SMALL FRACTIONAL PARTS OF THE SEQUENCE αn^k

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1. INTRODUCTION

Let k be a natural number, $k \geq 2$, and let $K = 2^{k-1}$. Denote by $\|\ldots\|$ the distance to the nearest integer. Let $N > c_1(k,\epsilon)$ where $\epsilon > 0$, then

$$\min_{1 \leq n \leq N} \|\alpha n^k\| < N^{-(1/K)+\epsilon} \tag{1}$$

for any real α. This was proved by Heilbronn [5] for $k = 2$ and extended to $k \geq 3$ by Danicic [2]. For extensions of (1) see W. M. Schmidt [6] and R. C. Baker [1]. Schmidt shows [6] that $-(1/K) + \epsilon$ can be replaced in (1) by the sharper exponent $-1/(8k^2 \log k + 4k^2 \log\log k + 11.2\, k^2)$ for $k \geq 14$ and $N > c_2(k)$.

It follows from (1) that

$$\|\alpha n^k\| < n^{-(1/K)+\epsilon} \tag{2}$$

for infinitely many natural numbers n. The exponent $-(1/K) + \epsilon$ in (2) can be replaced by $-(1/L)$, where $L = (8k(\log k + 1) \log (k \log k + 1))/\log k$. This is a special case of a theorem of I. M. Vinogradov [8, Chapter V]. However, it is by no means clear from Vinogradov's argument that $-(1/K) + \epsilon$ can be replaced by $-(1/L)$ in (1).

In the present note, I show that for any real α,

$$\min_{1 \leq n \leq N} \|\alpha n^k\| < N^{-(2/L)} \tag{3}$$

for $k \geq 9$ and $N > c_3(k)$. This is sharper than (1). The method of proof is adapted from Chapters IV and V of [8]. For $2 \leq k \leq 8$ the method of Heilbronn and Danicic is still the most effective.

All small Latin letters (except e and z) denote integers, and p denotes a prime variable. We write $e(z) = e^{2\pi i z}$ and $\theta = 1 - (1/k)$. Constants implied by '<<' depend at most on the quantities k, h and ϵ.

2. PERMISSIBLE EXPONENTS

Let $\lambda_1, \ldots, \lambda_h$ be real positive numbers, $1 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_h$. Suppose that the number of solutions of

$$m_1 x_1^k + \ldots + m_h x_h^k = m_1 y_1^k + \ldots + m_h y_h^k, \tag{4}$$

Received March 25, 1980.

Michigan Math. J. 28 (1981).

satisfying

$$P^{\lambda_1} < x_1, y_1 < 2P^{\lambda_1}, \ldots, P^{\lambda_h} < x_h, y_h < 2P^{\lambda_h}, \tag{5}$$

is at most

$$c_4(k,\lambda_1,\ldots,\lambda_h,\epsilon)\, P^{\lambda_1+\cdots+\lambda_h+\epsilon} M^h / |m_1 \ldots m_h| \tag{6}$$

whenever $\epsilon > 0$, $P \geq 1$, $0 \leq \log M << \log P$, and $m_1, \ldots, m_h$ are nonzero integers in $[-M,M]$. Then we say that $\lambda_1, \ldots, \lambda_h$ are *permissible exponents for* kth *powers.*

Note that permissible exponents are (in particular) *admissible* exponents in the sense of [3], [4].

THEOREM. *Let $\lambda_1, \ldots, \lambda_h$ be permissible exponents.*

Suppose further that $\Lambda = \lambda_1 + \ldots + \lambda_h - (k-1) > 0$. Then for $\epsilon > 0$, $N > c_5(k,\lambda_1,\ldots,\lambda_h,\epsilon)$ and any real α, we have

$$\min_{1 \leq n \leq N} \|\alpha n^k\| < N^{-(\Lambda/(4h+2\theta+\Lambda k^{-1}))+\epsilon}.$$

We require two lemmas.

LEMMA 1. Let $0 < \Delta < 1/2$ *and let* r *be a natural number. Then there is a function* $\psi(z)$ *of period one on the real line, having*

$$\psi(z) = 0 \qquad \text{for } \|z\| \geq \Delta, \tag{i}$$

$$\psi(z) = -\Delta + \sum_{m \neq 0} \gamma(m)\, e(mz), \tag{ii}$$

where

$$|\gamma(m)| < c_6(r) \min(\Delta, \Delta^{-r} m^{-r-1}). \tag{7}$$

Proof. This is a consequence of Lemma 12 of [8, Chapter I].

LEMMA 2. *Let $R, Q > 1$. Let α be real and suppose that*

$$|\alpha - a/q| \leq q^{-1} R^{-k}, \qquad (a,q) = 1, \qquad \text{where } 1 \leq q \leq R^k. \tag{8}$$

Let ϕ_n ($|n| < Q$) be complex numbers. Then

$$\sum_{|n|<Q} \sum_{R/4<p<R/2} \phi_n\, e(\alpha n p^k)$$
$$<< (Rq^{-1} + 1) q^{\epsilon} \left(\sum_{|n|<Q} |\phi_n|^2 \right)^{1/2} (Q^{1/2} + q^{1/2}) \min(R^{1/2}, q^{1/2}).$$

Proof. This is a variant of Lemma 2 of [7]; see also Lemma 2 of [8], Chapter IV.

Proof of the Theorem. We may suppose that ϵ is small as a function of $k, \lambda_1, \ldots, \lambda_h$. Let $N > c_6(k,\lambda_1,\ldots,\lambda_h,\epsilon)$. We define

$$\Delta = N^{-(\Lambda/(4h+2\theta+\Lambda k^{-1}))+\epsilon}$$

and we suppose that

$$\|\alpha n^k\| \geq \Delta \qquad (n = 1,\ldots,N). \tag{9}$$

We shall ultimately obtain a contradiction.

Let $\epsilon_1 = \epsilon/(5k)$. We write $M = [\Delta^{-(1+\epsilon_1)}]$ and

$$R = N^{1/2}M^{1/(2k)}, \qquad P = N^{1/2}M^{-1/(2k)}.$$

Then $R > P > 1$. (It is an easy consequence of the hypotheses of the theorem that $\Lambda \leq 1$, and so $\Delta > N^{-1/(4h)}$). Let $r = [2h/\epsilon_1] + 1$. Let $\psi(z)$ be as in Lemma 1 and define $\psi_0(z) = \psi(z) + \Delta$. Since $RP = N$, we have $\psi_0(p^k x_j^k) = \Delta$ whenever

$$\frac{1}{4}R < p < \frac{1}{2}R, \qquad P^{\lambda_j} < x_j < 2P^{\lambda_j}.$$

Consequently, $S_j(p) > \Delta P^{\lambda_j}/2$ $(R/4 < p < R/2, j = 1,\ldots,h)$, where

$$S_j(p) = \sum_{P^{\lambda_j} < x_j < 2P^{\lambda_j}} \psi_0(\alpha p^k x_j^k).$$

Let $H = \sum_{R/4<p<R/2} \prod_{j=1}^{h} S_j(p)$. Since R is large, we have

$$H > \Delta^h P^{\lambda_1+\ldots+\lambda_h} R^{1-\epsilon_1}. \tag{10}$$

On the other hand, we have

$$S_j(p) = \sum_{m_j \neq 0} \gamma(m_j) \sum_{P^{\lambda_j}<x_j<2P^{\lambda_j}} e(\alpha p^k m_j x_j^k),$$

so that

$$H = \sum_{m_1 \neq 0} \ldots \sum_{m_h \neq 0} \gamma(m_1) \ldots \gamma(m_h)\, T(m_1,\ldots,m_h), \tag{11}$$

where

$$T(m_1,\ldots,m_h) = \sum_{R/4<p<R/2} \sum_{P^{\lambda_1}<x_1<2P^{\lambda_1}} \cdots \sum_{P^{\lambda_h}<x_h<2P^{\lambda_h}} e(\alpha p^k (m_1 x_1^k + \ldots + m_h x_h^k)).$$

It is clear that

$$|T(m_1,\dots,m_h)| < RP^{\lambda_1+\dots+\lambda_h}. \tag{12}$$

From (7),

$$\sum_m |\gamma(m) << 1, \tag{13}$$

and

$$\sum_{|m|>M} |\gamma(m)| << (\Delta M)^{-r} << \Delta^{2h}. \tag{14}$$

It follows from (12), (13) and (14) that the contribution to the sum in (11) from those sets $m_1, \dots, m_h$, for which *any* of $|m_1|, \dots, |m_h|$ exceeds M, has modulus $<< \Delta^{2h} RP^{\lambda_1+\dots+\lambda_h}$. We deduce from (10), (11) that

$$\sum_{0<|m_1|\le M} \dots \sum_{0<|m_h|\le M} |\gamma(m_1) \dots \gamma(m_h)\, T(m_1,\dots,m_h)| > \Delta^h P^{\lambda_1+\dots+\lambda_h} R^{1-\epsilon_1}. \tag{15}$$

By Dirichlet's theorem there is a natural number $q \le R^k$ satisfying (8). If $q \le R$, then $\|\alpha q^k\| \le q^{k-1}\|\alpha q\| \le R^{(k-1)-k} < \Delta$, which contradicts (9). Thus we may suppose that $q > R$.

We rewrite $T(m_1,\dots,m_h)$ in the form

$$T(m_1,\dots,m_h) = \sum_{R/4<p<R/2}\ \sum_{|n|<<MP^k} \phi_n\, e(\alpha n p^k), \tag{16}$$

where ϕ_n is the number of sets $x_1, \dots, x_h$ satisfying $m_1 x_1^k + \dots + m_h x_h^k = n$. Thus $\sum_n |\phi_n|^2$ is the number of solutions of (4) satisfying (5). Since $\lambda_1, \dots, \lambda_h$ are permissible exponents and N is large, we have

$$\sum_n |\phi_n|^2 \le P^{\lambda_1+\dots+\lambda_h+\epsilon_1} M^h / |m_1 \dots m_h|. \tag{17}$$

In view of (16) and (17), we deduce from Lemma 2 that

$$\begin{aligned} T(m_1,\dots,m_h) &<< (Rq^{-1} + 1)q^{\epsilon_1}(P^{\lambda_1+\dots+\lambda_h+\epsilon_1}M^h)^{1/2} M^{1/2}P^{k/2}R^{1/2}/|m_1\dots m_h|^{1/2} \\ &<< R^{(1/2)+2k\epsilon_1} M^{(h+1)/2} P^{(k+\lambda_1+\dots+\lambda_h)/2}/|m_1\dots m_h|^{1/2}. \end{aligned}$$

Summing over $m_1, \dots m_h$, we obtain

$$\sum_{0<|m_1|\le M} \dots \sum_{0<|m_h|\le M} |\gamma(m_1)\dots\gamma(m_h)\, T(m_1,\dots,m_h)| \tag{18}$$

$$<< M^{(h+1)/2} R^{(1/2)+3k\epsilon_1} P^{(k+\lambda_1+\ldots+\lambda_h)/2} \left(\sum_{0<|m|\le M} |\gamma(m)| m^{-1/2} \right)^h .$$

We see from (7) that

$$\sum_{0<|m|\le M} |\gamma(m)| m^{-1/2} << \Delta \sum_{0<|m|\le M} m^{-1/2} << \Delta M^{1/2} << M^{-(1/2)+2\epsilon_1}. \tag{19}$$

Combining (15), (18) and (19), we obtain

$$\Delta^h P^{\lambda_1+\ldots+\lambda_h} R^{1-\epsilon_1} << M^{1/2} R^{(1/2)+3k\epsilon_1} P^{(k+\lambda_1+\ldots+\lambda_h)/2}.$$

Thus

$$\Delta^{h+(1/2)-(2-\Lambda)/4k} << N^{-(\Lambda/4)+4k\epsilon_1}.$$

This contradicts the definition of Δ, and the theorem is proved.

3. Proof of the Inequality (3). Let $\lambda_j = \theta^{j-1}$ $(j = 1,\ldots,h)$. Then $\lambda_1, \ldots, \lambda_h$ are permissible exponents for k-th powers. The proof is straightforward—the case $m_1 > 0, \ldots, m_h > 0$ is contained in Lemma 1 of [8], Chapter V. Now in the notation of the theorem,

$$\Lambda = 1 - k\theta^h. \tag{21}$$

Thus it suffices to show that there is a natural number h satisfying

$$\frac{1 - k\theta^h}{4h + 2\theta + \Lambda k^{-1}} > \frac{2}{L}.$$

We take h to be the least integer for which $k\,\theta^h < 1/(\log k + 1)$, or in other words

$$h = \left[\frac{\log(k \log k + k)}{-\log\theta} + 1 \right].$$

Write $\nu = 1/k$. Since $-\log\theta > \nu/(1 - (\nu/2))$, we have

$$h < k(1 - (\nu/2)) \log(k \log k + k) + 1.$$

Thus

$$\frac{\Lambda}{4h + 2\theta + \Lambda k^{-1}} > \frac{1 - k\theta^h}{4k(1 - (\nu/2)) \log(k \log k + k) + 6}$$

$$> \frac{\log k}{4k(\log k + 1)((1 - (\nu/2)) \log(k \log k + k) + 3\nu/2)}.$$

The proof is completed on noting that $\log(k \log k + k) \geq \log(9 \log 9 + 9) > 3$.

We outline the proof of a slightly stronger result. Let

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{k^2 - \theta^{h-3}}{k^2 + k - k\theta^{h-3}}, \qquad \lambda_3 = \frac{k^2 - k - 1}{k^2 + k - k\theta^{h-3}},$$

and let $\lambda_j = \lambda_3 \theta^{j-3} (j = 4, \ldots, h)$. By a straightforward extension of Lemma 3 of R. C. Vaughan [7], we can show that $\lambda_1, \ldots, \lambda_h$ are permissible exponents for k-th powers. Now we have

$$\Lambda = 1 - k\left(\frac{k^3 - 3k^2 + k + 2}{k^3 + k^2 - k^2\theta^{h-3}}\right)\theta^{h-3}$$

instead of (21). For example, we obtain $\min_{1 \leq n \leq N} \|\alpha x^9\| < N^{-1/159}$ for $N > c_6$, by taking $h = 31$. Further improvement is perhaps possible by adapting Theorem 2 of [3].

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