THE OBSTRUCTION TO THE FINITENESS OF THE TOTAL SPACE OF A FIBRATION

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INTRODUCTION

Given a space E, we say it is finitely dominated if it is a homotopy retract of a finite CW complex. That is, there is a finite CW complex K and maps $E \to K \to E$ so that the composition is homotopic to 1_E. Wall [20] showed that in that case the singular chain complex of the universal cover $\tilde{E}$ is chain equivalent to a finite, finitely generated projective chain complex P_i over the group ring $\mathbf{Z}\pi_1 E$. He further showed that $O(E) = \sum(-1)^i P_i \in \tilde{K}_0(\pi_1 E)$ is zero if and only if E has the homotopy type of a finite CW complex.

Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, where both the fiber F and base B are finitely dominated. Then E is also finitely dominated [17] and the question arises: how is $O(E)$ related to $O(\mathrm{B})$ and $O(F)$? For the trivial fibration $E = F \times B$, there is the product formula of Siebenmann [19] and Gersten [13],

$$O(E) = \chi(F)\cdot s_* O(B) + \chi(B)\cdot i_* O(F) + O(B) \otimes O(F),$$

where $\chi(F)$ is the Euler characteristic of F, and s_*, i_* are maps from $\tilde{K}_0(\pi_1 B)$, $\tilde{K}_0(\pi_1 F)$ into $\tilde{K}_0(\pi_1 E)$ induced by the corresponding maps of the fundamental groups. Lal [17] obtained the same formula for a fibration in which the base B has the homotopy type of a finite complex. The product formula breaks down, however, if $O(B) \neq 0$. Anderson [6] computed $p_* O(E) \in \tilde{K}_0(\pi_1 B)$ for some fibrations and showed that the orientation of the fibration had to be taken into account. I showed that $p_* O(E)$ depends only on B and F and the orientation [10], [11]. Then Pedersen and Taylor [18] gave an explicit formula for $p_* O(E)$.

In this paper we are interested in the actual computations of $O(E) \in \tilde{K}_0(\pi_1 E)$ rather than its image in $\tilde{K}_0(\pi_1 B)$. The only known result in this direction is Anderson [5]. He shows that $O(E) = 0$ for a principal S^1 bundle, $S^1 \to E \to B$, where $\pi_1 S^1$ injects into abelian $\pi_1 E$. I derive several formulae and show that under various conditions an analog of the product formula holds for orientable fibrations.

By an orientable fibration I shall mean a Hurewicz fibration $F \to E \to B$, which is a pullback of a fibration with a simply connected base space. That means given a space F, the orientable fibrations with fiber F are classified by a universal fibration $F \to EF \to BF$, where BF is simply connected. This notion of orientability

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is stronger than the more usual one due to Serre, where one merely requires $\pi_1 B$ to act trivially on $H_*(F)$.

Section 3 contains the major results (see 1.4 for the definition of a pseudo-abelian fibration):

THEOREM A. *Let $F \to E \to B$ be an orientable fibration with both F and B finitely dominated. Then $\pi_1 E = \pi_1 F \times \pi_1 B$ and $O(E) = O(F \times B)$, provided any one of the conditions below is satisfied.*

1. $\chi(F) \neq 0$ *and* $O(F) = 0$.
2. $\chi(F) \neq 0$ *and the center of $\pi_1 F$ has finite index in $\pi_1 F$.*
3. *$\pi_1 F$ has trivial center.*

Notice that the second half of condition 2. is satisfied if $\pi_1 F$ is either an abelian group or a finite group.

THEOREM B. *Let $F \xrightarrow{i} E \to B$ be an orientable pseudo-abelian fibration with F and B finitely dominated, such that $i : \pi_1 F \to \pi_1 E$ factors through a free abelian group. If $\chi(F) = 0$ and if $\pi_1 EF$ is a free abelian group, then $O(E) = 0$.*

THEOREM C. *Let $F \to E \to B$ be a Steenrod fiber bundle associated to a pseudo-abelian principal G bundle, where G is a connected, compact Lie group acting smoothly on the compact manifold F. If $\chi(F) = 0$ and B is finitely dominated, then $O(E) = 0$.*

There are also some results on non orientable fibrations, in particular Theorems 2.5, 2.7 and 3.5. The methods employed throughout the paper can just as well be applied to compute the Whitehead torsion of a fiber homotopy equivalence. Therefore all the theorems in this paper are true for *PL* fibrations if $\tilde{K}_0(\pi)$ is replaced by $Wh(\pi)$ and Wall finiteness obstruction is replaced by Whitehead torsion.

The results in this paper form part of my Ph.D. thesis. I would like to thank my advisor P. J. Kahn for his help and supervision.

1. ALGEBRAIC PRELIMINARIES

In this section we define a basic pairing in Proposition 1.2, which will enable us later (in 2.7 and 3.5) to get an explicit description of the chain complex of the total space in terms of the chain complexes of the base and fiber. We also introduce the concept of a pseudo-abelian map which comes up as a necessary condition in our main results. Proposition 1.7 shows that without this condition the basic pairing is useless.

Notation 1.1. Let π be a group. A module over the integral group ring $\mathbf{Z}\pi$ is called a π module.

2. Let $K_0(\pi)$ be the Grothendieck group of π modules, which admit a finite resolution by finitely generated projective π modules. Let $\tilde{K}_0(\pi)$ denote the reduced group. If ν is a subgroup of π, then let $G_\nu(\pi)$ be the Grothendieck group of π modules admitting a finite resolution by π modules, which are finitely generated projectives when restricted to ν. To indicate that a π module M admits such a resolution, and hence determines a class in $G_\nu(\pi)$, we may write $M \in G_\nu(\pi)$. Note that by a Theorem of Grothendieck (Bass [8], p. 407) $K_0(\pi)$ is naturally isomorphic

to the Grothendieck group of finitely generated projective modules. Similarly, $G_\nu(\pi)$ is naturally isomorphic to the Grothendieck group of π modules, which are finitely generated projectives, when viewed as ν modules.

3. Given a continuous map $f: A \to B$, it induces a map of fundamental groups also denoted by $f : \pi_1 A \to \pi_1 B$. This in turn induces a homomorphism $\tilde{K}_0(\pi_1 A) \to \tilde{K}_0(\pi_1 B)$ denoted by f_*.

PROPOSITION 1.2. *Let* $\nu \overset{i}{\rightarrowtail} \pi \overset{p}{\twoheadrightarrow} \sigma$ *be a short exact sequence of groups. Then there is a well defined pairing* $G_\nu(\pi) \otimes K_0(\sigma) \to K_0(\pi)$. *Let* $M \in G_\nu(\pi)$ *and* $N \in K_0(\sigma)$, *then the pairing is given by* $M \otimes p^* N$ *together with the diagonal* π *action. Here* $p^* N$ *denotes the* π *module obtained from* N *via the map* p.

Proof. Without loss of generality we can assume that M is ν projective and N is σ projective. Therefore tensoring over $\mathbf{Z}$ will preserve short exact sequences and direct sums. The only thing to prove then is that $M \otimes p^* \mathbf{Z}\sigma$ is a finitely generated projective π module. $M \otimes p^* \mathbf{Z}\sigma = M \otimes i_* \mathbf{Z} = i_*(i^* M \otimes \mathbf{Z}) = i_* i^* M$. The second equality is Frobenius reciprocity (Bass [8], p. 563). By hypothesis $i^* M$ is finitely generated projective and so $i_* i^* M$ will be.

Remark 1.3. This pairing was defined by Gersten [13] when $\pi = \nu \times \sigma$ and by Anderson [4] when π was a semidirect product of ν and σ.

Definition 1.4. An epimorphism $p : \pi \to \sigma$ of finitely presented groups will be called pseudo-abelian, if the kernel of p is finitely presented and if it intersects the commutator of π trivially. The resulting short exact sequence $\ker p \rightarrowtail \pi \twoheadrightarrow \sigma$ will be called a pseudo-abelian extension. A fibration $p : E \to B$ will be called pseudo-abelian if the map $p : \pi_1 E \to \pi_1 B$ is pseudo-abelian.

LEMMA 1.5. *Given a short exact sequency* $A \rightarrowtail \pi \twoheadrightarrow \sigma$, *the following are equivalent:*

1. $A \rightarrowtail \pi \twoheadrightarrow \sigma$ *is a pseudo-abelian extension.*
2. $A \rightarrowtail \pi \twoheadrightarrow \sigma$ *is a pullback of an abelian extension via the map* $\sigma \twoheadrightarrow \sigma_{ab}$.
3. A *injects into* π_{ab}.
4. A *is a central subgroup in* π *and* $H_2 p : H_2\pi \to H_2\sigma$ *is an epimorphism.*

Here π_{ab} denotes the abelianization of π.

Proof. We will prove here only 3. $\Leftrightarrow$ 4. The Serre spectral sequence of $A \rightarrowtail \pi \twoheadrightarrow \sigma$ gives rise to a diagram of exact sequences:

$$\begin{array}{ccccccccc} & & & & A & \rightarrowtail & \pi & \twoheadrightarrow & \sigma \\ & & & & \downarrow & & \downarrow & & \downarrow \\ H_2\pi & \to & H_2\sigma & \to & H_0(\sigma;A) & \to & \pi_{ab} & \twoheadrightarrow & \sigma_{ab} \end{array}$$

The composition $A \twoheadrightarrow H_0(\sigma;A) \to \pi_{ab}$ is an injection if, and only if, $A = H_0(\sigma;A)$ and $H_2\sigma \to H_0(\sigma;A)$ is the zero map. $H_0(\sigma;A) = A$ if and only if A is central in π.

Remark 1.6. The characterization 2. in Lemma 1.5 indicates why we call the extension pseudo-abelian. We often consider an extension as an element of $H^2(\sigma;A)$, in that case the pseudo-abelian extensions form the subgroup

$\operatorname{Ext}(H_1\sigma;A) \subseteq H^2(\sigma;A)$, where A is the trivial σ module. This is easily seen from Lemma 1.5 and the universal coefficient theorem

$$0 \to \operatorname{Ext}(H_1\sigma;A) \to H^2(\sigma;A) \to \operatorname{Hom}(H_2\sigma,A) \to 0.$$

PROPOSITION *1.7. Let $A \to \pi \twoheadrightarrow \sigma$ be a central extension of σ by A, where A is a finitely generated free abelian group. Then the following are equivalent.*

1. *$A \rightarrowtail \pi \twoheadrightarrow \sigma$ is a pseudo-abelian extension.*

2. *There is a non zero π module M, which is a projective finitely generated $\mathbf{Z}(A)$ module.*

3. *$G_A(\pi)$ has an infinite cyclic summand $\mathbf{Z}$.*

4. *The trivial π module $\mathbf{Z}$ belongs to $G_A(\pi)$ and it represents the zero element in $G_A(\pi)$.*

Proof. The implications 3. $\Rightarrow$ 2. and 4. $\Rightarrow$ 2. are obvious.

1. $\Rightarrow$ 3. The free abelian group A injects into π_{ab} and also into $\pi_{ab}/\text{torsion}$. Let M be a matrix, with integral entries, representing this monomorphism between finitely generated free abelian groups. Let N be a matrix with rational entries which is a left inverse to M. Let n be an integer, such that $n \cdot N$ is a matrix with integral entries. The matrix $n \cdot N$ defines a homomorphism $\pi_{ab}/\text{torsion} \to A$ such that the composite $A \to \pi_{ab}/\text{torsion} \to A$ is a monomorphism of index n. Denote by r the composite $\pi \to \pi_{ab}/\text{torsion} \to A$ and by i the injection $A \to \pi$. Then the map ri is a monomorphism of finite index. The maps i and r induce maps $r^*: K_0(A) \to G_A(\pi)$ and $i^*: G_A(\pi) \to K_0(A)$. The composite $i^*r^* = (ri)^*$ is multiplication by the finite index of the map ri. $K_0(A)$ is infinite cyclic so i^*r^* is a monomorphism. That gives 3.

1. $\Rightarrow$ 4. Note, that the trivial module $\mathbf{Z}$ represents the zero element in $K_0(A)$. That can be seen by looking at the cellular chains of the universal cover of a torus and from the fact that $\chi(\text{torus})$ is zero. Let $r^*: K_0(A) \to (G_A(\pi)$ be the homomorphism constructed in the preceeding argument. Then $\mathbf{Z} = r^*\mathbf{Z}$ is zero in $G_A(\pi)$.

2. $\Rightarrow$ 1. Let M be as specified. There is a homomorphism of π to the group of automorphisms of M, determined by the π action on M. A is central in π, and so π acts on M via the $\mathbf{Z}(A)$ automorphisms. Projective modules over $\mathbf{Z}(A)$ are known to be free, so once we fix a $\mathbf{Z}(A)$ basis of n elements for M, we get an isomorphism between the $\mathbf{Z}(A)$ automorphisms and $Gl(n,\mathbf{Z}(A))$, the group of $n \times n$ invertible matrices over the group ring $\mathbf{Z}(A)$. So we have a sequence of maps

$$A \to \pi \to Gl(n,\mathbf{Z}(A)) \xrightarrow{\det} \pm A \to A,$$

where det denotes the determinant map, and the composite is just multiplication by n, so A injects in π_{ab}.

Remark 1.8. If a finitely generated free abelian group A is a subgroup of finite index in π, then every finitely generated π module admits a resolution, of length at most (rank of A) + 2, by modules in $G_A(\pi)$. Therefore the natural map $G_A(\pi) \to G(\pi)$ is an isomorphism. Here $G(\pi)$ denotes the ordinary Grothendieck group of finitely generated π modules.

2. TWISTED PRODUCTS

Suppose that G is a compact Lie group or a discrete group and let $G \to X \to B$ be a principal left G bundle over a finitely dominated base B. For a right G space F define a left G action on $F \times X$ diagonally. The quotient space $F \underset{G}{\times} X$ is called the twisted product of F and X and the resulting fibration $F \to F \underset{G}{\times} X \to B$ is called a Steenrod bundle. We are interested in computing the Wall obstruction of $F \underset{G}{\times} X$, when F is a finite G-CW complex.

A finite G-CW complex is built up inductively by attaching finitely many equivariant cells c^n of the form $e^n \times G/H$, where $H \subset G$ is some closed subgroup (H is not fixed), by an equivariant map from $S^{n-1} \times G/H$. For a complete definition see [16] or [22]. Here G/H are the right cosets, G acts on G/H by right translation and e^n is an ordinary n-ball with the trivial G action. We will say that a G cell c^n is of type (H) if the space it represents is of the form $(e^n \times G/K, S^{n-1} \times G/K)$ with K conjugate to H.

Definition 2.1. Let $\chi_H(F) = \sum_n (-1)^n$ number of n-cells of F of type (H). We will call $\chi_H(F)$ an equivariant Euler characteristic of F. A routine argument [11], using cohomology with compact support, shows that $\chi_H(F)$ is a G invariant of the G space F. That is, $\chi_H(F)$ does not depend on a particular G triangulation of F.

LEMMA 2.2. *Let $\tilde{F}$ be the universal cover F. Then there exists a short exact sequence of topological groups $\pi_1 F \rightarrowtail \tilde{G} \twoheadrightarrow G$ such that $\tilde{F}$ is a finite $\tilde{G}$-CW complex and the projection $\tilde{F} \to F$ is a $\tilde{G}$ map.*

Proof. See Bredon [9] pp. 63–66 for the existence of $\tilde{G}$ acting on $\tilde{F}$.

To see that $\tilde{F}$ is a finite $\tilde{G}$-CW complex we will describe the $\tilde{G}$ cell of $\tilde{F}$ which covers a G cell of F. Let c be a G cell of type (H) and restrict the G action on F to an H action. H fixes a point x in the interior of c; pick a point $\tilde{x} \in \tilde{F}$, which lies over x. The choice of x and $\tilde{x}$ determines a monomorphism $H \rightarrowtail \tilde{G}$, which covers the inclusion $H \subset G$ and such that H, acting on $\tilde{F}$ via the map $H \to \tilde{G}$, fixes the point $\tilde{x}$. Let H' be the isomorphic image of H in $\tilde{G}$ under this map. Then the following diagram describes the G cell covering c:

$$\begin{array}{ccc} (e^n \times \tilde{G}/H', \quad S^{n-1} \times \tilde{G}/H') & \xrightarrow{\tilde{G}\text{ cell}} & (\tilde{F}^n, \tilde{F}^{n-1}) \\ \downarrow & & \downarrow \\ (e^n \times G/H, \quad S^{n-1} \times G/H) & \xrightarrow{G\text{ cell}} & (F^n, F^{n-1}) \end{array}$$

Here we assume that the cell c is an n dimensional G cell and F^n stands for the equivariant n-skeleton of F.

Notice that two G cells of the same isotropy type may very well lift to $\tilde{G}$

cells of different isotropy type, unless of course, G is a connected group.

LEMMA 2.3. *Let α, β be two $\tilde{G}$ equivariant maps $\tilde{G}/H' \to \tilde{F}$, and let $\bar{\alpha}, \bar{\beta}$ denote the induced maps $X/H = \tilde{G}/H' \times_{\tilde{G}} X \to \tilde{F} \times_{\tilde{G}} X = F \times_G X$. Then $\bar{\alpha}$ and $\bar{\beta}$ induce the same maps on fundamental groups.*

Remark 2.4. This means that $\bar{\alpha}, \bar{\beta} : \pi_1(X/H) \to \pi_1(F \times_G X)$ depend only on the conjugacy class H' and not on a particular inclusion of $\tilde{G}/H'$ in $\tilde{F}$. Therefore, let us denote by $j_{H'}$ the map on fundamental group induced by $G/H' \to \tilde{F}$.

Proof of 2.3. Consider the following diagram:

$$
\begin{array}{ccccc}
X & \rightarrow & \tilde{G}/H' \times X & \overset{\alpha \times 1}{\underset{\beta \times 1}{\rightrightarrows}} & \tilde{F} \times X \\
\downarrow \bmod H' & & \downarrow \bmod H' & & \downarrow \bmod H' \\
X/H & \overset{s}{\rightarrow} & \tilde{G}/H' \times_{H'} X & \overset{\hat{\alpha}}{\underset{\hat{\beta}}{\rightrightarrows}} & F \times_{H'} X \\
 & & \downarrow^{r} & & \downarrow \\
 & & \tilde{G}/H' \times_{\tilde{G}} X & \overset{\bar{\alpha}}{\underset{\bar{\beta}}{\rightrightarrows}} & \tilde{F} \times_{\tilde{G}} X = F \times_G X
\end{array}
$$

First let us observe that rs is a homeomorphism (Bredon [9], p. 81). This was the homeomorphism used in the statement of the Lemma to identify X/H and $\tilde{G}/H' \times_{\tilde{G}} X$. Then notice that $\hat{\alpha}s$ and $\hat{\beta}s$ define two cross-sections to the fibration $\tilde{F} \to \tilde{F} \times_{H'} X \to X/H' = X/H$. $\tilde{F}$ is simply connected so $\hat{\alpha}s$ and $\hat{\beta}s$ induce the same isomorphism on the fundamental groups.

THEOREM 2.5. *Let $j_{H'}: \pi_1(X/H) \to \pi_1(F \times_G X)$ be the map from Remark 2.4 and let G be a compact Lie group, then*

$$O(F \times_G X) = \sum \chi_{H'}(\tilde{F}) \cdot (j_{H'})_* O(X/H).$$

Proof. Proceed by induction on the number of $\tilde{G}$ cells of $\tilde{F}$ using the Sum Theorem of Siebenmann [19].

$$
\begin{aligned}
O((\tilde{F} \cup c^n) \times_{\tilde{G}} X) &= O(\tilde{F} \times_{\tilde{G}} X) + (j_{H'})_* O(e^n \times X/H) - (j_{H'})_* O(S^{n-1} \times X/H) \\
&= O(\tilde{F} \times_{\tilde{G}} X) + (-1)^n (j_{H'})_* O(X/H).
\end{aligned}
$$

Definition 2.6. If $G = \pi_1 B$ then the resulting fibration $F \to F \times_{\pi_1 B} \tilde{B} \to B$ is called a flat bundle.

THEOREM 2.7. *Let $F \to E \overset{p}{\to} B$ be a flat bundle with B finitely dominated, and let $C_*(\tilde{F})$ denote the cellular chains of $\tilde{F}$. Then $\chi = \sum (-1)^i C_i(\tilde{F}) \in G_{\pi_1 F}(\pi_1 E)$ and $O(E) = \chi \otimes p * O(B)$.*

Remark 2.8. This is a generalization of a result of Anderson [4]. He proves this theorem assuming that the $\pi_1 B$ action on F has a fixed point.

Proof. Lift the action of $\pi_1 B$ to the action of $\tilde{G}$ on $\tilde{F}$ as in Lemma 2.2. $\tilde{G}$ acts freely on $\tilde{F} \times \tilde{B}$ and the quotient is E. That means $\tilde{F} \times \tilde{B}$ is the universal cover of E and $\tilde{G} = \pi_1 E$. Therefore $\chi \in G_{\pi_1 F}(\pi_1 E)$.

$$\begin{aligned} O(E) &= \sum_n (-1)^n C_n(\tilde{E}) \\ &= \sum_{i,t} (-1)^{i+t} C_i(\tilde{F}) \otimes p^* C_t(\tilde{B}) \\ &= \sum_i (-1)^i C_i(\tilde{F}) \otimes p^* O(B) = \chi \otimes p^* O(B). \end{aligned}$$

Notice, that we used the pairing from Proposition 1.2.

Remark 2.9. If we compare the results from 2.5 and 2.7, we see that

$$\chi = \sum_{H \subset \pi_1 E} \chi_H(\tilde{F}) \cdot \mathbf{Z}((\pi_1 E/H))$$

when $\pi_1 B$ is a finite group. This can also be seen directly from the definition of χ.

3. ORIENTABLE FIBRATIONS

Given two orientable fibrations $F \to E \to B$ and $F \to \bar{E} \to \bar{B}$ and a map $\phi : \pi_1 \bar{E} \twoheadrightarrow \pi_1 E$ satisfying certain commutativity conditions, then $O(E) = \phi_* O(\bar{E})$. This statement (Lemma 3.2) will be the starting point for the computations in this section; we will apply 3.2 repeatedly. Whenever we succeed in computing the Wall obstruction for a particular fibration then Lemma 3.2 will enable us to extend the result to a larger class of fibrations with the same fiber. In particular, Theorem A comes from the knowledge of the Wall obstruction of the trivial fibration.

Before stating Lemma 3.2 we need the following definition.

Definition 3.1. Given a CW complex F, let $F \to EF \to BF$ be the universal (classifying) fibration for orientable fibrations with fiber F. This is a fibration with a *simply connected* base space BF, such that any orientable fibration $F \to E \to B$ is a pullback of $F \to EF \to BF$ via a map uniquely determined (up to homotopy) by the fiber homotopy type of $F \to E \to B$.

LEMMA 3.2. *Suppose we have two orientable fibrations $F \to E \to B$ and $F \to \bar{E} \to \bar{B}$ with the same connected finitely dominated fiber F and the same complex as the* 2 *skeleton of both B* and *$\bar{B}$. If there exists a map $\phi : \pi_1 \bar{E} \to \pi_1 E$ such that the following diagrams commute*

$$\begin{array}{ccc} \pi_1 F & = & \pi_1 F \\ \downarrow & & \downarrow \\ \pi_1 \bar{E} & \overset{\phi}{\twoheadrightarrow} & \pi_1 E \\ \downarrow & & \downarrow \\ \pi_1 \bar{B} & = & \pi_1 B \end{array} \quad \text{and} \quad \begin{tikzcd} \pi_1 \bar{E} \arrow[rr, two heads, "\phi"] \arrow[dr] & & \pi_1 E \arrow[dl] \\ & \pi_1 EF & \end{tikzcd}$$

and if $O(B) = O(\bar{B})$ *and* $\chi(B) = \chi(\bar{B})$ *then* $O(E) = \phi_* O(\bar{E})$.

Proof. See [11] for a complete proof or see [12] for an outline of the same proof.

LEMMA 3.3. *Let* $G_1(F)$ *be the image of the boundary map* $\pi_2 BF \to \pi_1 F$. *Then* $G_1(F)$ *is the trivial group if any one of the three conditions below is satisfied:*

1. $\chi F \neq 0$ *and* $O(F) = 0$.

2. $\chi F \neq 0$ *and the center of* $\pi_1 F$ *has a finite index in* $\pi_1 F$.

3. $\pi_1 F$ *has trivial center.*

Proof. The group $G_1(F)$ has been defined and studied by Gottlieb in [14] and [15]. Theorem 3.1 in [15] gives the first assertion. Theorem 1.4 in [14] states that $G_1(F)$ is a central subgroup of $\pi_1 F$. To show the second assertion let C be the center of $\pi_1 F$. C is finitely generated abelian so $C = \bigoplus_j \mathbf{Z} \oplus \bigoplus_i \mathbf{Z}/p_i^{l_i}\mathbf{Z}$, where p_i is a prime. consider the subgroup $H = \bigoplus_j Z \oplus \bigoplus_i Z/p_i Z \subset C$. Let $\hat{F}$ be the finite covering of F such that $\pi_1 \hat{F} = H$. Then Theorem 4.1 in [15] implies that $G_1(\hat{F})$ is trivial. $G_1(F) \cap \pi_1 \hat{F} \subset G_1(\hat{F})$ (Theorem 6.1 in [15]) and since any nontrivial central subgroup of $\pi_1 F$ intersects $\pi_1 \hat{F}$ nontrivially we get that $G_1(F)$ is trivial.

THEOREM A. *Let* $F \to E \to B$ *be an orientable fibration, with both* F *and* B *finitely dominated. If* $G_1(F)$ *is trivial then* $\pi_1 E = \pi_1 F \times \pi_1 B$ *and* $O(E) = O(F \times B)$.

Proof. If $G_1(F)$ is trivial then $\pi_1 F = \pi_1 EF$ (remember that BF is simply connected). The map $\pi_1 E \to \pi_1 EF = \pi_1 F$ defines a splitting of $\pi_1 E$ and therefore an isomorphism $\phi\colon \pi_1 F \times \pi_1 B \to \pi_1 E$. The map ϕ satisfies the commutativity conditions of Lemma 3.2, where $F \to \bar{E} \to B$ is taken to be just the product $F \times B$.

Theorem A handles most of the cases where $\chi F \neq 0$. To get some results for $\chi F = 0$, we will use the pairing from 1.2. We will adopt the arguments from 2.2 and 2.7, suitably generalized to apply to fibrations instead of bundles.

Given a fibration $F \xrightarrow{i} E \xrightarrow{p} B$, with a connected fiber F, it determines a lifting function, which in turn determines an action of the loop space ΩB on F. Let ν be the kernel of $p : \pi_1 E \to \pi_1 B$ and let $\nu \to \hat{F} \to F$ be the regular covering space determined by the epimorphism $\pi_1 F \twoheadrightarrow \nu$.

LEMMA 3.4. With the notation as above,

a) There exists a covering space (in the nonconnected sense) $\nu \to \widetilde{\Omega B} \to \Omega B$ *and an action of* $\widetilde{\Omega B}$ *on* $\hat{F}$ *covering the action of* ΩB *on* F.

b) *This action defines an action of $\pi_1 E$ on the homology of $\hat{F}$; hence $H_*\hat{F}$ is a $\pi_1 E$ module.*

Proof. It is not hard to verify that Theorem 9.3 on page 66 of Bredon [9] holds when the Lie group G is replaced by the loop space ΩB. To show, that the condition of 9.3 is satisfied, we have to verify, that for all $g \in \Omega B$ the induced map $g : F \to F$ lifts to a map $\hat{F} \to \hat{F}$. That is equivalent to showing that given any map $\alpha : S^1 \to F$ such that $S^1 \xrightarrow{\alpha} F \xrightarrow{i} E$ is nullhomotopic then $S^1 \xrightarrow{\alpha} F \xrightarrow{g} F \xrightarrow{i} E$ is also nullhomotopic. By looking at the principle homotopy bundle $\Omega B \to F \times PB \to E$, where PB is the path space of B, we see easily that $i\alpha$ nullhomotopic implies $ig\alpha$ is nullhomotopic. Therefore Theorem 9.3 gives us the desired covering $\nu \to \widetilde{\Omega B} \to \Omega B$ and an action of $\widetilde{\Omega B}$ on $\hat{F}$, which covers the original action.

The action of $\widetilde{\Omega B}$ on $\hat{F}$ defines an action of $\pi_0\widetilde{\Omega B}$ on $H_*\hat{F}$. The second part of the Lemma asserts only that $\pi_0\widetilde{\Omega B} = \pi_1 E$. To see this consider this diagram of principal hoomotopy bundles:

$$
\begin{tikzcd}
\nu \arrow[r] \arrow[d, equal] & \widetilde{\Omega B} \arrow[r] \arrow[d] & \Omega B \arrow[d] \\
\nu \arrow[r] & \hat{F} \times PB \arrow[r] \arrow[d] & F \times PB \arrow[d, dash] \\
 & E \arrow[r, equal] & E
\end{tikzcd}
$$

Let us also look at the homotopy exact sequences of these bundles:

$$
\begin{tikzcd}
 & \pi_1\widetilde{\Omega B} \arrow[r, tail] \arrow[d] & \pi_1\Omega B \arrow[r] \arrow[d] & \nu \arrow[r] \arrow[d, equal] & {} \\
 & \pi_1\hat{F} \arrow[r, tail] \arrow[d] & \pi_1 F \arrow[r, two heads] \arrow[d] & \nu \arrow[dl] & \\
 & \pi_1 E \arrow[r, equal] \arrow[d] & \pi_1 E \arrow[d] & & \\
{} \arrow[r] & \pi_0\widetilde{\Omega B} \arrow[r, two heads] \arrow[d] & \pi_0\Omega B \arrow[d] & & \\
 & \pi_0\hat{F} \arrow[r, equal] & \pi_0 F & &
\end{tikzcd}
$$

$\pi_1\hat{F} \to \pi_1 E$ is the trivial map since it factors as $\pi_1\hat{F} \to \pi_1 F \to \nu \to \pi_1 E$, therefore $\pi_1 E = \pi_0\widetilde{\Omega B}$ (we assumed F to be connected).

THEOREM 3.5. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration (not necessarily orientable), with fiber and base connected and finitely dominated, and let ν be the kernel of $\pi_1 E \to \pi_1 B$. If $H_*(\hat{F}) \in G_\nu(\pi_1 E)$, let*

$$\chi(\tilde{p}) = \sum_t (-1)^t H_t(\hat{F}) \in G_\nu(\pi_1 E),$$

then

$$O(E) = \chi(B) i_* O(F) + \chi(\tilde{p}) \otimes p^* O(B).$$

Proof. Throughout this proof we will use the following Lemma, which can be found in Bass [8] on p. 406: If a chain complex C_* and its homology $H_*(C)$ are finitely generated projective π modules, then $\Sigma(-1)^t C_t = \Sigma(-1)^t H_t(C)$ in $K_0(\pi)$.

Without loss of generality we can assume that B^n dominates B. Then

$$H_{n+1}(\tilde{B},\tilde{B}^n) \in K_0(\pi_1 B) = \mathbf{Z} \oplus \tilde{K}_0(\pi_1 B)$$

and this isomorphism splits $(-1)^{n+1} H_{n+1}(\tilde{B},\tilde{B}^n)$ as $(\chi(B) - \chi(B^n)) + O(B)$ in $K_0(\pi_1 B)$.

Let E_n denote the restriction of E over the n skeleton of B, $\tilde{E}_n$ is then the restriction of $\tilde{E}$ over $\tilde{B}^n$. From [21] we have $O(E) = O(E_n) + O(E,E_n)$ and from [17] we have $O(E_n) = \chi(B^n) i_* O(F)$, because E_n is a fibration over a finite CW complex B^n.

From the Serre's spectral sequence for the fiber pair $\hat{F} \to (\tilde{E},\tilde{E}_n) \xrightarrow{\tilde{p}} (\tilde{B},\tilde{B}^n)$, we obtain $H_{s+t}(\tilde{E},\tilde{E}_n) = H_t(\hat{F}) \otimes p^* H_s(\tilde{B},\tilde{B}^n)$, where $\pi_1 E$ acts diagonally on the tensor product. From 1.2 we see that $H_{s+t}(\tilde{E},\tilde{E}_n)$ is a finitely generated projective $\pi_1 E$ module and therefore

$$\begin{aligned} O(E,E_n) &= \sum_{s,t} (-1)^{s+t} H_{s+t}(\tilde{E},\tilde{E}_n) \\ &= \sum_{t} (-1)^t H_t(\hat{F}) \otimes (-1)^{n+1} p^* H_{n+1}(\tilde{B},\tilde{B}^n) \\ &= \chi(\tilde{p}) \otimes [(\chi(B) - \chi(B^n)) p^* \mathbf{Z}\pi_1 B + p^* O(B)] \\ &= (\chi(B) - \chi(B^n)) j_* j^* \chi(\tilde{p}) + \chi(\tilde{p}) \otimes p^* O(B). \end{aligned}$$

Here j is the inclusion $\nu \subset \pi_1 E$ and we used Frobenius reciprocity again, just as we did in 1.2. $H_t(\hat{F}) \in G_\nu(\pi_1 E)$, which means that $j^* H_t(\hat{F}) \in K_0(\nu)$ and therefore $j_* j^* \chi(\tilde{p}) = i_* O(F)$.

By adding $O(E,E_n)$ to $O(E_n)$ we obtain the conclusion of the Theorem.

Remark 3.6. In general, it is difficult to say when $H_*(\hat{F}) \in G_\nu(\pi_1 E)$. A necessary condition is that the Eilenberg-MacLane space $K(\nu;1)$ be finitely dominated. This can be seen from $H_0(\hat{F}) = \mathbf{Z} \in G_\nu(\pi_1 E)$, which implies that $\mathbf{Z} \in K_0(\nu)$ and that is equivalent to $K(\nu;1)$ being finitely dominated.

On the other hand, if ν is a free abelian group and $\pi_1 B$ is a finite group, then $H_*(\hat{F})$ always belongs to $G\nu(\pi_1 E)$. That is because $G_\nu(\pi_1 E) = G(\pi_1 E)$ (see 1.8) and $H_*(\hat{F})$ is finitely generated $\pi_1 E$ module (since it is a finitely generated ν module).

For an orientable fibration with free abelian ν, $H_*(\hat{F}) \in G_\nu(\pi_1 E)$ implies that the fibration is pseudo-abelian, see 1.7.

We return to the orientable fibrations. An orientable fibration $F \to E \to B$ is a pullback from the universal fibration $F \to EF \to BF$ via a map $k : B \to BF$. Let c denote the induced map $E \to EF$.

LEMMA 3.7. *If $F \to E \to B$ is an orientable fibration and $\hat{F}$ is the regular covering space from 3.4 then $H_*\hat{F}$ is also a $\pi_1 EF$ module. The $\pi_1 E$ action on $H_*\hat{F}$ constructed in 3.4 comes from this $\pi_1 EF$ action via the epimorphism $c : \pi_1 E \twoheadrightarrow \pi_1 EF$.*

Proof. ΩB acts on F via the map $\Omega k : \Omega B \to \Omega BF$. If we apply the generalized Theorem 9.3 of [9], we see that there is a covering (not necessarily connected) $\nu \to \widetilde{\Omega BF} \to \Omega BF$ and an action of $\widetilde{\Omega BF}$ on $\hat{F}$ covering the action of ΩBF on F. This time the conditions of 9.3 are trivally verifiable since ΩBF is connected. We get a diagram of covering spaces:

$$\begin{array}{ccc}
\nu & = & \nu \\
\downarrow & & \downarrow \\
\widetilde{\Omega B} & \to & \widetilde{\Omega BF} \\
\downarrow & & \downarrow \\
\Omega B & \xrightarrow{k} & \Omega BF
\end{array}$$

and $\widetilde{\Omega B}$ acts on $\hat{F}$ via the map $\widetilde{\Omega B} \to \widetilde{\Omega BF}$. Consider the homotopy exact sequences of these coverings:

$$\begin{array}{ccccccc}
\pi_1 \Omega B & \to & \nu & \to & \pi_0 \widetilde{\Omega B} & \twoheadrightarrow & \pi_0 \Omega B \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\pi_1 \Omega BF & \to & \nu & \to & \pi_0 \widetilde{\Omega BF} & \to & 1
\end{array}$$

The cokernel of $\pi_1 \Omega BF = \pi_2 BF \to \nu$ is just $\pi_1 EF$, therefore $\pi_0 \widetilde{\Omega BF} = \pi_1 EF$, and $\pi_1 E$ acts on $H_*(\hat{F})$ via the map $c : \pi_1 E \to \pi_1 EF$.

LEMMA 3.8. *Let $F \to E \xrightarrow{p} B$ be an orientable pseudo-abelian fibration with F and B connected and finitely dominated, such that the kernel of p is a free abelian group. If $\pi_1 EF$ is free abelian and $\chi(F) = 0$ then $O(E) = 0$.*

Proof. We will first show that there exists a commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail, "j"] \arrow[dr, "m"'] & \pi_1 E \arrow[r, "c"] \arrow[d, "r"] & \pi_1 EF \\
& \bar{A} \arrow[ur, "s"'] &
\end{tikzcd}$$

where $\bar{A}$ is a free abelian group of the same rank as A, j and m are injections and r and s are surjections. $\pi_1 EF$ is an abelian group, so c will factor through the abelianization of $\pi_1 E$. Let $h : \pi_1 E \to H_1 E$ be the Hurewicz map and let $c' : H_1 E \to \pi_1 EF$ be such that $c = c'h$. The fibration is pseudo-abelian so $hj : A \to H_1 E$ is still an injection and $H_1 E$ splits up as a direct sum of $\bar{A} \oplus D$, where $\bar{A}$ is a free abelian group containing $hj(A)$ as a subgroup of finite index, see proof of 1.7. Let i' be the inclusion of $\bar{A}$ into H_1E, cj is surjective and therefore so will be $c'i'$. Let f be a right inverse to $c'i'$, that is let $f: \pi_1 EF \to \bar{A}$ be a map such that $c'i'f = 1_{\pi_1 EF}$, here we used the assumption that $\pi_1 EF$ is a free abelian group.

Now define $r': H_1E \to \bar{A}$ by $r'(\bar{a}, d) = \bar{a} + fc'(0,d)$ and let $r: \pi_1 E \to \bar{A}$ be just the composite $r'h$. Let $m: A \to \bar{A}$ be given by hj followed by the projection onto $\bar{A}$ and let $s: \bar{A} \to \pi_1 EF$ be just $c'i'$. Then it is an easy check that $m = rj$ and $c = sr$.

The maps in the diagram induce homorphisms

$$K_0(\bar{A}) \xrightarrow{r^*} G_A(\pi_1 E) \xrightarrow{j^*} K_0(A)$$

and the composition $j^* r^* = m^*$ is a monomorphism. From 3.7 we know that $H_t(\hat{F})$ is a $\pi_1 EF$ module, and we want to show that $c^* H_t(\hat{F}) \in G_A(\pi_1 E)$. $m^* s^* H_t(\hat{F})$ is finitely generated (because the chains of $\hat{F}$ are finitely generated projective $\mathbf{Z}(A)$ modules) and so $s^* H_t(\hat{F})$ is finitely generated. Every finitely generated $\mathbf{Z}(\bar{A})$ module has a finite, finitely generated projective resolution and so $s^* H_t(\hat{F}) \in K_0(\bar{A})$. Therefore $c^* H_t(\hat{F}) = r^* s^* H_t(\hat{F}) \in G_A(\pi_1 E)$.

$m^* s^* \left(\sum (-1)^t H_t(\hat{F}) \right) = j^* \chi(\tilde{p}) = \chi(F) = 0.$ m^* is a monomorphism so

$$s^* \left(\sum (-1)^t H_t(\hat{F}) \right) = 0, \chi(\tilde{p}) = r^* s^* \left(\sum (-1)^t H_t(\hat{F}) \right) = 0,$$

and the Lemma follows from 3.5.

Let F and B be two connected CW complexes and let $\pi_1 F \twoheadrightarrow A$ be a fixed epimorphism onto an abelian group A. Consider all orientable fibrations $F \to E \to B$, where the map $\pi_1 F \to \pi_1 E$ decomposes as the epimorphism $\pi_1 F \twoheadrightarrow A$ followed by an inclusion $A \rightarrowtail \pi_1 E$. Equivalence classes of such fibrations lie in the kernel K of the following natural map:

$$\begin{array}{ccc} [B,BF] \to [B, K(\pi_2 BF; 2)] = H^2(B; \pi_2 BF) & \to & H^2(B;A) \\ & & \downarrow \\ & & \mathrm{Hom}(\pi_2 B; A) \end{array}$$

BF is simply connected so there is a canonical map from BF to the Eilenberg-MacLane space $K(\pi_2 BF; 2)$. The next map is induced by the homomorphism $\pi_2 BF \to \pi_1 F \twoheadrightarrow A$ and the vertical line comes from the Serre's exact sequence for a covering space $H^2(\pi_1 B; A) \rightarrowtail H^2(B;A) \to \mathrm{Hom}(\pi_2 B; A)$. From the last description we see that the image of K in $H^2(B;A)$ lies in $H^2(\pi_1 B; A)$, and this induces a map $\sigma: K \to H^2(\pi_1 B; A)$.

LEMMA 3.9. *Let $k \in K \subset [B, BF]$ represent an orientable fibration $F \to E \to B$ such that A is the kernel of $\pi_1 E \twoheadrightarrow \pi_1 B$. Then $A \rightarrowtail \pi_1 E \twoheadrightarrow \pi_1 B$ is equivalent* *to the extension of* $\pi_1 B$ *by* A *determined by* $\sigma(k) \in H^2(\pi_1 B;A)$.

LEMMA 3.10. *Let* $F \xrightarrow{i} E \rightarrow B$ *be an orientable pseudo-abelian fibration over a two dimensional CW complex B. If* $i : \pi_1 F \rightarrow \pi_1 E$ *factors through a finitely generated abelian group A, then there exists an orientable pseudo-abelian fibration* $F \rightarrow \bar{E} \rightarrow B$ *such that A is the kernel of* $\pi_1 \bar{E} \rightarrow \pi_1 \bar{B}$ *and there exists* $\phi : \pi_1 \bar{E} \twoheadrightarrow \pi_1 E$ *satisfying the conditions of Lemma 3.2.*

Remark 3.11. The proofs for 3.9 and 3.10 will be given in the Appendix.

THEOREM B. *Let* $F \xrightarrow{i} E \rightarrow B$ *be an orientable pseudo-abelian fibration, with F and B finitely dominated, such that* $i : \pi_1 F \rightarrow \pi_1 E$ *factors through a free abelian group A. If* $\pi_1 EF$ *is free abelian and* $\chi(F) = 0$ *then* $O(E) = 0$.

Proof. Let B^2 be the two skeleton of B and let $F \rightarrow E_2 \rightarrow B^2$ be the restriction of E over B^2. Let $F \rightarrow \bar{E} \rightarrow B^2$ be the fibration from Lemma 3.10 corresponding to A and $F \rightarrow E_2 \rightarrow B^2$. Let $\bar{B}$ be a CW complex dominated by $B^2 \vee VS^3$, such that $\bar{B}^2 = B^2$, $\chi(\bar{B}) = \chi(B)$ and $O(\bar{B}) = O(B)$. $\bar{B}$ can be obtained from $B^2 \vee VS^3$ by attaching cells of dimension 4 and higher, see [20] Theorem F. Take the pullback of $F \rightarrow E \rightarrow B^2$ via the map $\bar{B} \rightarrow B^2 \vee VS^3 \rightarrow B^2$. The resulting fibration over $\bar{B}$ has the same maps on fundamental groups as $F \rightarrow \bar{E} \rightarrow B^2$. From Lemma 3.8 we get that $O(\bar{E}) = 0$ and from Lemma 3.2 we get that $O(E)$ is a homomorphic image of $O(\bar{E})$ so $O(E) = 0$.

Example 3.12. The obvious question is: Are there any spaces F where $\pi_1 EF$ is free abelian? If $G_1(F)$ is the Gottlieb subgroup from 3.3, then $\pi_1 EF = \pi_1 F / G_1(F)$. Theorem 2.1 in [15] states that $G_1(X \times Y) = G_1(X) \times G_1(Y)$, which implies for $F = X \times Y$ that $\pi_1 EF = \pi_1 EX \times \pi_1 EY$. If X is an H space then $G_1(X) = \pi_1 X$ and so $\pi_1 EX = 0$. Another example: Let X be an H space and let Y be a simply connected complex or a finite complex with $\pi_1 Y$ free abelian and $\chi(Y) \neq 0$. In either case $G_1(Y) = 0$ and $\pi_1 E(X \times Y) = \pi_1 EY = \pi_1 Y =$ free abelian.

LEMMA 3.13. *Let* $F \rightarrow E \rightarrow B$ *be a Steenrod bundle associated to a principal pseudo-abelian T bundle,* $T \rightarrow X \rightarrow B$, *where T is a torus and F is a finite T-CW complex. If B is finitely dominated and* $\chi(F) = 0$, *then* $O(E) = 0$.

Proof. First, let us consider the case of $F = T/H$, where H is a proper closed subgroup of T. $F \times_T X = T/H \times_T X = X/H$ and X/H is the total space of another principal torus bundle $T/H \rightarrow X/H \rightarrow B$. Let ν be the kernel of $\pi_1 X \rightarrow \pi_1 B$ and let σ be the kernel of $\pi_1(X/H) \rightarrow \pi_1 B$. We will show that $\sigma \rightarrow H_1(X/H) \rightarrow H_1 B$ is a short exact sequence and then we can apply Theorem B to the pseudo-abelian fibration $T/H \rightarrow X/H \rightarrow B$ to get $O(X/H) = 0$, since $\pi_1 T$ is a free abelian group and $\pi_1 ET = 0$.

There is a natural map of short exact sequences:

$$\begin{array}{ccccc} \nu & \rightarrowtail & \pi_1 X & \rightarrow & \pi_1 B \\ \downarrow & & \downarrow & & \| \\ \sigma & \rightarrowtail & \pi_1(X/H) & \twoheadrightarrow & \pi_1 B \end{array}$$

Look at the homology exact sequences:

$$\begin{array}{ccccccc} H_2(\pi_1 B) & \to & H_1(\nu) & \rightarrowtail & H_1 X & \twoheadrightarrow & H_1 B \\ \| & & \downarrow & & \downarrow & & \| \\ H_2(\pi_1 B) & \to & H_1(\sigma) & \to & H_1(X/H) & \twoheadrightarrow & H_1 B \end{array}$$

By hypothesis $\nu = H_1(\nu) \rightarrowtail H_1 X$, therefore $H_2(\pi_1 B) \to H_1 \nu$ is the zero map. Then, $H_2(\pi_1 B) \to H_1(\sigma)$ is the zero map and $\sigma = H_1(\sigma) \rightarrowtail H_1(X/H)$.

Now we can consider the general case of F being any finite T-CW complex. If all the isotropy subgroups of T are proper, then by Theorem 2.5 we get

$$O(E) = O(F \times_T X) = \Sigma \chi_H c(F)(j_H)_* O(X/H) = 0.$$

If, however, T fixes a point in F then the map $T \to F$ induces the trivial homomorphism $\pi_1 T \to \pi_1 F$. Let $F \to F \times_T ET \to BT$ be the classifying fibration for associated T bundles with fiber F. $\pi_2 BT = \pi_1 T \to \pi_1 F$ is zero, so $\pi_1(F \times_T ET) = \pi_1 F$ and by the same argument as in the proof of Theorem A we see that $O(E) = O(F \times B) = 0$.

THEOREM 3.14. *Let $F \to E \to B$ be a Steenrod bundle associated to a principal pseudo-abelian G-bundle $G \to X \to B$, where G is a connected, compact Lie group and F is a finite G-CW complex of Euler characteristic zero.* If B *is finitely dominated, then* $O(E) = 0$.

Proof. *Step* 1. Assume first that $\pi_1 G$ injects into $\pi_1 X$. We will compute $O(E)$ by changing the fibration somewhat, but leaving it the same over the 2 skeleton of B. The new fibration $F \to \bar{E} \to B$ will have $O(\bar{E})$ equal to $O(E)$ by 3.2 and it will admit a reduction to a pseudo-abelian torus bundle. Applying 3.13 we get $O(E) = O(\bar{E}) = 0$.

Let T be a maximal torus of G, the inclusion of T into G induces a map of classifying spaces $BT \to BG$. consider the following diagram:

$$\begin{array}{ccccc} [B,BG] & \to & [B, K(\pi_2 BG, 2)] & = & H^2(B;\pi_1 G) \\ \uparrow & & & & \cup \\ [B,BT] & = & H^2(B;\pi_1 T) \supset \operatorname{Ext}(H_1 B;\pi_1 T) & \twoheadrightarrow & \operatorname{Ext}(H_1 B;\pi_1 G) \end{array}$$

The first horizontal map comes from the natural inclusion of the simply connected space BG in the Eilenberg-MacLane space $K(\pi_2 BG; 2)$. The map on Ext is a surjection because $\pi_1 T$ surjects onto $\pi_1 G$. Let $k \in [B,BG]$ be the classifying map for the principal G bundle $X \to B$. By hypothesis $H_1 G \rightarrowtail H_1 X$, so the map from $H_2 B$ to $H_1 G$ is the zero map, and therefore the image of k in $H^2(B;\pi_1 G)$ lies in $\operatorname{Ext}(H_1 B;\pi_1 G)$. Therefore, there exists an element $r \in \operatorname{Ext}(H_1 B;\pi_1 T)$ mapping onto the same element in $\operatorname{Ext}(H_1 B;\pi_1 6)$ as k did. Let $\bar{k}$ be the image of r in $[B,BG]$. Then $\bar{k}$ determines a fibration $F \to \bar{E} \to B$, which by Lemma 3.9 has the same maps on fundamental groups as $F \to E \to B$ had because $\sigma(\bar{k}) = \sigma(k)$. So we can apply Lemma 3.2 to get $O(\bar{E}) \doteq O(E)$. By a standard argument the map $r : B \to BT$ defines a reduction of $F \to \bar{E} \to B$ to a torus bundle.

Step 2. We proceed just like we did in the proof of Theorem B. Let $\bar{B}$ be the space from that proof and let $A = \pi_1 G$. Construct the principal pseudo-abelian bundle $G \to X \to B^2$ from Lemma 3.10 and let $G \to Y \to \bar{B}$ be the pullback via the map $\bar{B} \to B^2$. Then the bundles $F \to E \to B$ and $F \to F \times_G Y \to \bar{B}$ satisfy conditions of Lemma 3.2 and, therefore, there is a map ϕ such that $\phi_* O(F \times_G Y) = O(E)$. From Step 1. we know that $O(Fx_G Y) = 0$ and so $O(E) = 0$.

THEOREM C. Let $F \to E \to B$ be a Steenrod bundle associated to a principal pseudo-abelian G bundle, where G is a connected, compact Lie group acting smoothly on the compact manifold F. If $\chi F = 0$ and B is finitely dominated, then $O(E) = 0$.

Proof. F has the G-homotopy type of a finite G complex, see for example [22]. Therefore we can apply 3.14.

APPENDIX

LEMMA 3.9. *Let $k \in K$ represent an orientable fibration $f \to E \to B$ such that A is the kernel of $\pi_1 E \to \pi_1 B$. Then $A \rightarrowtail \pi_1 E \twoheadrightarrow \pi_1 B$ is equivalent to the extension of $\pi_1 B$ by A determined by $\sigma(k) \in H^2(\pi_1 B; A)$.*

Proof. We will use $\sigma(k) \in H^2(\pi_1 B; A)$ to denote three things: an equivalence class of extensions of $\pi_1 B$ by A, or a group representing this class, or a homotopy class of maps from $K(\pi_1 B;1)$ to $K(A;2)$. The geometric description of the map $\sigma : K \to [K(\pi_1 B;1), K(A;2)]$ is the following. Given $k : B \to BF$, then $\sigma(k)$ is the map making the next diagram commutative.

$$\begin{tikzcd}
B \arrow[r, "k"] \arrow[d] & BF \arrow[d] \\
K(\pi_1 B;1) \arrow[r, "\sigma(k)"] & K(A;2).
\end{tikzcd}$$

Let $PK \to K(A;2)$ be the path space fibration and let $Q \to BF$ and $P \to B$ be pullbacks of this fibration given by the following diagram:

$$\begin{tikzcd}
K(A;1) \arrow[r, equal] \arrow[d] & K(A;1) \arrow[r, equal] \arrow[d] & K(A;1) \arrow[d] \\
P \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & PK \arrow[d] \\
B \arrow[r, "k"] & BF \arrow[r] & K(A;2).
\end{tikzcd}$$

We also have this pullback diagram:

$$\begin{tikzcd}
K(A;1) \arrow[r, equal] \arrow[d] & K(A;1) \arrow[r, equal] \arrow[d] & K(A;1) \arrow[d] \\
P \arrow[r] \arrow[d] & K(\sigma(k);1) \arrow[r] \arrow[d] & PK \arrow[d] \\
B \arrow[r] & K(\pi_1 B;1) \arrow[r, "\sigma(k)"] & K(A;2).
\end{tikzcd}$$

When we look at the exact sequences of fundamental groups of this last diagram, we see immediately that $A \rightarrowtail \pi_1 P \twoheadrightarrow \pi_1 B$ is equivalent to $\sigma(k)$ as extensions of $\pi_1 B$ by A. So all we have to do is to show that $A \rightarrowtail \pi_1 p \twoheadrightarrow \pi_1 B$ is equivalent to $A \rightarrowtail \pi_1 E \twoheadrightarrow \pi_1 B$.

Let $F \to K(\pi_1 F; 1) \to K(A; 1)$ be the map which realizes our fixed epimorphism $\pi_1 F \twoheadrightarrow A$ as the map of the fundamental groups and consider this diagram:

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & K(A;1) \arrow[d] \\
EF \arrow[d] & Q \arrow[d] \\
BF \arrow[r, equal] & BF
\end{tikzcd}$$

Claim. There is a map from $EF \to Q$ making the diagram commute.

Proof. The map $EF \to BF \to K(\pi_2 BF; 2) \to K(A; 2)$, when thought of as an element in $H^2(EF; A)$, is the image of an element in $H^1(F; A)$ via the map $H^1(F; A) \to H^2(BF; A) \to H^2(EF; A)$. It then follows, from the Serre's exact sequence in cohomology, that the map $EF \to K(A; 2)$ is null homotopic and therefore there is a map $EF \to PK$ making the following diagram commute.

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & K(A;1) \arrow[d] \\
EF \arrow[r] \arrow[d] & PK \arrow[d] \\
BF \arrow[r] & K(A;2)
\end{tikzcd}$$

The claim follows once we realize that Q is the pullback by $BF \to K(A; 2)$.

This determines a unique map from $E \to P$ such that the following diagram commutes.

$$\begin{tikzcd}
& P \arrow[rr] \arrow[ddl] & & Q \arrow[ddl] \\
E \arrow[ur, dashed] \arrow[rr] \arrow[d] & & EF \arrow[ur] \arrow[d] & \\
B \arrow[rr] & & BF &
\end{tikzcd}$$

Looking at the exact sequence of fundamental groups, we see that $A \rightarrowtail \pi_1 E \twoheadrightarrow \pi_1 B$ is indeed equivalent to $A \rightarrowtail \pi_1 p \twoheadrightarrow \pi_1 B$.

LEMMA 3.10. *Let $F \to E \to B$ be an orientable pseudo-abelian fibration over a two-dimensional CW complex B. Let ν be the kernel of $\pi_1 E \twoheadrightarrow \pi_1 B$ and let $\pi_1 F \twoheadrightarrow A \twoheadrightarrow \nu$ be surjections, such that the map of fundamental groups induced by $F \to E$ is the composition $\pi_1 F \twoheadrightarrow A \twoheadrightarrow \nu \rightarrowtail \pi_1 E$. If A is a finitely generated* *abelian group, then there exists an orientable pseudo-abelian fibration $F \to \bar{E} \to B$ such that:*

1. *A is a kernel of $\pi_1\bar{E} \twoheadrightarrow \pi_1B$, and*

2. *there exists a surjection $\phi : \pi_1\bar{E} \twoheadrightarrow \pi_1E$ satisfying the conditions of Lemma* 3.2.

Proof.

Step 1. The universal orientable fibration $F \to EF \to BF$ induces an exact sequence $\pi_2BF \to \pi_1F \to \pi_1EF$. BF is simply connected so there is the canonical inclusion $BF \to K(\pi_2BF; 2)$. Let $G_1(F)$ be the image of the boundary map $\pi_2BF \to \pi_1F$. Then the map $\pi_2BF \twoheadrightarrow G_1(F)$ induces a map on the corresponding Eilenberg-MacLane spaces $K(\pi_2BF; 2) \to K(G_1(F); 2)$. Let $k : B \to BF$ be a classifying map for the given fibration $F \to E \to B$, and let

$$\delta : B \to BF \to K(\pi_2BF; 2) \to K(G_1(F); 2).$$

The composition $\pi_2B \xrightarrow{\pi_2\delta} G_1(F) \to \pi_1F$ is the boundary map in the exact sequence of homotopy groups $\pi_2B \to \pi_1F \to \pi_1E \to \pi_1B$; and $H_2B \xrightarrow{\pi_2\delta} G_1(F) \to H_1F$ is the boundary map in Serre's exact sequence in homology for an oriented fibration $H_2B \to H_1F \to H_1E \to H_1B$.

Consider the following commutative diagram:

$$\begin{tikzcd}
\pi_2B \arrow[rr] \arrow[dr, "\pi_2\delta"] \arrow[dd] & & \pi_1F \arrow[r, two heads] \arrow[dd] & \nu \\
& G_1(F) \arrow[ur] & & \\
H_2B \arrow[ur, "H_2\delta"] \arrow[rr] & & H_1F \arrow[uur, dashed, "\exists!"'] &
\end{tikzcd}$$

By hypothesis, ν is an abelian group so the map $\pi_1F \to \nu$ factors through the abelianization H_1F of π_1F. It is clear from the diagram above that $H_2B \to H_1F \to \nu$ is the zero map if and only if the image of $H_2\delta$ is the same as the image of $\pi_2\delta$. $H_2B \to \nu$ is the zero map if and only if ν injects into H_1E if and only if $\nu \to \pi_1E \twoheadrightarrow \pi_1B$ is a pseudo-abelian extension. Therefore, an orientable fibration is pseudo-abelian if and only if $\operatorname{im} H_2\delta = \operatorname{im} \pi_2\delta$.

Step 2. Let ω be the kernel of the map $\pi_1F \twoheadrightarrow A$, and let D be the kernel of the map $A \twoheadrightarrow \pi_1EF$. Then there are the following short exact sequences.

$$\begin{tikzcd}
\omega \arrow[r, equal] \arrow[d, tail] & \omega \arrow[d] & \\
G_1(F) \arrow[r, tail] \arrow[d] & \pi_1F \arrow[r, two heads] \arrow[d] & \pi_1EF \arrow[d, equal] \\
D \arrow[r, tail] & A \arrow[r, tail] & \pi_1EF.
\end{tikzcd}$$

Step 3. Ultimately, we want to construct a fibration $F \to \bar{E} \to B$ satisfying the conditions of the Lemma. The boundary map $\bar{\delta}: B \to K(G_1(F); 2)$ corresponding to this new fibration will have to satisfy the following:

$$\operatorname{im} H_2 \bar{\delta} = \operatorname{im} \pi_2 \bar{\delta} = \omega \subset \operatorname{im} \pi_2 \delta = \operatorname{im} H_2 \delta.$$

The first equality is equivalent (by Step 1) to $F \to \bar{E} \to B$ being pseudo-abelian. The second equality will insure that A will be the kernel of $\pi_1 \bar{E} \twoheadrightarrow \pi_1 B$. In this step we will construct one such map $\bar{\delta}$.

Let r be an endomorphism of the finitely generated abelian group $\operatorname{im} H_2 \delta$, whose image is precisely $\omega \subset \operatorname{im} H_2 \delta$. Such an r is easy to construct. We can write ω as a direct sum of cyclic groups C_i and $\operatorname{im} H_2 \delta$ as a direct sum of cyclic groups $C_i^!$, in such a way that C_i imbeds in $C_i^!$ with index n_i. Then let r be defined on each summand $C^!$ as just multiplication by n_i. Now let α be the composition $H_2 B \twoheadrightarrow \operatorname{im} H_2 \delta \xrightarrow{r} \operatorname{im} H_2 \delta$. α has the property that image α = image $\alpha h = \omega$, where $h: \pi_2 B \to H_2 B$ is the Hurewicz homomorphism. If we think of α as a map from $H_2 B$ to ω then it can be realized by a topological map $\beta: B \to K(\omega; 2)$ because $H^2(B; \omega)$ surjects onto $\operatorname{Hom}(H_2 B, \omega)$. Let $\bar{\delta}$ be the composite

$$B \to K(\omega, 2) \to K(G_1(F); 2).$$

Step 4. We want to find an element $\bar{k} \in [B, BF]$ such that:

1. $\bar{k}$ maps to the element $\pi_2 \bar{\delta} \in \operatorname{Hom}(\pi_2 B; G_1 F)$ via the maps

$$[B, BF] \twoheadrightarrow H^2(B; \pi_2 BF) \twoheadrightarrow H^2(B; G_1 F) \to \operatorname{Hom}(\pi_2 B; G_1(F)),$$

which by the previous step will guarantee the correct kernel of $\pi_1 \bar{E} \twoheadrightarrow \pi_1 B$. And

2. $\sigma(\bar{k}) \in \operatorname{Ext}(H_1 B; A) \subset H^2(\pi_1 B; A)$ and $\sigma(\bar{k})$ maps to the pseudo-abelian extension $\nu \rightarrowtail \pi_1 E \twoheadrightarrow \pi_1 B$ via the surjection $\operatorname{Ext}(H_1 B; A) \twoheadrightarrow \operatorname{Ext}(H_1 B; \nu)$, which, by Lemma 3.9, will guarantee the existence of $\phi: \pi_1 \bar{E} \twoheadrightarrow \pi_1 E$.

Consider the following diagram, where both straight lines are exact.

$$\begin{tikzcd}
 & {[B;BF]} \arrow[d] \arrow[r, two heads] & H^2(B;\pi_2 BF) \arrow[dl, two heads] \\
\operatorname{Ext}(H_1B;G_1(F)) \arrow[r, tail] \arrow[d, equal] & H^2(B;G_1(F)) \arrow[r, two heads] & \operatorname{Hom}(H_2B;G_1(F)) \\
\operatorname{Ext}(H_1B;G_1(F)) \arrow[r, two heads] & \operatorname{Ext}(H_1B;D) \arrow[d] & \\
 & \operatorname{Ext}(H_1B;A) \arrow[r, two heads] \arrow[d] & \operatorname{Ext}(H_1B;\nu) \arrow[dl] \\
 & \operatorname{Ext}(H_1B;\pi_1 EF) &
\end{tikzcd}$$

$[B, BF]$ surjects onto $H^2(B; G_1(F))$ because B is a two dimensional CW complex.

Also notice, that the map on Ext induced by $\nu \twoheadrightarrow \pi_1 EF$ sends the extension $\nu \rightarrowtail \pi_1 E \twoheadrightarrow \pi_1 B$ into the zero element. That is obvious from the diagram

$$\begin{tikzcd}
\nu \arrow[r, tail] \arrow[d] & \pi_1 E \arrow[r, two heads] \arrow[d] \arrow[dl, "c"'] & \pi_1 B \arrow[d, equal] \\
\pi_1 EF \arrow[r, tail] & ? \arrow[r, two heads] & \pi_1 B
\end{tikzcd}$$

because c defines a splitting of the bottom short exact sequence.

By a simple diagram chase, there is an element $x \in \mathrm{Ext}(H_1 B; G_1(F))$ mapping onto $\nu \rightarrowtail \pi_1 E \twoheadrightarrow \pi_1 B$ in $\mathrm{Ext}(H_1 B; \nu)$. Let $\bar{k} \in [B, BF]$ be any element which maps onto $x + \bar{\delta} \in H^2(B; G_1(F)$. It is a straightforward verification to see that both conditions 1 and 2 are satisfied.

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