THE EXTENSION OF CR FUNCTIONS TO ONE SIDE OF A SUBMANIFOLD OF $\mathbf{C}^n$

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1. Introduction. There has been a lot of research on the problem of extending CR functions on a submanifold of $\mathbf{C}^n$ to CR functions on a submanifold with one higher dimension. The most comprehensive work in this area has been done by Hill and Taiani [2]. In that paper and later papers ([3] and [4]) by the same authors, they show that if the Leviform of a submanifold M is nonzero, then there is a local submanifold $\tilde{M}$ of one higher dimension with boundary $(\tilde{M})=M$ such that CR functions on M extend to CR functions on $\tilde{M}$. In their papers, the manifold $\tilde{M}$ is roughly one-third as smooth as M. In this paper, we present an easier proof of their result with the improvement that our $\tilde{M}$ is roughly one-half as smooth as M. In the Hill and Taiani papers, the normal direction to M which lies tangent to $\tilde{M}$ lies in the image of the Leviform of M. Here, we also make the improvement to allow this normal direction to be in the convex hull of the image of the Leviform of M.

As in [4], we obtain $\tilde{M}$ using the technique of analytic discs. However, our approach differs in that we sweep out $\tilde{M}$ with the centers of analytic discs and then we use the CR approximation result of Baouendi and Trèves [1] to actually obtain the CR extension. Our approach avoids the delicate problem of examining how the discs attach to the original manifold M.

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2. Statement of the result. We shall think of the Leviform of M at a point $p\in M$ as a map from the holomorphic tangent space of M at p into the space of vectors which are normal to M at p. If M is graphed over its tangent space at p, then the Leviform can be identified with the restriction of the complex hessian of the (vector valued) graphing function of M at p to the holomorphic tangent space of M at p.

Here is the precise statement of our theorem.

THEOREM 2.1. *Let M be a generic* CR*-submanifold of $\mathbf{C}^n$ of class at least C^{2k+1} $(k\geqslant 1)$. Let $p\in M$ and $v\neq 0$ a normal vector which lies in the convex hull of the image of the Leviform of M at p. Then there is a generic submanifold $\tilde{M}$ with boundary such that*

(i) *$\tilde{M}$ is of class C^k and* $\dim_{\mathbf{R}}\tilde{M}=(\dim_{\mathbf{R}}M)+1$,

(ii) *the boundary of $\tilde{M}$ is an open neighborhood of p in M,*

(iii) *the tangent cone of $\tilde{M}$ at p is spanned by v and the tangent space of M at p,*

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(iv) *Given a neighborhood $\omega \ni p$ in M, there is a neighborhood $\tilde{\omega} \ni p$ relative to $\tilde{M}$ such that each* CR *function on ω of class C^j $(j \geq 0)$ extends to a* CR *function on $\tilde{\omega}$ of class $C^{\tilde{j}}$ where $\tilde{j} = \min\{j, k\}$.*

The proof of the above theorem will also show that the interior of $\tilde{M}$ is at least of class C^{2k}.

3. Analytic discs. An analytic disc is a holomorphic map $A: D \to \mathbf{C}^n$ (D = open unit disc in $\mathbf{C}$) which is continuous on $\bar{D}$. Without apologies, the analytic disc shall be identified with its image in $\mathbf{C}^n$. The boundary of an analytic disc A is by definition the restriction of A to $S^1 :=$ the unit circle. The boundary of A shall also be identified with its image in $\mathbf{C}^n$. The desired $\tilde{M}$ will be swept out by the centers of analytic discs (i.e. $A(\zeta=0)$) with boundaries contained in M.

Let us review the construction of an analytic disc with boundary in M. Suppose $u: S^1 \to \mathbf{R}^d$ is a continuous function. Then u is the boundary values of a unique harmonic function $U: \bar{D} \to \mathbf{R}^d$ ($U = (U_1, \ldots, U_d)$ with each U_j harmonic). The Hilbert transform of u (denoted $Tu: S^1 \to \mathbf{R}^d$) is the boundary values of the function $V = (V_1, \ldots, V_d): \bar{D} \to \mathbf{R}^d$ where each V_j is the unique harmonic conjugate of U_j with $V_j(0) = 0$. Thus the function $u + iTu: S^1 \to \mathbf{C}^d$ is the boundary values of a unique analytic disc $G = U + iV: \bar{D} \to \mathbf{C}^d$ with $V(\zeta=0) = 0$.

Let us assume the submanifold M of $\mathbf{C}^n$ (in Theorem 1.1) has real codimension $d > 1$ ($d = 1$ is the classical Hans Lewy extension phenomenon [5]). Suppose $n = d + m$ and that $\mathbf{C}^{d+m}$ has coordinates (z, w), $z = x + iy \in \mathbf{C}^d$, $w \in \mathbf{C}^m$. We can assume the given point $p \in M$ is the origin and that locally $M = \{(x+iy, w); x = h(y, w)\}$ where $h: \mathbf{R}^d \times \mathbf{C}^m \to \mathbf{R}^d$ is of class at least C^{2k+1} with $h(0) = 0$ and $\nabla h(0) = 0$.

If $v: S^1 \to \mathbf{R}^d$, $W: S^1 \to \mathbf{C}^m$, we define $H(v, W): S^1 \to \mathbf{R}^d$ by $H(v, W)(e^{i\phi}) = h(v(e^{i\phi}), W(e^{i\phi}))$.

Now suppose $W: \bar{D} \to \mathbf{C}^m$ is a given analytic disc and let $y \in \mathbf{R}^d$ be a given vector. If $v: S^1 \to \mathbf{R}^d$ satisfies Bishop's equation, i.e.

$$v(e^{i\phi}) = T(H(v, W))(e^{i\phi}) + y, \tag{3.1}$$

then by the above discussion on the Hilbert transform, the function

$$H(v, W)(\cdot) + iv(\cdot): S^1 \to \mathbf{C} \tag{3.2}$$

is the boundary values of a unique analytic disc $G = U + iV: \bar{D} \to \mathbf{C}^d$ with $V(\zeta=0) = y$. Since the boundary values of G are given by (3.2), it is clear that $\operatorname{Re} G(\zeta) = h(\operatorname{Im} G(\zeta), W(\zeta))$ for $|\zeta| = 1$. Thus the boundary of the analytic disc $A := (G, W): \bar{D} \to \mathbf{C}^{d+m}$ must lie in M.

We need the following existence and uniqueness result on the solution to Bishop's equation.

THEOREM 3.3. *Suppose the function h is of class C^{2k+1} with $h(0) = 0$, $\nabla h(0) = 0$. There is a neighborhood $\mathcal{U}$ of the origin in the space $C^1(S^1)$ and a neighborhood $\mathcal{Y} \subset \mathbf{R}^d$ of the origin and a map $v: \mathcal{U} \times \mathcal{Y} \to C^0(S^1)$ of class C^{2k}*

such that for each $W \in \mathcal{U}$, $y \in \mathcal{Y}$, *the function* $v(W,y)(\cdot)$ *is the unique solution to Bishop's equation* (3.1). *Moreover*

$$|v(W,y)|_{C^0(S^1)} \leqslant C(|W|_{C^1(S^1)} + |y|) \tag{3.4}$$

where $|\cdot|_{C^k(S^1)}$ *is the usual* C^k*-norm for functions on* S^1 *and where C is a fixed constant.*

This result is contained in Theorem 5.1 in [2]. However, Theorem 5.1 in [2] contains much more information (which is unnecessary for us) than the result stated above. The above result can be easily shown as follows. Let $W^1(S^1)$ be the first Sobolev space of functions on S^1 with the usual Sobolev norm $|\cdot|_{W^1(S^1)}$. Define the function $F\colon W^1(S^1) \times W^1(S^1) \times \mathbf{R}^d \to W^1(S^1)$ by

$$F(v, W, y) = v - T(H(v, W)) - y.$$

Now, tangential derivatives to S^1 commute with T. Since h is C^{2k+1}, it is then easy to show that F is a C^{2k} map (in the sense of Banach spaces). Note that we lose a derivative since the domain and range of F involve first Sobolev spaces. Clearly $F(0,0,0)=0$. Since $\nabla h(0)=0$, the Banach space derivative $\partial F(0,0,0)/\partial v$ is the identity map $I\colon W^1(S^1) \to W^1(S^1)$. The result now follows from the implicit function theorem. The estimate (3.4) follows from $v(0,0)=0$ and Sobolev's estimate $|f|_{C^0(S^1)} \leqslant C|f|_{W^1(S^1)}$.

4. Construction of $\tilde{M}$. To construct $\tilde{M}$, it will be useful to arrange holomorphic coordinates $(z=x+iy, w)$ for $\mathbf{C}^d \times \mathbf{C}^m$ in a standard way so that

$$\frac{\partial^2 h}{\partial y_i \partial y_j}(0) = 0 \quad 1 \leqslant i,\ j \leqslant d \qquad \text{and} \qquad \frac{\partial^2 h}{\partial y_i \partial w_j}(0) = 0 \quad 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant m.$$

As stated in the introduction, the Leviform at 0 can be identified with the following map $\mathcal{L}\colon \mathbf{C}^m \to \mathbf{R}^d$ given by

$$\mathcal{L}(\alpha) := \sum_{j,k=1}^{m} \frac{\partial^2 h}{\partial w_j \partial \bar{w}_k}(0) \alpha_j \bar{\alpha}_k$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbf{C}^m$. The convex hull of the image of $\mathcal{L}$ is the set of all vectors of the form $\sum_{l=1}^{e} s_l \mathcal{L}(\tilde{\alpha}^l)$ where $e>0$, $0 \leqslant s_l \leqslant 1$ with $\sum_{l=1}^{e} s_l = 1$ and $\tilde{\alpha}^l \in \mathbf{C}^m$. Since $s_l \mathcal{L}(\tilde{\alpha}^l) = \mathcal{L}(\sqrt{s_l}\,\tilde{\alpha}^l)$, the hypothesis of Theorem 2.1 on the Leviform of M at 0 means that for some vector $\nu \in \mathbf{R}^d$, there exist vectors $\alpha^1, \ldots, \alpha^e \in \mathbf{C}^m$ with

$$\nu = \sum_{l=1}^{e} \mathcal{L}(\alpha^l) \tag{4.1}$$

Let $W\colon \mathbf{C}^m \times \mathbf{R} \times \bar{D} \to \mathbf{C}^m$ be defined by

$$W(w,t)(\zeta) := w + t(\alpha^1 \zeta + \alpha^2 \zeta^3 + \alpha^3 \zeta^5 + \cdots + \alpha^e \zeta^{2e-1}) \tag{4.2}$$

where $w \in \mathbf{C}^m$, $t \in \mathbf{R}$, $\zeta \in \bar{D}$. W is an analytic disc which also depends smoothly on parameters w and t. Clearly there is an $\epsilon_0 > 0$ such that if $\max\{|w|, |y|, |t|\} < \epsilon_0$, then $y\in\mathcal{Y}$ and $W(w,t)(\cdot)|_{S^1}$ belongs to $\mathcal{U}\subset C^1(S^1)$ where $\mathcal{U}$ and $\mathcal{Y}$ are as in Theorem 3.3. Using Theorem 3.3, we obtain a solution $v(y, w, t)(e^{i\phi})$ to Bishop's equation (3.1) which depends on the parameters y, w, t in a C^{2k} fashion. Thus for each y, w, t, we obtain an analytic disc $G := U+iV: \bar{D}\to \mathbf{C}^d$ such that if $A := (G, W): \bar{D}\to \mathbf{C}^{d+m}$, then the boundary of A is contained in M and $\operatorname{Im} G(\zeta=0) = V(\zeta=0) = y$. Since V depends on y, w, t in a C^{2k} fashion, and since

$$U(y,w,t)(\zeta)=H(V(y,w,t),W(w,t))(\zeta)\quad \text{for } |\zeta|=1,$$

clearly U also depends on y, w, t in a C^{2k} fashion.

The manifold $\tilde{M}$ will be swept out by the map

$$(y,w,t)\to A(y,w,t)(\zeta=0)=\begin{pmatrix} U(y,w,t)(\zeta=0)+iy \\ w\end{pmatrix}$$

for $\max\{|y|,|w|,|t|\}<\epsilon_0$. Since $U(y,w,t)(\zeta)$ is harmonic in $\zeta\in D$, we have

$$\text{(4.3)}\qquad \begin{aligned} U(y,w,t)(\zeta=0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(y,w,t)(e^{i\phi})\,d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(V(y,w,t)(e^{i\phi}), W(w,t)(e^{i\phi}))\,d\phi \end{aligned}$$

where the last equality uses the fact that the boundary of A is contained in M.

Let us Taylor expand $U(y,w,t)(\zeta=0)$ in t about $t=0$. From (4.2), clearly $W(w,t=0)(\zeta)=w$. By the uniqueness part of Theorem 3.3, clearly

$$V(y,w,t=0)(\zeta)=y,\quad |\zeta|=1.$$

Therefore from (4.3), the constant term in the Taylor expansion (i.e., $U(y,w,t=0)(\zeta=0)$) is just $h(y,w)$.

The coefficient of the linear term in t is

$$\text{(4.4)}\qquad \begin{aligned} \frac{\partial}{\partial t}\{U(y,w,t)(\zeta=0)\}\Big|_{t=0} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial h}{\partial y}(y,w)\cdot\frac{\partial V}{\partial t}(e^{i\phi})\Big|_{t=0}d\phi \\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\operatorname{Re}\left\{\frac{\partial h}{\partial w}(y,w)\cdot\frac{\partial W}{\partial t}(e^{i\phi})\right\}d\phi. \end{aligned}$$

Here and below, the parameters y, w, t occurring in V and W will often be suppressed. In the above, $(\partial h/\partial y)(y,w)$ represents the $d\times d$ matrix $(\partial h_i/\partial y_j)(y,w)$ and $(\partial h/\partial w)(y,w)$ represents the $d\times m$ matrix $(\partial h_i/\partial w_j)(y,w)$. Now $(\partial V/\partial t)(\zeta)|_{t=0}$ is harmonic in $\zeta\in D$. Since $V(\zeta=0)=y$, $\forall t$, clearly $(\partial V/\partial t)(\zeta=0)|_{t=0}=0$. Thus, the first integral on the right side of (4.4) vanishes by the mean value property for harmonic functions. Similarly, the second integral on the right of (4.4) vanishes because $(\partial W/\partial t)(\zeta)$ is holomorphic in ζ and vanishes at $\zeta=0$ (see (4.2)). Thus the linear term in t vanishes. In fact, the Taylor expansion in t of $U(y,w,t)(\zeta=0)$ contains only even powers of t according to the next lemma.

LEMMA 4.5. $U(y,w,-t)(\zeta=0)=U(y,w,t)(\zeta=0)$. *In particular,*

$$\frac{\partial^{2j-1}}{\partial t^{2j-1}}\{U(y,w,t)(\zeta=0)\}\Big|_{t=0}=0 \quad \text{for} \quad j=1,2,3,\ldots,k.$$

Proof. Fix $-\pi\leqslant\theta\leqslant\pi$ and define $P_\theta: C^0(S^1)\to C^0(S^1)$ by the following. If $v: S^1\to\mathbf{R}^d$, then $(P_\theta v)(\zeta)=v(e^{i\theta}\zeta)$, $|\zeta|=1$. We claim that $P_\theta\circ T=T\circ P_\theta$. This is an easy consequence of the fact that if $(U+iV)(\zeta)$ is holomorphic on D with $V(\zeta=0)=0$, then $(\tilde{U}+i\tilde{V})(\zeta):=(U+iV)(e^{i\theta}\zeta)$ is holomorphic on D with $\tilde{V}(\zeta=0)=0$.

In addition, if $v: S^1\to\mathbf{R}^d$ and $W: S^1\to\mathbf{C}^m$, then clearly $P_\theta\circ H(v,W)=H(P_\theta v, P_\theta W)$ where H was defined in Section 3. Thus, if we apply P_θ to Bishop's equation (3.1) we obtain $P_\theta v=T(H(P_\theta v, P_\theta W))+y$. We see that $P_\theta v$ is the unique solution to Bishop's equation (3.1) with W replaced by $P_\theta W$. We now let $\theta=\pi$. Clearly $P_\pi v(y,w,t)(\zeta)=v(y,w,t)(e^{i\pi}\zeta)$. In addition, $P_\pi W(w,t)(\zeta)=W(w,t)(-\zeta)=W(w,-t)(\zeta)$ where the last equality uses the fact that only odd powers of ζ occur in the formula for W (see (4.2)). However, we note that $v(y,w,-t)(\zeta)$ is the unique solution to Bishop's equation with $W=W(w,-t)(\zeta)$. Therefore we obtain $v(y,w,t)(e^{i\pi}\zeta)=v(y,w,-t)(\zeta)$. Now we easily compute

$$\begin{aligned} U(y,w,-t)(\zeta=0)&=\frac{1}{2\pi}\int_{-\pi}^{\pi} h(v(y,w,-t)(e^{i\phi}), W(w,-t)(e^{i\phi}))\,d\phi\\ &=\frac{1}{2\pi}\int_{-\pi}^{\pi} h(v(y,w,t)(e^{i(\phi+\pi)}), W(w,t)(e^{i(\phi+\pi)}))\,d\phi\\ &=U(y,w,t)(\zeta=0)\end{aligned}$$

where the last equality uses the fact that the integrand is periodic of period 2π. This completes the lemma. □

Now we continue the proof of Theorem 2.1. Let $\dot{\mathbf{R}}^+=\{s\in\mathbf{R},\ s\geqslant 0\}$. Define $f: \mathbf{R}^d\times\mathbf{C}^m\times\dot{\mathbf{R}}^+\to\mathbf{R}$ by

$$f(y,w,s)=U(y,w,\sqrt{s})(\zeta=0).$$

In view of Lemma 4.5 and since U is C^{2k} in y, w, t, and easy argument which we omit shows that f is of class C^k up to $s=0$ (f is of class C^{2k} for $s>0$). Now define $F: \mathbf{R}^d\times\mathbf{C}^m\times\dot{\mathbf{R}}^+\to\mathbf{C}^{d+m}$ by

$$\begin{aligned} F(y,w,s)&=A(y,w,\sqrt{s})(\zeta=0)\\ &=\begin{pmatrix} U(y,w,\sqrt{s})(\zeta=0)+iy\\ w\end{pmatrix}=\begin{pmatrix} f(y,w,s)+iy\\ w\end{pmatrix},\end{aligned}$$

and we let $\tilde{M}=\{F(y,w,s);\ s\geqslant 0 \text{ and } \max\{|y|,|w|,\sqrt{s}\}<\epsilon_0\}$. We shall show that M satisfies (i)–(iv) in Theorem 2.1.

Since the constant term in the expansion of $U(y,w,t)(\zeta=0)$ is $h(y,w)$, it is clear that the map $(y,w)\to F(y,w,s=0)=(h(y,w)+iy, w)$ for $\max\{|y|,|w|\}<\epsilon_0$ parameterizes an open set in M which forms the boundary of M. Thus (ii) holds.

To show (i) and (iii) and that M is a manifold with boundary, we shall examine the linear term in s of $F(y=0, w=0, s)$, which in turn means that we must examine the quadratic term in t of $U(y=0, w=0, t)(\zeta=0)$. We differentiate (4.3) twice with respect to t. At the beginning of this section, we arranged coordinates so that

$$\frac{\partial^2 h}{\partial y_i \partial y_j}(0)=\frac{\partial^2 h}{\partial y_i \partial w_j}(0)=0.$$

In addition,

$$\int_{-\pi}^{\pi} \operatorname{Re}\left\{\frac{\partial W_j}{\partial t}(e^{i\phi})\frac{\partial W_k}{\partial t}(e^{i\phi})\right\} d\phi=0, \quad 1 \leqslant j,\ k \leqslant m$$

because the integrand is the boundary values of a holomorphic function on D which vanishes at the origin. Therefore

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial t^2}\{U(y=0, w=0, t)(\zeta=0)\}\Big|_{t=0} \\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j,k=1}^{m}\frac{\partial^2 h}{\partial w_j \partial \bar{w}_k}(0)\frac{\partial W_j}{\partial t}(e^{i\phi})\overline{\frac{\partial W_k}{\partial t}}(e^{i\phi})\, d\phi \\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{L}\left(\frac{\partial W}{\partial t}(e^{i\phi})\right) d\phi.
\end{aligned}$$

Since

$$\frac{\partial W}{\partial t}(\zeta)=\alpha^1\zeta+\alpha^2\zeta^3+\cdots+\alpha^e\zeta^{2e-1},$$

an easy computation shows that the above integral becomes $\sum_{j=1}^{e}\mathcal{L}(\alpha^j)=\nu$. Therefore

$$\begin{aligned}
f(y=0, w=0, s)&=U(y=0, w=0, \sqrt{s})(\zeta=0) \\
&=\nu s+\mathcal{O}(s^2).
\end{aligned}$$

In view of the above and the fact that $f(y, w, s=0)=h(y, w)$ with $\nabla h(0)=0$, clearly,

$$(dF)(y=0, w=0, s=0)=\left[\begin{array}{c|c|c} \nu & & \\ \hline & I_d & \\ \hline & & I_{2m} \end{array}\right]$$

where dF represents the real Jacobian in y, w, s and $I_j=j\times j$ identity matrix. Thus $(dF)(0,0,0)$ has maximal rank and (i) and (iii) are also satisfied.

Finally to show (iv), we use the following CR approximation theorem of Baouendi and Trèves (cf. Theorem 2.1 in [1]).

THEOREM 4.6. *Suppose M is a* CR *submanifold of $\mathbf{C}^n$ of class C^N $(N \geqslant 2)$. Let $p \in M$ and $\omega \ni p$ an open set in M. Then there exists an open set ω_1 in M with $p \in \omega_1 \subset \omega$ such that each* CR *function of class C^j $(j<N)$ on ω can be approximated in the C^j norm on ω_1 by a sequence of entire functions.*

The above theorem is proved for the sup norm case ($j=0$) in [1], but the version stated here with derivatives is no harder to prove (cf. also [3]).

Let $\omega \ni 0$ be the given open neighborhood in (iv) in Theorem 2.1. Let ω_1 be the corresponding neighborhood in Theorem 4.6. From (4.2), it is clear that $|W(w,t)(\cdot)|_{C^1(S^1)} \leq C(|w|+|t|)$. From the estimate (3.4) we have

$$|v(y,w,t)(\cdot)|_{C^0(S^1)} \leq C(|y|+|w|+|t|).$$

The function U must also satisfy the above estimate since $U(y,w,t)(e^{i\phi}) = h(v(y,w,t)(e^{i\phi}), W(w,t)(e^{i\phi}))$. Therefore, there must exist an $\epsilon>0$ such that the boundary of the analytic disc $\zeta \to A(y,w,\sqrt{s})(\zeta)$ lies in ω_1 provided that $\max\{|y|,|w|,\sqrt{s}\} < \epsilon$. Thus the map $(y,w,s) \mapsto F(y,w,s) = A(y,w,\sqrt{s})(\zeta=0)$ for $\max\{|y|,|w|,\sqrt{s}\}<\epsilon$, $s \geq 0$, parameterizes an open set $\tilde{\omega}$ relative to $\tilde{M}$ such that each point in $\tilde{\omega}$ lies in the center of an analytic disc with boundary in ω_1.

Now suppose g is a CR function on ω of class C^j, $j<2k+1$. By Theorem 4.6, there is a sequence of entire functions $\{G_m\}_{m=1}^{\infty}$ which converges to g in the C^j norm on ω_1. Let D^α, $|\alpha| \leq j$, be a differential operator on $\mathbf{C}^n$ with constant coefficients. Since M is generic, the Cauchy Riemann equations imply that the sequence $\{D^\alpha G_m|_{\omega_1}\}$ is uniformly convergent on ω_1. Since each point in $\tilde{\omega} \subset \tilde{M}$ lies in the image of an analytic disc with boundary in ω_1, the maximum principle implies that the sequence $\{G_m|_{\tilde{\omega}}\}$ converges to a function G on $\tilde{\omega}$ in the $C^{\tilde{j}}$ norm where $\tilde{j} = \min\{j,k\}$. Clearly G is CR and of class $C^{\tilde{j}}$ and $G|_{\omega_1} = g$. This proves (iv) and concludes the proof of Theorem 2.1. □

REMARK. We lose roughly one-half of the derivatives because we compose A with $\sqrt{s}$ to get our parameterization (F) of $\tilde{M}$. We have to compose with $\sqrt{s}$ to make sure the differential of F has maximal rank. In order to lose fewer derivatives, one must construct an (apparently very different) analytic disc so that its differential with respect to its parameter set has maximal rank.

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