C_0-FREDHOLM OPERATORS. III

Hari Bercovici

An operator T, acting on a Hilbert space, is said to be of class C_0 (cf. [6]) if T is a completely nonunitary contraction and $u(T)=0$ for some nonzero function u in H^∞. An operator T is said to have property (P) (cf. [1, part II]) if the equalities $\ker X=\{0\}$ and $\ker X^*=\{0\}$ are equivalent for every operator X in the commutant $\{T\}'$ of T. In the two preceding papers ([1]) we studied the multiplicative semigroup $\Phi(T',T)$ of C_0-Fredholm operators, associated with a given pair (T',T) of operators of class C_0. We recall that $\Phi(T',T)$ consists of those operators X intertwining T' and T $(T'X=XT)$ with the following properties:

(A) $T|\ker X$ and $T_{\ker X^*}(=(T^*|\ker X^*)^*)$ have property (P); and

(B) the mapping $X_*:\mathfrak{M}\mapsto(X\mathfrak{M})^-$, $\mathfrak{M}\in\operatorname{Lat}(T_{(\ker X)^\perp})$ is an isomorphism of $\operatorname{Lat}(T_{(\ker X)^\perp})$ onto $\operatorname{Lat}(T|(\operatorname{ran}X)^-)$.

Here, as usual, $(\mathfrak{M})^-$ stands for the closure of the set $\mathfrak{M}$. When $T'=T$, we use the notation $\Phi(T)$ for $\Phi(T',T)$ and note that $\Phi(T)$ is contained in $\{T\}'$. If T is the zero operator on $\mathcal{H}$, then $\Phi(T)$ coincides with the familiar class of Fredholm operators on $\mathcal{H}$.

In [1, part I, Lemma 3.3] we proved that the operators $T|\ker X$ and $T_{\ker X^*}$ are quasisimilar provided that T is of class C_0 and X belongs to the bicommutant $\{T\}''$ of T (cf. also [9]). We also know from [1, part II] that property (P) is a quasisimilarity invariant in the class C_0. Therefore, in order to verify that an operator X in $\{T\}''$ is C_0-Fredholm, it suffices to verify (B) and half of (A). In this paper we prove that condition (B) is a consequence of (A) for X in $\{T\}''$, thus establishing the following result.

THEOREM 1. *Let T be an operator of class C_0 and let $X\in\{T\}''$. Then X is C_0-Fredholm if and only if $T|\ker X$ has property* (P).

We had previously noted (cf. [1, part I, Proposition 3.5]) that Theorem 1 is true in case $\ker X=\{0\}$. Observe that property (B) is not satisfied for every X in $\{T\}''$ even in the case of nilpotent operators T; this follows from the discussion given below of C_0-Fredholm operators in the case when T is an algebraic operator (cf. Example 9 below).

Let T be an arbitrary operator of class C_0, and let m denote the minimal function of T. It follows from results of [3] and [2] that for every X in $\{T\}''$ there exist functions u,v in H^∞ such that v and m are relatively prime $(v\wedge m=1)$ and

$$v(T)X=u(T)$$

or, using the notation of [6], $X=(u/v)(T)$. It follows from the proof of the main theorem in [8] that v can be chosen independently of X.

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LEMMA 2. *If T is an operator of class C_0 acting on $\mathcal{H}$ and $X=(u/v)(T)$ is in $\{T\}''$ $(v\wedge m=1)$ then X is in $\Phi(T)$ if and only if $u(T)$ is in $\Phi(T)$.*

Proof. We certainly have $v(T)\in\{T\}''$ and it follows from [6, Chapter III] that $v(T)$ is one-to-one. Therefore, by [1, part I, (3.9)], $(v(T)\mathfrak{M})^-=\mathfrak{M}$ for every $\mathfrak{M}$ in $\mathrm{Lat}(T)$. The relation $v(T)X=u(T)$ shows then that $\ker X=\ker u(T)$ and

$$(u(T)\mathfrak{M})^-=(v(T)X\mathfrak{M})^-=(v(T)(X\mathfrak{M})^-)^-=(X\mathfrak{M})^- \tag{1}$$

for every $\mathfrak{M}$ in $\mathrm{Lat}(T)$. In particular $(u(T)\mathcal{H})^-=(X\mathcal{H})^-$ so that $\ker X^*=\ker u(T)^*$. We can already see that X has property (A) if and only if $u(T)$ has property (A). As for property (B), relation (1) is easily seen to imply that $X_*=u(T)_*$. Indeed, if $\mathfrak{M}$ is in $\mathrm{Lat}(T_{(\ker X)^\perp})=\mathrm{Lat}(T_{(\ker u(T))^\perp})$, we have

$$\mathfrak{M}+\ker X\in \mathrm{Lat}\, T$$

and therefore

$$\begin{aligned} X_*(\mathfrak{M})=(X\mathfrak{M})^-&=(X(\mathfrak{M}+\ker X))^-\\ &=(u(T)(\mathfrak{M}+\ker X))^-\\ &=(u(T)(\mathfrak{M}+\ker u(T)))^-\\ &=(u(T)\mathfrak{M})^-=u(T)_*(\mathfrak{M}).\end{aligned}$$

The lemma is proved. □

Lemma 2 shows that we only have to consider operators X of the form $u(T)$ in the proof of Theorem 1.

We will need the characterization given in [1, part II] of operators of class C_0 with the property (P). Recall that, by results of [3] and [2], every operator T of class C_0 is quasisimilar with a unique operator S (called the Jordan model of T) of the form $S=\bigoplus_\alpha S(\theta_\alpha)$, where $\{\theta_\alpha\}_\alpha$ is a family of inner functions, indexed by the ordinal numbers, satisfying the following conditions:

(i) θ_α divides θ_β whenever $\alpha\geqslant\beta$;

(ii) $\theta_\alpha=\theta_\beta$ whenever $\mathrm{card}(\alpha)=\mathrm{card}(\beta)$; and

(iii) $\theta_{\alpha_0}=1$ for some α_0 (and therefore $\theta_\alpha=1$ for all $\alpha\geqslant\alpha_0$).

Then we have the following result from [1].

THEOREM 3. *Let T be an operator of class C_0, and let $S=\bigoplus_\alpha S(\theta_\alpha)$ be its Jordan model. Then T has property* (P) *if and only if $\bigwedge_{\alpha<\omega}\theta_\alpha=1$.*

Here, as usual, ω denotes the first transfinite ordinal.

We also recall the fact that, if T and S are quasisimilar operators of class C_0 and $u\in H^\infty$, then $T\,|\,[\mathrm{ran}\, u(T)]^-$ [resp. $T\,|\ker u(T)$] and $S\,|\,[\mathrm{ran}\, u(S)]^-$ [resp. $S\,|\ker u(S)$] are quasisimilar (cf., e.g., [1]).

In the following lemma we will use an arithmetic property of H^∞. A function u in H^∞ is absolutely continuous with respect to $v\in H^\infty$ if $u\wedge w\neq 1$ implies $v\wedge w\neq 1$ for every w in H^∞. Given a function u in H^∞ and an inner function m, there exists a decomposition $m=m_{ac}m_s$ such that m_{ac} and m_s are inner functions, m_{ac} is absolutely continuous with respect to u (in symbols, $m_{ac}\prec u$), and $m_s\wedge u=1$ (cf. [4]).

LEMMA 4. *Let T be an operator of class C_0 with minimal function m and assume that $T|\ker u(T)$ has property* (P) *for some u in H^∞. If we write $m = m_{ac}m_s$, with $m_{ac} \prec u$ and $m_s \wedge u = 1$, then the operator $T|\ker m_{ac}(T)$ also has property* (P).

Proof. It follows from the remarks above that we may assume that $T = \bigoplus_\alpha S(\theta_\alpha)$ is a Jordan operator; of course we have $\theta_0 = m$ in this case. Observe that

$$T|\ker u(T) = \bigoplus_\alpha S(\theta_\alpha)|\ker u(S(\theta_\alpha)) \tag{2}$$

and $S(\theta_\alpha)|\ker u(S(\theta_\alpha))$ is unitarily equivalent to $S(u \wedge \theta_\alpha)$. Consequently $T|\ker u(T)$ is unitarily equivalent to the Jordan operator $\bigoplus_\alpha S(u \wedge \theta_\alpha)$. By Theorem 3, the condition that $T|\ker u(T)$ have property (P) can be translated into

$$\bigwedge_{\alpha<\omega} (u \wedge \theta_\alpha) = u \wedge \left(\bigwedge_{\alpha<\omega} \theta_\alpha \right) = 1$$

and this implies, by the definition of absolute continuity, that

$$\bigwedge_{\alpha<\omega} (m_{ac} \wedge \theta_\alpha) = m_{ac} \wedge \left(\bigwedge_{\alpha<\omega} \theta_\alpha \right) = 1.$$

Relation (2) applied to $u = m_{ac}$ shows now that $T|\ker m_{ac}(T)$ must have property (P). The lemma is proved. □

COROLLARY 5. *Under the conditions of Lemma* 4, *$T|[\operatorname{ran} m_s(T)]^-$ and $T_{[\ker m_s(T)]^\perp}$ also have property* (P).

Proof. Since $m_{ac}(T)m_s(T) = m(T) = 0$ it follows that

$$[\operatorname{ran} m_s(T)]^- \subset \ker m_{ac}(T).$$

Thus $T|[\operatorname{ran} m_s(T)]^-$ is a restriction of $T|\ker m_{ac}(T)$, and therefore Theorem 3 combined with Corollary 2.9 of [2, part II] imply the desired condition about $T|[\operatorname{ran} m_s(T)]^-$. A similar argument shows that $T_{[\ker m_s(T)]^\perp}$ is a compression of $T_{\ker m_{ac}(T)^*}$ and we know that $T_{\ker m_{ac}(T)^*}$ is quasisimilar to $T|\ker m_{ac}(T)$. Again the conclusion is that $T_{[\ker m_s(T)]^\perp}$ has property (P). □

We are now ready to prove our main result.

Proof of Theorem 1. Assume that T is acting on the Hilbert space $\mathcal{H}$. By Lemma 2 we have to prove only that $u(T)$ is in $\Phi(T)$ whenever u is a function in H^∞ such that $T|\ker u(T)$ has property (P). Write the minimal function m of T as $m = m_{ac} \cdot m_s$ with $m_{ac} \prec u$ and $m_s \wedge u = 1$. Let us set $\mathcal{H}' = \ker m_s(T)$ and $\mathcal{H}'' = [\ker m_{ac}(T)]^\perp$; obviously then $\mathcal{H}' \in \operatorname{Lat}(T)$ and $\mathcal{H}'' \in \operatorname{Lat}(T^*)$. If $J: \mathcal{H}' \to \mathcal{H}$ denotes the inclusion operator and $P: \mathcal{H} \to \mathcal{H}''$ the orthogonal projection, we have

$$TJ = J(T|\mathcal{H}') \quad \text{and} \quad T_{\mathcal{H}''}P = PT. \tag{3}$$

Note moreover that J and P are C_0-Fredholm operators. Indeed, they have closed ranges so condition (B) is verified trivially,

$$\ker J=\{0\},\quad \ker J^*=[\ker m_s(T)]^\perp,\quad \ker P^*=\{0\},\quad \ker P=\ker m_{ac}(T)$$

so that property (A) is verified by Lemma 4 and Corollary 5. Relations (3) also imply

$$(4)\qquad u(T)J=Ju(T\mid\mathcal{H}')\quad\text{and}\quad u(T_{\mathcal{H}''})P=Pu(T).$$

The minimal functions of $T\mid\mathcal{H}'$ and $T_{\mathcal{H}''}$ both obviously divide m_s (in fact they equal m_s by [6, Chapter III]) and then the relation $u\wedge m_s=1$ implies that $u(T\mid\mathcal{H}')$ and $u(T_{\mathcal{H}''})$ are C_0-Fredholm operators (cf. the remark following the statement of Theorem 1). Therefore, Theorem 8.5 of [1, part II] and (4) imply that $u(T)J$ and $Pu(T)$ are C_0-Fredholm operators. Finally, an application of Proposition 8.8 of [1, part II] (with $A=u(T)$, $B=J$, and $C=P$) shows that $u(T)$ is a C_0-Fredholm operator. The theorem is proved. □

We will discuss now a few facts about the class $\Phi(T)$ in case T is an algebraic operator. Note that an algebraic operator, that is, an operator T that satisfies the relation $p(T)=0$ for some nonzero polynomial p, is also an operator of class C_0 if $\|T\|<1$. In this case the minimal function of T is a Blaschke product with the same zeros as the minimal polynomial of T. If T is an algebraic operator but $\|T\|\geq 1$, then $T/2\|T\|$ is a C_0-operator. Therefore all results concerning quasi-similarity classifications and C_0-Fredholm operators will apply to arbitrary algebraic operators, provided that we consistently replace minimal functions by minimal polynomials.

Let us use the notation $\mathcal{F}(\mathcal{H})$ for the set of all Fredholm operators acting on the Hilbert space $\mathcal{H}$.

PROPOSITION 6. *For every algebraic operator T on $\mathcal{H}$ we have*

$$\Phi(T)=\{T\}'\cap\mathcal{F}(\mathcal{H}\}.$$

Proof. The inclusion $\{T\}'\cap\mathcal{F}(\mathcal{H})\subset\Phi(T)$ is obvious. Indeed, if $X\in\mathcal{F}(\mathcal{H})$, the range of X is closed so that condition (B) is trivially satisfied. As for (A), $T\mid\ker X$ and $T_{\ker X^*}$ are operators on finite dimensional spaces, and every operator acting on a finite dimensional space has property (P).

Conversely, assume that $X\in\mathcal{F}(T)$ so that X commutes with T. For every h in $(X\mathcal{H})^-$ the cyclic space $\mathcal{H}_h=\bigvee_{n\geq 0}T^n h$ is finite dimensional (because T is algebraic) and $\mathcal{H}_h\subset(X\mathcal{H})^-$. Property (B) implies the existence of a subspace $\mathcal{M}$ of $\mathcal{H}$ such that $(X\mathcal{M})^-=\mathcal{H}_h$. Since $\mathcal{H}_h$ is finite dimensional, $X\mathcal{M}$ is finite dimensional so that $X\mathcal{M}=(X\mathcal{M})^-=\mathcal{H}_h$. In particular, $h\in X\mathcal{H}$ for every h in $(X\mathcal{H})^-$ and it follows that X has closed range. We have to prove only that $\ker X$ and $\ker X^*$ are finite dimensional, and this will follow from condition (A) once we prove that every algebraic operator T having property (P) necessarily acts on a finite dimensional space. To prove this last statement there is no loss of generality in assuming that $\|T\|<1$ so that T is of class C_0. Let $S=\bigoplus_\alpha S(\theta_\alpha)$ be the Jordan model of T. Thus θ_0 is a finite Blaschke product and the condition $\bigwedge_{\alpha<\omega}\theta_\alpha=1$ (which is equivalent to T having property (P)) is equivalent to $\theta_n=1$ for some $n<\omega$. Thus, if T has property (P), S is a finite sum

$$S=S(\theta_0)\oplus S(\theta_1)\oplus\cdots\oplus S(\theta_{n-1})$$

where $\theta_0, \theta_1, \dots, \theta_{n-1}$ are finite Blaschke products. To finish the proof it suffices to remark that the operator $S(\theta)$ is acting on a space of finite dimension k if and only if θ is a Blaschke product with k factors. □

REMARK 7. The proof given above of the fact that X has closed range if $X \in \Phi(T)$ used only property (B) and the fact that the cyclic subspaces of an algebraic operator are finite dimensional. It is interesting to note that, by a result of Sherman [5], any operator whose cyclic spaces are finite dimensional is algebraic. The fact that T is algebraic is essential to Proposition 6, as shown by the following example.

EXAMPLE 8. *If T is a nonalgebraic operator of class C_0, there exists X in $\Phi(T) \cap \{T\}''$ such that X does not have closed range.*

Proof. As noted before Lemma 2, there exists a function v in H^∞ such that $v \wedge m = 1$ (m = the minimal function of T) and every operator X in $\{T\}''$ can be represented as $X = (u/v)(T)$ for some u in H^∞. Since T is not algebraic, m is not a finite Blaschke product and therefore we can find an inner function p such that $p \wedge m = 1$ but

$$\inf\{|p(\lambda)| + |m(\lambda)| : |\lambda| < 1\} = 0. \tag{5}$$

Indeed, it suffices to define p as a Blaschke product whose zeros $\{\alpha_n\}_{n \geqslant 0}$ satisfy the condition $m(\alpha_n) \neq 0$ and $\lim_{n \to \infty} |m(\alpha_n)| = 0$. We now set $X = (pv)(T)$; obviously $X \in \{T\}''$ and, since $pv \wedge m = 1$, X is a quasiaffinity. By Theorem 1 we have $X \in \Phi(T)$. If x would have closed range, it would follow that X^{-1} is a bounded operator in $\{T\}''$, so that $X^{-1} = (u/v)(T)$ for some u in H^∞. Equivalently, $(pv)(T)u(T) = v(T)$ or $v(T)((pu)(T) - I) = 0$, which implies $(pu-1)(T) = 0$ because $v(T)$ is a quasiaffinity. Therefore m must divide $pu - 1$, say $pu - 1 = mg$, $g \in H^\infty$ so that $pu - gm = 1$ in contradiction with (5). This contradiction shows that X cannot have closed range.

Finally we give the example promised after the statement of Theorem 1.

EXAMPLE 9. *There exists an operator T of class C_0 and X in $\{T\}''$ such that X does not satisfy condition* (B).

Proof. Let A on $\mathcal{H}$ be a noninvertible quasiaffinity such that $\|A\| < 1$ (e.g., $A = \operatorname{diag}(1/(n+1))$ on l^2), and denote by T the operator on $\mathcal{H} \oplus \mathcal{H}$ whose matrix is given by

$$T = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}.$$

Clearly $T^2 = 0$ and T does not have closed range. Set $X = T$ and note that X cannot have property (B) by Remark 7. □

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Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139