover each point of Ω the same number of times, I_1 and I_2 have identical valence functions. Both can be shown to be continuous (in fact, piecewise analytic) on **T**. However, we observe that there is no homeomorphism ψ of **T** with $I_1 \circ \psi \equiv I_2$; for as one proceeds in a positive direction about **T**, the values a, b, and c (see Figure 7) are taken in the order a, c, b, c by I_1 but in the order a, c, c, b by I_2.

Additional examples might be helpful in understanding the valence questions raised in this section. For instance, if f and g are rational and $v_f \equiv v_g$, must property (P) hold? Clearly property (P) implies $v_f \equiv v_g$. Is the same true of (Q)? Which homeomorphisms ψ can arise in (Q)?

6. In this section we discuss applications to Toeplitz operators. Denote by L^2 the functions which are measurable and square integrable with respect to Lebesgue measure on **T**; by $H^2 \subset L^2$, the Hardy space consisting of those functions whose negative Fourier coefficients vanish; and by $P: L^2 \to H^2$, the orthogonal projection. If ϕ is a bounded measurable function on **T** then the *Toeplitz operator* $\mathbf{T}_\phi: H^2 \to H^2$ with *symbol* ϕ is defined by $\mathbf{T}_\phi(h) = P(\phi h)$. If ϕ is the boundary function for a bounded analytic function on $\mathcal{U}$, then $\mathbf{T}_\phi$ is an *analytic* Toeplitz operator; whereas, if ϕ is the restriction to **T** of an entire or rational function, then $\mathbf{T}_\phi$ is an *entire* or *rational* Toeplitz operator, respectively.

First we consider questions of similarity, where compositions have frequently played a role. For analytic Toeplitz operators, Theorem 3 adds condition (iv) to a result proven by Carl Cowen in [5]. Here, a function of *finite valence* is one which assumes no value infinitely often.

THEOREM 4. *Suppose f and g are bounded analytic functions of finite valence on* $\mathcal{U}$. *The following are equivalent:*

(i) $\mathbf{T}_f$ *and* $\mathbf{T}_g$ *are similar,*
(ii) $\mathbf{T}_f$ *and* $\mathbf{T}_g$ *are unitarily equivalent,*
(iii) *f and g have property* (P), *and*
(iv) *f and g have property* (Q), *with distinguished homeomorphism* ψ.

Apply this to our functions F and G. We have seen that property (P) fails, so $\mathbf{T}_F$ and $\mathbf{T}_G$ are not similar, even though they have identical valence functions. They also illustrate another phenomenon. An analytic Toeplitz operator $\mathbf{T}_f$ is subnormal and has the corresponding Laurent operator L_f as its minimal normal dilation, where L_f is defined on L^2 by $L_f(h)=fh$. In a private communication, Warren Wogen pointed out the following: Since F and G are continuous on $\mathbf{T}$ and take each value the same number of times there, L_F and L_G are unitarily equivalent by the multiplicity theory for normal operators (see [1]). We conclude that, though the subnormal operators $\mathbf{T}_F$ and $\mathbf{T}_G$ are not even similar, their minimal normal dilations L_F and L_G are unitarily equivalent.

In the study of similarity of Toeplitz operators, property (Q) may have some advantages over (P) since it concerns only the boundary values of f and g and it relates them directly. Observe, however, that the usefulness of (Q) depends to a certain extent on analyticity. Though (P) and (Q) both make sense for essentially bounded, measurable functions f and g on $\mathbf{T}$, and (P) implies (Q) holds with a distinguished homeomorphism ψ, the converse fails. This can be shown with simple examples using differences of characteristic functions. In the case of rational functions, there is enough regularity for Theorem 3 to remain true (see the comment immediately preceding its proof). Along with Theorem 1 of [5], this implies: *If f and g are rational and* (Q) *holds for a distinguished homeomorphism, then* $\mathbf{T}_f$ *and* $\mathbf{T}_g$ *are similar.* The situation regarding the converse is not clear. However, an example obtained by the methods we used earlier may provide insight.

We construct F_1 in the same manner as F (see Section 2), only the curve γ is replaced by the curve γ_1, the image of the circle C under $z\to z^3$, and the 4-fold Blaschke product B is replaced by the 3-fold Blaschke product

$$b(z)=z^2(2z-1)/(2-z), \quad z\in\mathcal{U}.$$

Construct G_1 likewise, using z^3 in place of b. The path γ_1 consists of the three nested loops shown in Figure 8. The analysis of the mapping properties of these functions proceeds exactly as before. Their valence functions are identical, and $k_{F_1}=1=k_{G_1}$. Were property (P) to hold, F_1 and G_1 would necessarily be related by $F_1\circ\sigma\equiv G_1$, for some Möbius transformation σ of $\mathcal{U}$. This is not possible, since F_1 has two simple branch points while G_1 has only a branch point of order 2. From Theorem 4, we conclude that $\mathbf{T}_{F_1}$ and $\mathbf{T}_{G_1}$ are not similar.

One easily checks by Schwarz reflection that G_1 is rational while F_1 is algebraic. Their derivatives do not vanish on $\mathbf{T}$ and they map $\mathbf{T}$ one-to-one onto γ_1,

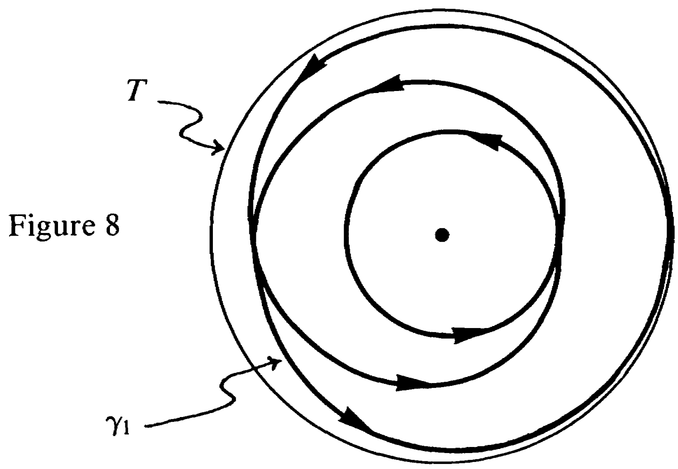


Figure 8

with the exception of the three points where γ_1 crosses itself. The functions F_1 and G_1 therefore provide a counterexample to a result of Doug Clark in [3]. Specifically, Theorem 5 of [3] fails if we let our G_1 play the role of Clark's function F and we let our F_1 play the role of his τ. Using his notation, the error in the proof concerns the extension of F to the set Ω with values in the image surface $\mathcal{R}$ of τ. As one can verify in our example, the branch structure of τ may obstruct this continuation.

It is instructive to observe what happens when such a continuation is possible: To be precise, suppose F and τ satisfy the hypotheses of [3], Case IV, and suppose F extends to a map F^* on some region $\Omega \subseteq \mathcal{U}$ with values in $\mathcal{R}$ and mapping $\partial\Omega$ to $\partial\mathcal{R}$. It is not difficult to see that if τ^* is the lifting of the map τ to $\mathcal{R}$, then $f = (\tau^*)^{-1} \circ F^*$ is a single-valued analytic function mapping Ω to $\mathcal{U}$ and $\partial\Omega$ to $\mathbf{T}$; that is, f is an inner function on the (possibly multiply-connected) region Ω. The hypotheses on F and τ imply that $\mathbf{T} \subseteq \partial\Omega$. Defining ψ to be the restriction of f to $\mathbf{T}$, we have $F \equiv \tau \circ \psi$ on $\mathbf{T}$. Summarizing, we see that property (Q) holds for F and τ, with a homeomorphism ψ which is the restriction to $\mathbf{T}$ of an inner function on Ω. This suggests a new class of homeomorphisms of $\mathbf{T}$ which may be useful in the study of similarity of rational Toeplitz operators.

Lastly, we comment on commutants. The commutant $\{\mathbf{T}_\phi\}'$ of $\mathbf{T}_\phi$ is the algebra of operators on H^2 which commute with $\mathbf{T}_\phi$, and has been an object of considerable study (see, for example, [4]). Deddens and Wong posed several questions about commutants in [6]. In [2], Baker, Deddens, and Ullman were able to give many affirmative answers for entire Toeplitz operators using Theorem 1. They proved that when f is entire, $\{\mathbf{T}_f\}' = \{\mathbf{T}_{z^k}\}'$, where $k = k_f$. Thomson [8] went further by showing that the answers are still affirmative when f is analytic on $\bar{\mathcal{U}}$. Again, $\{\mathbf{T}_f\}' = \{\mathbf{T}_b\}$, for some n-fold Blaschke product b. However, Thomson's work required different methods since it was not known whether Theorem 1 could be generalized—in particular, whether $n = k_f$. Our function F (from Section 2) shows that the alternate methods of Thomson are necessary. Indeed, one has

$$\{\mathbf{T}_F\}' = \{\mathbf{T}_z\}' = \{\mathbf{T}_h : h \in H^\infty\},$$

even though $k_F = 2$.

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