THE FAILURE OF L^p ESTIMATES FOR HARMONIC MEASURE IN CHORD-ARC DOMAINS

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Let D be a bounded domain in the complex plane whose boundary ∂D is the image of a simple closed rectifiable curve. For $\zeta, \zeta' \in \partial D$, denote by $\sigma(\partial D; \zeta, \zeta')$ the length of the shorter arc of ∂D with endpoints ζ and ζ'. D is said to be a *chord-arc* domain if there is a constant C such that for every $\zeta, \zeta' \in \partial D$,

$$\sigma(\partial D; \zeta, \zeta') \leqslant C|\zeta - \zeta'|.$$

For each $p > 0$, the Hardy class H^p is the collection of analytic functions F on the unit disc in the complex plane satisfying

$$\sup_{r<1} \mu_p(r,F) < \infty, \quad \text{where} \quad \mu_p(r,F) = \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta.$$

Our purpose is to comment on a theorem of Lavrentiev [8], namely

THEOREM 1. *For any constant C, there exists $p > 0$ such that if f is a conformal mapping of the unit disc onto D and D is a chord-arc domain with constant C, then $1/f' \in H^p$.*

This result has received considerable attention recently ([1], [4], [6], [9]) by virtue of its link to real-variable lemmas of John-Nirenberg type and to the boundedness of the Cauchy integral on curves. In the closely related special case in which ∂D is given locally as the graph of a Lipschitz function, the corresponding conformal mapping f satisfies $1/f' \in H^1$ independent of the Lipschitz constant. The same is true of another simple example, a logarithmic spiral. For these reasons, Jerison and Kenig [6] and Baernstein [1] asked if Theorem 1 is valid with some exponent p independent of C, and in particular for $p = 1$. We will show here that it is not.

THEOREM 2. *For any $p > 0$, there exists a chord-arc domain D for which $1/f' \notin H^p$ for any conformal mapping f of the unit disc onto D.*

Jones and Zinsmeister have independently given another proof of this theorem [7].

Background and notation. We will reformulate Theorem 2 in terms of harmonic measure and state some standard results needed in the proof.

A well known consequence of Jensen's inequality is that

(1) *$\mu_p(r,F)$ is an increasing function of r for $r < 1$* [10: p. 273].

If $F \in H^p$, then $F(e^{i\theta}) = \lim_{r\to 1} F(re^{i\theta})$ exists for almost every $\theta \in [0, 2\pi)$ and

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Received October 8, 1982.
This work was supported in part by NSF Grant MCS-8202127.
Michigan Math. J. 30 (1983).

$\sup_{r<1} \mu_p(r,F)=\mu_p(1,F)<\infty$. It is also well known that a conformal mapping f of the unit disc $\Delta=\{z\in\mathbf{C}: |z|<1\}$ onto D extends to a homeomorphism of $\bar{\Delta}$ to $\bar{D}$ [10: p. 290]. By theorems of Privalov and F. and M. Riesz [10: p. 293], $f'\in H^1$ and

$$\sup_{r<1} \mu_1(r,f')=\sigma(\partial D)/2\pi<\infty;$$

$$d\sigma(\zeta)=|f'(e^{i\theta})|\,d\theta, \quad \text{where} \quad \zeta=f(e^{i\theta}).$$

Here σ denotes arc length measure on ∂D, and $d\sigma$ denotes the arc length element on ∂D. We will also use the notations σ and $d\sigma$ for length on curves other than ∂D.

The harmonic measure for D at z is the measure $\omega^z(D;\ )$ on ∂D such that for any continuous function h on ∂D, $\int_{\partial D} h(\zeta)\,d\omega^z(D;\zeta)$ is the value at z of the harmonic function in D with boundary values h. If $z_0=f(0)$, then $\omega^{z_0}(D;E)=(1/2\pi)\int_{f^{-1}(E)} d\theta$ for any measurable $E\subset\partial D$. Hence, the density of harmonic measure with respect to arc length measure exists. In fact, let $k_{z_0}(D,\zeta)=(1/2\pi)|f'(e^{i\theta})|^{-1}$, where $\zeta=f(e^{i\theta})$, then $k_{z_0}(D,.)\in L^1(d\sigma)$ and $d\omega^{z_0}(D;\zeta)=k_{z_0}(D,\zeta)\,d\sigma(\zeta)$.

For almost every $\zeta\in\partial D$,

$$k_z(D,\zeta)=\lim \frac{\omega^z(D,I)}{\sigma(I)}$$

as the arc I shrinks to ζ. Therefore, when it exists, $k_z(D,\zeta)$ is a positive harmonic function. We state here versions of the maximum principle and Harnack's inequality.

(2) *If $D\subset D'$, then for almost every $\zeta\in\partial D\cap\partial D'$ and every $z\in D$, $k_z(D',\zeta)\geqslant k_z(D,\zeta)$.*

(3) *Suppose that $r>0$ and $D\supset z_0+2r\Delta$. For any $z,z'\in z_0+r\Delta$ and almost every $\zeta\in\partial D$, $10^{-1}k_z(D,\zeta)\leqslant k_{z'}(D,\zeta)\leqslant 10k_z(D,\zeta)$.*

(For a set E we will use the notations $rE=\{rz: z\in E\}$ and $z_0+E=\{z_0+z: z\in E\}$.) As a result, because D is connected, for any $z_1,z_2\in D$ there is a constant C depending on z_1,z_2 and D such that

(4) $C^{-1}k_{z_1}(D,\zeta)\leqslant k_{z_2}(D,\zeta)\leqslant Ck_{z_1}(D,\zeta)$ *for almost every $\zeta\in\partial D$.*

The domain D is said to have the *Smirnov property* if

$$\log f'(re^{i\theta})=\int_0^{2\pi} P_r(\theta-\chi)\log f'(e^{i\chi})\,d\chi,$$

where $P_r(\theta)=(1/2\pi)(1-r^2)/(1-2r\cos\theta+r^2)$ is the Poisson kernel. (In other words, $\log f'$ is the Poisson integral of its boundary values.) Because the Poisson formula extends to the boundary, a conformal mapping f of Δ onto a Smirnov domain satisfies an improvement of (1).

$\mu_p(r,1/f')$ is an increasing function of r for $r\leqslant 1$.

A change of variable implies that

$$\mu_p(1,\ 1/f')=(2\pi)^p\int_{\partial D} k_{z_0}(D,\zeta)^{1+p}\,d\sigma(\zeta).$$

This equality and the preceding remarks yield

PROPOSITION 1. *Fix $p>0$. Suppose that ∂D is a rectifiable Jordan curve and that f is a conformal mapping of Δ onto D. If $1/f'\in H^p$, then $k_{z_0}(D,.)\in L^{1+p}(d\sigma)$. Conversely, if D has the Smirnov property and $k_{z_0}(D,.)\in L^{1+p}(d\sigma)$, then $1/f'\in H^p$.*

PROPOSITION 2. *Suppose that ∂D is rectifiable and $1/f'\in H^p$ for some $p>0$, then D has the Smirnov property.*

Proof. Since $1/f'\in H^p$ and $f'\in H^1$,

$$\sup_{r<1}\mu_q(r,\log|f'|)<\infty \quad \text{for any } q<\infty.$$

It follows from M. Riesz' theorem that the harmonic conjugate $\arg f'(z)$ satisfies $\sup_{r<1}\mu_q(r,\arg f')<\infty$. Hence $\log f'\in H^q$ for any q and the Smirnov property follows using [10: p. 258, pp. 149–150].

The purpose for which Lavrentiev obtained Theorem 1 was the following corollary to Theorem 1 and Proposition 2.

COROLLARY. *Every chord-arc domain is a Smirnov domain.*

The example we will construct is based on the following theorem of Keldysh and Lavrentiev [5].

THEOREM 3. *There exists a domain (with rectifiable boundary) that is not a Smirnov domain.*

In light of Proposition 1 and the Corollary above, Theorem 2 is equivalent to

THEOREM 2′. *For any $p>0$ there exists a chord-arc domain D such that*

$$\int_{\partial D} k_z(D,\zeta)^{1+p}\,d\sigma(\zeta)=\infty.$$

The construction. The construction proceeds in two steps. The first step is to use Theorem 3 to obtain a piecewise linear domain Ω for which the density of harmonic measure has large $L^{1+p}(d\sigma)$ norm. The second step is to build D out of the union of affine linear transformations of Ω by an iterative procedure.

LEMMA. *For any $p>0$ and any $B<\infty$ there is a bounded domain $\Omega\subset\mathbf{C}$ with boundary given by the union of $\{z:\operatorname{Im}z=2,\ -2\leqslant\operatorname{Re}z\leqslant 2\}$, $\{z:0\leqslant\operatorname{Im}z\leqslant 2,\ \operatorname{Re}z=\pm 2\}$, and a piecewise linear simple curve $\Gamma:[-2,2]\to\mathbf{C}$ satisfying*

(a) $\sigma(\Gamma(-2,2))\leqslant 10$.

(b) $\Gamma(s)=s$ *for* $-2\leqslant s\leqslant -1$ *and* $1\leqslant s\leqslant 2$, $\Gamma(-1,1)\subset\{z:\frac{1}{4}<|z|<1,\ \operatorname{Im}z<0\}$.

(c) $\int_{\Gamma(-1,1)} k_0(\Omega,\zeta)^{1+p}\,d\sigma(\zeta)\geqslant B$.

Proof. By Theorem 3 there is a domain Ω_1 that does not have the Smirnov property. By an affine linear transformation we can assume that there exists $s>0$ such that $3s\Delta \subset \Omega_1 \subset 10^{-2}\Delta$ and $\sigma(\partial\Omega_1)\leqslant 1$. Let f_1 be the conformal mapping of Δ onto Ω_1 such that $f_1(0)=0$. By Proposition 2, $\sup_{r<1}\mu_p(r,1/f_1')=\infty$. Since $\mu_p(r,1/f_1')$ is increasing for $r<1$, there exists r_0 sufficiently close to 1 for which $f_1(r_0\Delta)\supset 2s\Delta$ and $\mu_p(r_0,1/f_1')\geqslant B'$. The constant B' will be chosen later depending only on s and B. Notice also that

$$\sigma(\partial f_1(r_0\Delta))=2\pi\mu_1(r_0,f_1')\leqslant 2\pi\mu_1(1,f_1')=\sigma(\partial\Omega_1)\leqslant 1.$$

Choose a region Ω_2 with boundary a piecewise linear simple closed curve sufficiently close to $\partial f_1(r_0\Delta)$ that $s\Delta\subset\Omega_2\subset(1/50)\Delta$, $\sigma(\partial\Omega_2)\leqslant 2$, and

$$\mu_p(1,1/f_2')\geqslant B'/10, \tag{5}$$

where f_2 is a conformal mapping of Δ onto Ω_2 such that $f_2(0)=0$. We can rewrite (5) as

$$\int_{\partial\Omega_2} k_0(\Omega_2,\zeta)^{1+p}\,d\sigma(\zeta)\geqslant\frac{(2\pi)^pB'}{10}.$$

Choose a sub arc γ of $\partial\Omega_2$ of length $s/10$ such that

$$\int_{\gamma} k_0(\Omega_2,\zeta)^{1+p}\,d\sigma(\zeta)\geqslant\frac{(2\pi)^pB's}{100}. \tag{6}$$

Let L be an isometry of $\mathbf{C}$ sending the origin to the point $-i/2$ and such that

$$\operatorname{Im} z<-i/2 \quad \text{for all } z\in L(\gamma).$$

Let

$$R_1=\{z: 0<\operatorname{Im} z\leqslant 2 \text{ and } -2<\operatorname{Re} z<2\}$$

$$R_2=\{z: -\tfrac{1}{2}<\operatorname{Im} z\leqslant 0 \text{ and } -1<\operatorname{Re} z<1\}.$$

Define $\Omega=R_1\cup R_2\cup L(\Omega_2)$. As the union of regions with piecewise linear boundary, Ω clearly has a piecewise linear boundary. The integral in (6) is unchanged by an isometry, hence by the maximum principle (2)

$$\int_{L(\gamma)} k_{-i/2}(\Omega,\zeta)^{1+p}\,d\sigma(\zeta)\geqslant\int_{L(\gamma)} k_0(L(\Omega_2),\zeta)^{1+p}\,d\sigma(\zeta)\geqslant\frac{(2\pi)^pB's}{100}.$$

The region Ω contains the unions of the discs $\frac{1}{2}\Delta$ and $-i/2+s\Delta$. Therefore, by Harnack's inequality (4) there exists $\eta>0$ depending only on s such that

$$\int_{L(\gamma)} k_0(\Omega,\zeta)^{1+p}\,d\sigma(\zeta)\geqslant\eta^{1+p}\int_{L(\gamma)} k_{-i/2}(\Omega,\zeta)^{1+p}\,d\sigma(\zeta).$$

If we choose B' sufficiently large that $(2\pi)^pB's\eta^{1+p}/100>B$, then the proof of the lemma is complete. □

In preparation for the iteration, here are two remarks concerning $k_z(D,\zeta)$. First, let $L(z)=az+z_0$, where a and z_0 are complex numbers, be an affine linear mapping of the z-plane. Harmonic measure is preserved by any such mapping, i.e., $\omega^{L(z)}(L(D); L(E)) = \omega^z(D; E)$ for any measurable $E \subset \partial D$. Also, $\sigma(L(E)) = |a|\sigma(E)$. Hence,

(7) $$k_{L(z)}(L(D), L(\zeta)) = |a|^{-1} k_z(D, \zeta).$$

Second, there is an absolute constant A such that if $\zeta \in \partial D$ and $(\zeta + 2r\Delta) \cap \partial D$ consists of a single line segment, then

(8) $$A^{-1} k_z(D, \zeta') < k_z(D, \zeta'') < A k_z(D, \zeta')$$

for $\zeta', \zeta'' \in \partial D \cap (\zeta + r\Delta)$, $z \in D \backslash (\zeta + \frac{3}{2} r\Delta)$. (For ζ' on an open line segment of ∂D, $k_z(D, \zeta')$ has a unique representative that is continuous in ζ'.) In order to prove (8), note that for the regions

$$S_1 = \{z: |z| < 2 \text{ and } \operatorname{Im} z > 0\}, \qquad S_2 = \{z: |z| > 2 \text{ or } \operatorname{Im} z > 0\}$$

harmonic measure can be calculated explicitly. In fact, S_1 is the image under linear fractional transformation of a right-angle sector, and S_2 is the image of the complement of a right-angle sector. In particular, there is an absolute constant A such that $k_z(S_2, \zeta') < \sqrt{A}\, k_z(S_1, \zeta'')$ whenever $\zeta', \zeta'' \in [-1, 1]$ and $|z| = \frac{3}{2}$, $\operatorname{Im} z > 0$. The hypothesis on D implies that there exists an affine linear mapping $L(z) = az + \zeta$ such that $|a| = r$ and $L(S_1) \subset D \subset L(S_2)$. It follows from the maximum principle that (8) is valid for $z \in D \cap \partial(\zeta + \frac{3}{2} r\Delta)$. Moreover, if $|\zeta' - \zeta| < r$ and $\zeta' \in \partial D$, then for $z \in D \backslash (\zeta + \frac{3}{2} r\bar{\Delta})$,

$$k_z(D, \zeta') = \int_{D \cap \partial(\zeta + (3/2) r\Delta)} k_{z'}(D, \zeta')\, d\omega^z(D \backslash \zeta + \tfrac{3}{2} r\bar{\Delta}; z').$$

Therefore, (8) is valid in full.

Denote $E_t = \{\zeta \in \Gamma(-1, 1): \zeta + 10^{10} t\Delta \cap \partial\Omega$ consists of a single line segment$\}$. As t tends to 0, E_t increases to $\Gamma(-1, 1)$ minus the (finite) set of endpoints of the segments of $\Gamma(-1, 1)$. Hence for sufficiently small $t > 0$,

$$\int_{E_t} k_0(\Omega, \zeta)^{1+p}\, d\sigma(\zeta) > \frac{B}{10}.$$

Choose a finite collection of intervals $I_j = (z_j + t\Delta) \cap \Gamma(-1, 1)$, $j = 1, \ldots, M$, such that $z_j \in E_t$, $\bigcup_{j=1}^{M} (z_j + 10^8 t) \supset E_t$ and $|z_j - z_k| > 10^5 t$, whenever $j \neq k$. Then remark (8) implies that

$$\omega^0(\Omega, I_j) = \int_{I_j} k_0(\Omega, \zeta)\, d\sigma(\zeta) \geqslant A^{-1} \sup\{k_0(\Omega, \zeta) : \zeta \in (z_j + 10^8 t\Delta) \cap \partial\Omega\}\, \sigma(I_j).$$

Hence,

(9) $$\sum_{j=1}^{M} \left(\frac{\omega^0(\Omega; I_j)}{\sigma(I_j)} \right)^{1+p} \sigma(I_j) \geqslant 10^{-8} A^{-(1+p)} \int_{E_t} k_0(\Omega, \zeta)^{1+p}\, d\sigma(\zeta)$$
$$\geqslant 10^{-9} A^{-(1+p)} B.$$

The chord-arc domain D is constructed as follows.

Let $D_1=\Omega$. Let $\Omega_j=L_j(\Omega)$, where $L_j(z)=a_jz+z_j$ is an affine linear mapping such that $|a_j|=10t$ and $L_j(\Gamma(-1,1))$ is exterior to D_1. This specifies L_j uniquely. Denote $I=\{z: z \text{ is real and } -1/10<z<1/10\}$. Notice that $I_j=L_j(I)$. Let $D_2=D_1\cup\bigcup_{j=1}^{M}\Omega_j$. We define by induction $L_{j_1\ldots j_l}=L_j\circ L_{j_1\ldots j_{l-1}}$ where $j_\alpha=1,\ldots,M$ and $\alpha=1,\ldots,l$. Denote $\Omega_{j_1\ldots j_l}=L_{j_1\ldots j_l}(\Omega)$ and inductively $D_{l+1}=D_l\cup\bigcup_{j_1\ldots j_l}\Omega_{j_1\ldots j_l}$. Let $I_{j_1\ldots j_l}=L_{j_1\ldots j_l}(I)$ and $z_{j_1\ldots j_l}=L_{j_1\ldots j_l}(0)$. Finally, let $D=\bigcup_{l=1}^{\infty}D_l$.

Because the points z_j are at distances greater than 10^5t from each other, $M\leqslant(10^5t)^{-1}\sigma(\Gamma(-1,1))<10^{-4}t^{-1}$. Furthermore, $\sigma(L_j(\Gamma(-1,1)))\leqslant 100t$. Hence, $\sigma(\partial D_2)-\sigma(\partial D_1)\leqslant M(100t)\leqslant 10^{-2}$. Similarly, $\sigma(\partial D_{l+1})-\sigma(\partial D_l)\leqslant 10^{-2l}$. Thus, $\sigma(\partial D)\leqslant\sigma(\partial D_1)+\sum_{l=1}^{\infty}10^{-2l}$ is finite. Because $z_j\in E_t$, the same reasoning applies to any segment of $\partial\Omega$. Consequently, $\sigma(\partial D;\ \zeta,\zeta')\leqslant 20\sigma(\partial\Omega;\ \zeta,\zeta')$ for any pair of points ζ and ζ' belonging to $\partial D\cap\partial\Omega$. The region $\Omega=D_1$ is a chord-arc domain because its boundary consists of finitely many segments. In order to prove that D is a chord-arc domain we consider three cases.

Case 1. *ζ and ζ' belong to $\partial D\cap\partial D_1$.* If C is the chord-arc constant for Ω then

$$\sigma(\partial D;\ \zeta,\zeta')\leqslant 20\sigma(\partial\Omega;\ \zeta,\zeta')\leqslant 20C|\zeta-\zeta'|.$$

Case 2. *$\zeta\in(\partial D_1)\cap\partial D$ and $\zeta'\in(\partial D_2\backslash\partial D_1)\cap\partial D$.*

(a) *$\zeta'\in\partial\Omega_j$ and $\zeta\in\partial\Omega_j$.* This reduces by an affine linear transformation to Case 1.

(b) *$\zeta\in(\partial D_1)\backslash\partial\Omega_j$ and $\zeta'\in L_j(\Gamma(-1,1)$.* Then $|\zeta-\zeta'|>10^{-2}\sigma(\partial\Omega_j)$. We also have $|z_j-\zeta|\leqslant 10|\zeta-\zeta'|$. By the method of Case 1,

$$\sigma(\partial D;\ \zeta,\zeta')\leqslant 20(\sigma(\partial\Omega_j)+\sigma(\partial\Omega;\ z_j,\zeta)).$$

Hence, $\sigma(\partial D;\ \zeta,\zeta')\leqslant 10^4C|\zeta-\zeta'|$, where C is the chord-arc constant for Ω.

Case 3. *$\zeta\in(\partial D_1)\cap\partial D$ and $\zeta'\in(\partial D_{k+1}\backslash\partial D_k)\cap\partial D$ for some $k\geqslant 2$.* The proof here is essentially the same as in Case 2(b).

Because $z_j\in E_t$, the shorter arc of ∂D between any two points of ∂D can be transformed by an affine linear mapping to a configuration like one of the three preceding cases.

$$\begin{aligned}\omega^0(D_l;I_{j_1\ldots j_l})&=\int_{\partial D_{l-1}}\omega^z(D_l;I_{j_1\ldots j_l})\,d\omega^0(D_{l-1};z)\\&\geqslant\int_{I_{j_1\ldots j_{l-1}}}\omega^z(D_l;I_{j_1\ldots j_l})\,d\omega^0(D_{l-1};z)\\&\geqslant\frac{1}{10}\,\omega^{z_{j_1\ldots j_{l-1}}}(D_l;I_{j_1\ldots j_l})\int_{I_{j_1\ldots j_{l-1}}}d\omega^0(D_{l-1};z)\\&\geqslant\frac{1}{10}\,\omega^{z_{j_1\ldots j_{l-1}}}(\Omega_{j_1\ldots j_{l-1}},I_{j_1\ldots j_l})\,\omega^0(D_{l-1};I_{j_1\ldots j_{l-1}}),\end{aligned}$$

by Harnack's inequality (3) and the maximum principle (2). Therefore, using (9) and (7)

$$\sum_{j_l=1}^{M}\left(\frac{\omega^0(D_l; I_{j_1\dots j_l})}{\sigma(I_{j_1\dots j_l})}\right)^{1+p}\sigma(I_{j_1\dots j_l})$$
$$\geqslant 10^{-(1+p)}\left(\sum_{j_l=1}^{M}\frac{\omega^{z_{j_1\dots j_{l-1}}}(\Omega_{j_1\dots j_{l-1}}; I_{j_1\dots j_l})}{\sigma(I_{j_1\dots j_l})}\omega^0(D_{l-1}; I_{j_1\dots j_{l-1}})\right)^{1+p}\sigma(I_{j_1\dots j_l})$$
$$\geqslant 10^{-(1+p)}\omega^0(D_{l-1}; I_{j_1\dots j_{l-1}})^{1+p}10^{-9}A^{-(1+p)}B(10t)^{-p(l-1)}$$
$$=10^{-(1+p)-9}A^{-(1+p)}B\sigma(I)^p\left(\frac{\omega^0(D_{l-1}; I_{j_1\dots j_{l-1}})}{\sigma(I_{j_1\dots j_{l-1}})}\right)^{1+p}\sigma(I_{j_1\dots j_{l-1}}).$$

Choose B so that $10^{-(1+p)-9}A^{-(1+p)}B5^{-p}>10$. Then repeating the inequality above for $l-1$, $l-2$, etc. we find that

$$\sum_{j_1\dots j_l=1}^{M}\left(\frac{\omega^0(D_l; I_{j_1\dots j_l})}{\sigma(I_{j_1\dots j_l})}\right)^{1+p}\sigma(I_{j_1\dots j_l})\geqslant 10^l.$$

Let $J=\{z: z$ is real and $1<z<6/5\}$. Denote $J_{j_1\dots j_l}=L_{j_1\dots j_l}(J)$. Notice that $J_{j_1\dots j_l}\subset\partial D$. The same proof as above shows that

$$\sum_{j_l=1}^{M}\left(\frac{\omega^0(D; J_{j_1\dots j_l})}{\sigma(J_{j_1\dots j_l})}\right)^{1+p}\sigma(J_{j_1\dots j_l})\geqslant 10\left(\frac{\omega^0(D_{l-1}; I_{j_1\dots j_{l-1}})}{\sigma(I_{j_1\dots j_{l-1}})}\right)^{1+p}\sigma(I_{j_1\dots j_{l-1}}).$$

Hence,

$$\sum_{j_1\dots j_l=1}^{M}\left(\frac{\omega^0(D; J_{j_1\dots j_l})}{(J_{j_1\dots j_l})}\right)^{1+p}\sigma(J_{j_1\dots j_l})\geqslant 10^l.$$

Let $E_l=\bigcup_{j_1\dots j_l}J_{j_1\dots j_l}$. By Hölder's inequality $\int_{E_l}k_0(D,\zeta)^{1+p}\,d\sigma(\zeta)\geqslant 10^l$. Since l can be arbitrarily large, the proof is complete. □

Final remarks. When $p=1$, the domain Ω can be constructed easily without recourse to Theorem 3. We need only make use of the exterior of a very narrow sector of the plane. For instance, for $x\geqslant 0$, define

$$\begin{aligned} g(x) &= -1/4-x/4\epsilon, && 0\leqslant x\leqslant\epsilon \\ &= -1/2-\epsilon+x, && \epsilon\leqslant x\leqslant 1/2+\epsilon \\ &= 0, && 1/2+\epsilon\leqslant x, \end{aligned}$$

and define $g(x)=g(-x)$ for $x<0$. Let $\Omega=\{z\in\mathbf{C}: -2<\operatorname{Re}z<2$ and $g(\operatorname{Re}z)<\operatorname{Im}z<2\}$. Then Ω satisfies the lemma when ϵ is sufficiently small. Thus, in this case the construction is very explicit.

As a result of the scale invariance of the chord-arc condition, Lavrentiev was able to localize his theorem as follows. For some $p>0$, the inequality

$$\left(\frac{1}{\sigma(I)}\int_I k(\zeta)^{1+p}\,d\sigma(\zeta)\right)^{1/(1+p)}<A\frac{1}{\sigma(I)}\int_I k(\zeta)\,d\sigma(\zeta) \tag{10}$$

holds for any arc I of ∂D with a constant A independent of I. This condition is the same as the condition that arc length measure $d\sigma$ belongs to the Muckenhoupt class A_q with respect to harmonic measure $kd\sigma$, in which $1/(p+1)+1/q=1$ [3].

This stronger form of Lavrentiev's theorem is the one that is important to the Cauchy integral on curves.

In the first proof of bounds for the Cauchy integral on Lipschitz curves, A. P. Calderón [2] makes use of the L^∞ bound on arg $f'(z)$ in order to prove (10) with $p=1$. This technique also appears in Lavrentiev's work and [10: p. 295, Example 1]. It would be interesting to understand better the interplay between the chord-arc condition and bounds on arg $f'(z)$ in their contribution to the validity of (10).

REFERENCES

1. A. Baernstein, *Curves satisfying the chord-arc condition,* Abstracts Amer. Math. Soc. 792-30-77, Jan. 1982.
2. A. P. Calderón, *Cauchy integrals on Lipschitz curves and related operators,* Proc. Nat. Acad. Sci. U.S.A. 74 (1977), no. 4, 1324–1327.
3. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals,* Studia Math. 51 (1974), 241–250.
4. R. Coifman and Y. Meyer, *Le théorème de Calderón par les "méthodes de variable réelle",* C.R. Acad Sci. Paris Ser. A–B 289 (1979), A425–A428.
5. P. Duren, *The theory of H^p spaces,* Academic Press, New York, 1970.
6. D. Jerison and C. Kenig, *Hardy spaces, A_∞, and singular integrals on chord-arc domains,* Math Scand., to appear.
7. P. Jones and M. Zinsmeister, to appear.
8. M. Lavrentiev, *Boundary problems in the theory of univalent functions,* Math. Sb. (N.S.) 1(43) 1936, 815–844. English translation, Amer. Math. Soc. Transl., Ser. 2, 32 (1963), 1–35.
9. C. Pommerenke, *Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation,* Comment. Math. Helv. 52 (1977), 591–602.
10. A. Zygmund, *Trigonometric series,* Vol. I. Cambridge Univ. Press, New York, 1959.

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