L^p MULTIPLIERS ON THE HEISENBERG GROUP

Leonede de Michele and Giancarlo Mauceri

1. INTRODUCTION

The Heisenberg group is the simplest example in the class of stratified groups. On these groups one can define a one-parameter family of anisotropic dilations and an homogeneous norm. Hence it is possible to extend to them many of the standard constructions of Euclidean spaces: singular integrals, homogeneous differential operators, Lipschitz classes etc. ([4], [10], [9]). However, except for a few instances where the representations of the group play a peripheral role, the noncommutative Fourier transform has not been so far a tool in this kind of harmonic analysis. More recently an attempt to make the Fourier transform on the Heisenberg group a usable tool in the study of the Schwartz space and of homogeneous differential operators has been made by Geller [6].

In this framework, using the theory of singular integrals on homogeneous spaces developed by Coifman, De Guzman and Weiss [2], we extend to the three-dimensional Heisenberg group the classical multiplier theorem of Hörmander.

We recall that Hörmander's theorem is stated in the following way [8].

THEOREM 1. *Let M be a function of a class* C^k in $\mathbf{R}^n \setminus \{0\}$, $k \geq \frac{n}{2} + 1$. *Assume that*

i) $M \in L^\infty(\mathbf{R}^n)$

ii) $$\sup_{R \in [0,+\infty)} R^{2|\alpha|-n} \int_{R<|\xi|\leq 2R} |\partial^\alpha M(\xi)|^2 d\xi \leq C$$

for all differential monomials ∂^α of order $|\alpha| \leq k$. Then the linear operator T_M *defined by*

$$T_M f(x) = \int e^{-2i\pi\langle x,\xi\rangle} M(\xi) \hat{f}(\xi)\, d\xi$$

is bounded on $L^P(R^n)$, $1 < p < \infty$.

This theorem has already been extended to $SU(2)$ by Coifman and Weiss [1] and to the group of Euclidean motions by Rubin [11].

In the next section we review some basic tools of the harmonic analysis on

Received April 14, 1978. Revision received November 3, 1978.

This paper was written while the authors were visiting the University of California, Berkeley, with a NATO fellowship.

Michigan Math. J. 26 (1979).

H and we state our main result. In the section 3 we prove the multiplier theorem and in the last section we give some applications to several multipliers on H.

This paper grew out of several discussions with Guido Weiss and Ronald Coifman during a visit of the authors to Washington University. It is a pleasure to thank them both for their warm hospitality and their suggestions.

2. PRELIMINARIES

The Heisenberg group H is the Lie group whose underlying manifold is $\mathbf{R} \times \mathbf{C}$ and whose composition law is given by

$$(t,z)(t',z') = (t + t' + 2\mathrm{Im} z\bar{z}', z + z') \qquad t,t' \in \mathbf{R},\ z,z' \in \mathbf{C}.$$

The Haar measure on H coincides with the Lebesgue measure dV on $\mathbf{R} \times \mathbf{C}$. H is endowed with a one parameter family of anisotropic dilations $\{\delta_\varepsilon : \varepsilon > 0\}$ defined by $\delta_\varepsilon(t,z) = (\varepsilon^2 t, \varepsilon z)$. A function f on H will be called homogeneous of degree d if $f \circ \delta_\varepsilon = \varepsilon^d f$ for every $\varepsilon > 0$. The Heisenberg Lie algebra $\mathfrak{h}$ is the algebra of the left invariant vector fields on H generated by $T = \partial_t$, $Z = \partial_z + i\bar{z}\partial_t$, $\bar{Z} = \partial_{\bar{z}} - iz\partial_t$. The only nonzero commutation relation is $[Z,\bar{Z}] = -2iT$. The irreducible unitary representations of H split into two classes. The one-dimensional representations are just the usual characters of $\mathbf{C}$ lifted to H. Since these representations form a set of zero Plancherel measure we will not be concerned with them further. The infinite dimensional representations are classified by a parameter $\lambda \in \mathbf{R}^*$, the set of nonzero real numbers, and may be realized as follows. For every $\lambda > 0$ let $\mathscr{H}_\lambda$ be the Bargmann space of holomorphic functions F on $\mathbf{C}$ such that

$$\|F\|^2 = (2\lambda/\pi) \int_C |F(\zeta)|^2 \exp(-2\lambda|\zeta|^2)\, d\zeta\, d\bar{\zeta} < \infty$$

Then $\mathscr{H}_\lambda$ is an Hilbert space and an orthonormal basis for $\mathscr{H}_\lambda$ is given by the monomials $F_{\lambda,\alpha}(\zeta) = (\sqrt{2\lambda}\,\zeta)^\alpha/\sqrt{\alpha!}$, $\alpha = 0, 1, \ldots$. For $\lambda \in \mathbf{R}^*$ the representation Π_λ acts on $\mathscr{H}_{|\lambda|}$ by

$$\Pi_\lambda(t,z)F(\zeta) = \exp(i\lambda t + 2\lambda(\zeta z - |z|^2/2)F(\zeta - \bar{z}) \qquad \text{if } \lambda > 0$$
$$\Pi_\lambda(t,z)F(\zeta) = \exp(i\lambda t - 2\lambda(\zeta\bar{z} - |z|^2/2)F(\zeta - z) \qquad \text{if } \lambda < 0.$$

For $f \in L^1(H)$ set $\hat{f}(\lambda = \int_H f(t,z)\Pi_\lambda(t,z)\,dV$, where the integral is defined in the weak sense. Then the operator valued function $\lambda \to \hat{f}(\lambda)$, $\lambda \in \mathbf{R}^*$ is the Fourier transform of f. For $(\lambda, m, \alpha) \in \mathbf{R} \times \mathbf{Z} \times \mathbf{N}$ define as in [6], the partial isometry $W_\alpha^m(\lambda)$ on $\mathscr{H}_{|\lambda|}$ by

$$W_\alpha^m(\lambda)F_{\beta,\alpha} = (-1)^m \delta_{\alpha+m,\beta} F_{\alpha,\lambda} \quad \text{for} \quad m \geq 0, \lambda > 0$$
$$W_\alpha^m(\lambda)F_{\beta,\alpha} = \delta_{\alpha,\beta} F_{\alpha+m,\lambda} \quad \text{for} \quad m < 0, \lambda > 0$$

and for negative λ by $W_\alpha^m(\lambda) = [W_\alpha^m(-\lambda)]^*$. Since $\{W_\alpha^m(\lambda): (m,\alpha) \in \mathbf{Z} \times \mathbf{N}\}$ is a orthonormal basis for the Hilbert-Schmidt operators on $\mathscr{H}_{|\lambda|}$, given a function $f \in L^2(H)$ such that $f(t,z) = \sum_{m,\alpha} f_m(t,r)\, e^{im\theta}$, $z = re^{i\theta}$, we have

$$\hat{f}(\lambda) = \sum_{m,\alpha} \mathscr{R}_f(\lambda, m, \alpha)\, W_\alpha^m(\lambda)$$

where

$$\mathscr{R}_f(\lambda, m, \alpha) = \int_H f_m(t, |z|)\, e^{i\lambda t}\, \ell_\alpha^{|m|}(2|\lambda|\,|z|^2)\, dV \tag{1}$$

and $\ell_\alpha^{|m|}$ is the Laguerre function of type $|m|$ and degree α [6]. Denoting by $\|\hat{f}(\lambda)\|_{HS}$ the Hilbert-Schmidt norm of $\hat{f}(\lambda)$ we have the Plancherel formulae

$$\|f\|_2^2 = \pi^2 \int_{-\infty}^{+\infty} \|\hat{f}(\lambda)\|_{HS}^2 |\lambda|\, d\lambda \qquad f \in L^1 \cap L^2(H)$$

that enables us to extend the Fourier transform as an isometry from $L^2(H)$ onto the Hilbert space of the operator valued functions $\lambda \to A(\lambda)$, $\lambda \in \mathbf{R}^*$ such that

i) $A(\lambda)$ is an Hilbert-Schmidt operator on $\mathscr{H}_\lambda$, for almost every $\lambda \in \mathbf{R}^*$.

ii) $(A(\lambda)\xi, \eta)$ is a measurable function of λ for every ξ, $\eta \in \mathscr{H}_\lambda$

iii) $\|A\|_2^2 = \pi^2 \int_{-\infty}^{+\infty} \|A(\lambda)\|_{HS}^2 |\lambda|\, d\lambda < +\infty$.

Definition. A left invariant multiplier of $L^p(H)$, $1 \le p \le \infty$, is an operator valued function $M: \lambda \to M(\lambda)$, $\lambda \in \mathbf{R}^*$ such that

a) for every $\lambda \in \mathbf{R}^*$, $M(\lambda)$ is a bounded operator on $\mathscr{H}_\lambda$

b) the operator T_M defined by $(T_M f)\hat{}\,(\lambda) = M(\lambda)\hat{f}(\lambda)$, $f \in L^1 \cap L^p(H)$, extends to a bounded operator on $L^p(H)$.

It is well known that M is a left multiplier of $L^p(H)$ for some $1 \le p \le \infty$ if and only if T_M is a right invariant operator on $L^p(H)$. From Plancherel formula it follows immediately that M is a left L^2-multiplier if and only if $\sup_{\lambda \in \mathbf{R}^*} \|M(\lambda)\| < +\infty$. Since H is amenable, by a result of Herz [7], every left multiplier of $L^p(H)$, $1 \le p \le \infty$, is a left multiplier of $L^2(H)$. We also remark that everything we say for left multipliers may be translated for right multipliers similarly defined, because the group is unimodular.

Our next step is to define certain difference-differential operators on the dual of the Heisenberg group which play the role of the differential monomials in Hörmander's theorem. These operators will be defined as the Fourier transform of the multiplication by homogeneous polynomials on H. Given a polynomial P in the variables $t, z, \bar{z}$ let Δ_p be the operator on the space of functions $\mathscr{R} = \mathscr{R}(\lambda, m, \alpha)$,

$(\lambda, m, \alpha) \in \mathbf{R}^* \times \mathbf{Z} \times \mathbf{N}$ such that $\mathscr{R} = \mathscr{R}_f$ for some f in the Schwartz space $\mathscr{S}(H)$, defined by

$$(Pf)\hat{\ }(\lambda) = \sum_{m,\alpha} \Delta_p \mathscr{R}_f(\lambda, m, \alpha)\, W_\alpha^m(\lambda)$$

By means of the Laguerre transform formula (1) and the classical relations between Laguerre functions, it is not hard to obtain an explicit expression for the operator Δ_p. It is obviously enough to derive it for the operators Δ_t, Δ_z, $\Delta_{\bar z}$. Introducing the translation operators

$$\begin{aligned} \sigma_k \mathscr{R}(\lambda, m, \alpha) &= \mathscr{R}(\lambda, m+k, \alpha) \\ \tau_k \mathscr{R}(\lambda, m, \alpha) &= \mathscr{R}(\lambda, m, \alpha + k) \qquad (\lambda, m, \alpha) \in \mathbf{R}^* \times \mathbf{Z} \times \mathbf{N},\ k \in \mathbf{Z}, \end{aligned}$$

where the undefined terms in the above formulas are intended to be zero, one has

$$\begin{aligned} \Delta_t &= -i\partial_\lambda - \frac{i}{2\lambda}\left(\tau_0 + \sqrt{\alpha(\alpha+|m|)}\,\tau_{-1} - \sqrt{(\alpha+1)(\alpha+|m|+1)}\,\tau_1\right) \\ \Delta_z &= -(2|\lambda|)^{-1/2}\left(\sqrt{\alpha+1}\,\tau_1 - \sqrt{\alpha+|m|}\,\tau_0\right)\sigma_{-1} \\ \Delta_{\bar z} &= (2|\lambda|)^{-1/2}\left(\sqrt{\alpha+|m|+1}\,\tau_0 - \sqrt{\alpha}\,\tau_{-1}\right)\sigma_1. \end{aligned}$$

By the above formulas we can extend the operators Δ_p as formal difference-differential operators acting on functions defined on $\mathbf{R}^* \times \mathbf{Z} \times \mathbf{N}$. If

$$M(\lambda) = \sum_{m,\alpha} B(\lambda, m, \alpha)\, W_\alpha^m(\lambda)$$

we denote by $\Delta_p M(\lambda)$ the operator $\sum_{m,\alpha} \Delta_p B(\lambda, m, \alpha)\, W_\alpha^m(\lambda)$.

To reproduce on H the decomposition of the dual of $\mathbf{R}^n$ into dyadic annuli, which amounts to a partition of the identity into spectral projections of the Laplace operator, we consider the subelliptic Laplacian $\mathscr{L}_o = -\frac{1}{2}(Z\bar Z + \bar Z Z)$. The Fourier transform of $\mathscr{L}_o$ is the operator valued function

$$\hat{\mathscr{L}}_0(\lambda) = \sum_\alpha [(2\alpha+1)|\lambda|]\, W_\alpha^0(\lambda),\ \lambda \in \mathbf{R}^*.$$

Thus we introduce the partition of the identity $I = \sum_{k \in Z} \Pi_{2^k R}$, $R > 0$, where Π_s is the spectral projection of $\mathscr{L}_o$ corresponding to the multiplier

$$\hat\Pi_s(\lambda) = \sum_{s < (2\lambda+1)|\lambda| \le 2s} W_\alpha^0(\lambda).$$

The multiplier theorem can now be stated in the following way.

THEOREM 2. *Let M be an operator valued function such that*

(2) $$\sup_{\lambda \in \mathbf{R}^*} \|M(\lambda)\| < +\infty$$

(2′) $$\sup_{R \in [0,+\infty)} R^{\deg P - 2} \int_{-\infty}^{+\infty} \|\Delta_P M(\lambda) \hat{\Pi}_R(\lambda)\|_{HS}^2 |\lambda| \, d\lambda < +\infty$$

for every normalized monomial P, homogeneous on H of degree ≤ 4. *Then M is a multiplier of* $L^p(H)$, $1 < p \leq 2$, *and is weak type* $(1,1)$.

Remark. The corresponding theorem for $L^p(H)$, $2 \leq p < +\infty$, follows from the facts that the boundedness of the operator T_M on $L^p(H)$ is equivalent to the boundedness of its adjoint T_M^* on $L^q(H)$, $p^{-1} + q^{-1} = 1$, and that T_M is the operator defined by the multiplier $N(\lambda) = [M(\lambda)]^*$.

3. PROOF OF THE MULTIPLIER THEOREM

According to the theory developed in [2, ch.3] the main steps in the proof of the multiplier theorem are two. The first one is the construction of a well behaved approximate identity $\{\phi_r : 0 < r < \infty\}$, which allows us to decompose the operator T_M as the sum of a series of singular integrals on H. The second step consists in proving uniform estimates for the L^p norms of the partial sums of the series. To obtain this one has to prove the following inequality:

(3) $$\int_H |T_M \psi_r(t,z)|^2 |\rho(t,z)|^2 dV < Cr \qquad 0 < r < \infty$$

where $\psi_r = \phi_r - \phi_{r/2}$ and $\rho(t,z) = t^2 + |z|^4$ is the usual pseudodistance on the homogeneous space H.

LEMMA 1. *Let* $\mathscr{R}_{\phi_r}(\lambda, 0, \alpha) = \exp[-r(2\alpha+1)^2\lambda^2]$, $\lambda \in \mathbf{R}^*$, $\alpha \in \mathbf{N}$. *Then* $\hat{\phi}_r(\lambda) = \sum_\alpha \mathscr{R}_{\phi_r}(\lambda, 0, \alpha) W_\alpha^0(\lambda)$ *is the Fourier transform of a function* ϕ_r *in* $\mathscr{S}(H)$. *Moreover* $\{\phi_r : 0 < r < \infty\}$ *is an approximate identity on H satisfying*

(i) $\int_H |\phi_r(t,z)| (1 + \rho(t,z)/r)^\eta dV < C$

(ii) $\int_H \phi_r dV = 1$

(iii) $\phi_r * \phi_s = \phi_s * \phi_r$

(iv) $\int_H |\phi_r((t,z)(t_0,z_0)^{-1}) - \phi_r(t,z)| \, dV < C(\rho(t_0,z_0)/r)^\eta$

(v) $\phi_r(t,z) = \phi_r(-t,-z)$ for some $\eta > 0$.

Proof. By [6, Th. 1] $\hat{\phi}_r(\lambda$ is the Fourier transform of a function $\phi_r \in \mathscr{T}(H)$. It follows immediately from the Fourier transform formula (1) that

$$\phi_r(t,z) = r^{-1}\phi_1(r^{-1/2}t, r^{-1/4}z).$$

From this (i) follows for every $\eta > 0$ and (iv) will follow for $\eta \leq 1/4$ once we have proved it for $r = 1$. Let $L \in \mathfrak{h}$ the normalized generator of the one parameter subgroup through $(0,z)^{-1}$. We have:

$$\int_H |\phi_1((t,z)(0,z_0)^{-1}) - \phi_1(t,z)|\,dV = \int_H \int_0^{|z_0|} |L\phi_1((t,z)(\exp(sL))|\,ds\,dV$$
$$= \rho(0,z_0)^{1/4}\|L\phi_1\|_1$$

by the right invariance of the Haar measure and the fact that $\rho(0,z_0) = |z_0|^4$. In general, if $g = (t,z)$, we choose a unit vector $z' \in \mathbf{C}$ and write $g = g_0 g_1 g_2 g_1^{-1} g_2^{-1}$ where $g_0 = (0,z)$, $g_1 = (0, i\sqrt{t}z'/2)$, and $g_2 = (0, \sqrt{t}z'/2)$. Since $\rho(g_0) = |z^4| \leq \rho(g)$ and $\rho(g_1) = \rho(g_2) = \dfrac{|t|^2}{16} \leq \rho(g)$, we can write $\phi_r((t,z)(t_0,z_0)^{-1}) - \phi_r(t,z)$ as a sum of five differences and apply the result just established to complete the proof of (iv). Since $\lim_{\lambda\to 0} \mathscr{R}_{\phi_r}(\lambda, 0, \alpha) = 1$ by (1) and the Lebesgue dominated convergence theorem we have (ii). Properties (iii) and (v) follows from the facts that the $\hat{\phi_r}(\lambda)$ are simultaneously diagonalized and $\hat{\phi}_r(\lambda) = \hat{\phi}_r(-\lambda)$.

Now, since for $f \in \mathscr{S}(H)$ $\mathscr{R}_{\rho f}(\lambda, m, \alpha) = \Delta_\rho \mathscr{R}_f(\lambda, m, \alpha)$, we have the weighted Plancherel formula

$$\int_H |\rho f|^2\,dV = \pi^2 \int_{-\infty}^{+\infty} \sum_{m,\alpha} |\Delta_\rho \mathscr{R}_f(\lambda, m, \alpha)|^2 |\lambda|\,d\lambda.$$

Thus (3) is implied by

$$\int_{-\infty}^{+\infty} \sum_{m,\alpha} |\Delta_\rho \mathscr{R}_{T_M\phi_r}(\lambda, m, \alpha)|^2 |\lambda|\,d\lambda \leq \mathrm{C}r, \qquad 0 < r < \infty \tag{4}$$

To prove (4) we need a Leibniz formula for the operator Δ_ρ. Set $\delta(t,z) = t + i|z|^2$, then:

$$\begin{aligned}\rho(t,z) = {}& \rho((t,z)(t',z')^{-1}) + \rho(t',z') + \delta((t,z)(t',z')^{-1})\bar{\delta}(t',z') \\ & + \bar{\delta}((t,z)(t',z')^{-1})\delta(t',z') - 2i(z-z')\delta((t,z)(t',z')^{-1})\bar{z}' \\ & - 2i(z-z')\bar{z}'\delta(t',z') + 2i\overline{(z-z')}\delta((t,z)(t',z')^{-1})\bar{z}' \\ & + 2i\overline{(z-z')}z'\bar{\delta}(t',z') + 4|z-z'|^2|z'|^2\end{aligned}$$

for all (t,z), $(t',z') \in H$. Hence

$$\begin{aligned}\Delta_p(M(\lambda)\hat{\psi}_r(\lambda)) = {} & (\Delta_p M(\lambda))\hat{\psi}_r(\lambda) + M(\lambda)(\Delta_p\hat{\psi}_r(\lambda)) + \Delta_\delta M(\lambda)\Delta_{\bar\delta}\hat{\psi}_r(\lambda) \\ & + \Delta_{\bar\delta}M(\lambda)\Delta_\delta\hat{\psi}_r(\lambda) - 2i\Delta_{z\delta}M(\lambda)\Delta_{\bar z}\hat{\psi}_r(\lambda) \\ & - 2i\Delta_z M(\lambda)\Delta_{\bar z\delta}\hat{\psi}_r(\lambda) + 2i\Delta_{\bar z\delta}M(\lambda)\Delta_z\hat{\psi}_r(\lambda) \\ & + 2i\Delta_{\bar z}M(\lambda)\Delta_{z\bar\delta}\hat{\psi}_r(\lambda) + 4\Delta_{|z|^2}M(\lambda)\Delta_{|z|^2}\hat{\psi}_r(\lambda).\end{aligned}$$

Condition (2′) in the hypothesis of the multiplier theorem may be used now to prove that the integral of the Hilbert-Schmidt norm squared of each summand in the Leibniz formula is bounded by Cr. For this we need the following lemma.

LEMMA 2. *For every polynomial P homogeneous of degree ≤ 4 one has the following estimate*

(5) $\sup\{|\Delta_p\mathscr{R}_{\psi_r}(\lambda,m,\alpha)|^2 : m\in\mathbf{Z}, R<(2\alpha+1)|\lambda|<2R\} < C_p r^2 R^{4-\deg P} f_P(rR^2)$

where $f_P \in L^1(\mathbf{R})$. Moreover

$$(6)\qquad |\mathscr{R}_{\psi_r}(\lambda,0,\alpha)| \leq \begin{cases} C_0 r(2\alpha+1)^2|\lambda|^2 & \text{for } r\lambda^2(2\alpha+1)^2 \leq 1 \\ 1 & \text{for } r\lambda^2(2\alpha+1)^2 > 1\end{cases}.$$

Proof. The last inequality is obvious. As for (5) we prove it only for

$$P(t,z,\bar z) = |z|^2,$$

because the proof carries over to the other cases with only minor modifications. We have

$$\begin{aligned}\Delta_{|z|^2}\mathscr{R}_{\phi_r}(\lambda,0,\alpha) &= (2|\lambda|)^{-1}\,[(2\alpha+1)\exp(-r\lambda^2(2\alpha+1)^2) \\ &\quad - (\alpha+1)\exp(-r\lambda^2(2\alpha+3)^2) - \exp(-r\lambda^2(2\alpha-1)^2)] \\ &= (2|\lambda|)^{-1}\exp(-r\lambda^2(2\alpha+1)^2)\,[(2\alpha+1) \\ &\quad - (\alpha+1)\exp(-8r\lambda^2(\alpha+1)) - \alpha\exp(8r\lambda^2\alpha)].\end{aligned}$$

Setting $(2\alpha+1)|\lambda| = \sigma$, we have

$$\begin{aligned}\Delta_{|z|^2}\mathscr{R}_{\phi_r}(\lambda,0,\alpha) = (4\lambda^2)^{-1}\exp(-r\sigma^2)\,[2\sigma &- (\sigma+|\lambda|) \\ &\times \exp(-4r|\lambda|(\sigma+|\lambda|)) - (\sigma-|\lambda|) \\ &\times \exp(4r|\lambda|(\sigma-|\lambda|))].\end{aligned}$$

Since $4|\lambda||\sigma-|\lambda|| < 8/9\,\sigma^2$, using the Taylor formula it is easily seen that:

$$|\Delta_{|z|^2}\mathscr{R}_{\phi_r}(\lambda,0,\alpha)| \leq Cr\sigma\exp(-r\sigma^2)\,[1 + r\sigma^2 + r\sigma^2\exp(8/9\,r\sigma^2)].$$

The conclusion for $\Delta_{|z|^2}\mathscr{R}_{\psi_r}$ readily follows.

Now consider

$$I=\int_{-\infty}^{+\infty}\sum_{m,\alpha}|\Delta_\rho \mathscr{R}_{T_M\psi_r}(\lambda,m,\alpha)|^2|\lambda|\,d\lambda.$$

Since $\hat{\psi}_r(\lambda)$ is a diagonal matrix

$$\begin{aligned}
I &= \int_{-\infty}^{+\infty}\sum_{\alpha}\|\Delta_\rho M(\lambda)\,W_\alpha^0(\lambda)\|_{HS}^2\,|\mathscr{R}_{\psi_r}(\lambda,0,\alpha)|^2|\lambda|\,d\lambda \\
&= \sum_{j<0}\int_{-\infty}^{+\infty}\|\Delta_\rho M(\lambda)\hat{\Pi}_{2^{j}(r)^{-1/2}}(\lambda)\|_{HS}{}^2|\mathscr{R}_{\psi_r}(\lambda,0,\alpha)|^2|\lambda|\,d\lambda \\
&\quad+\sum_{j\geq 0}{}'\int_{-\infty}^{+\infty}\|\Delta_\rho M(\lambda)\hat{\Pi}_{2^{j}(r)^{-1/2}}(\lambda)\|_{HS}^2|\mathscr{R}_{\psi_r}(\lambda,0,\alpha)|^2|\lambda|\,d\lambda \\
&= I_1+I_2.
\end{aligned}$$

Inequality (6), combined with the hypothesis (2′) for $P=\rho$, yields

$$I_1\leq C_1\sum_{j<0}2^{2j}r=C_2r.$$

On the other hand $I_2=\sum_{J>0}2^{-j}r=C_3r$. Next consider

$$J=\int_{-\infty}^{+\infty}\sum_{m,\alpha}|\Delta_P M\Delta_Q\hat{\psi}_r(\lambda,m,\alpha)|^2|\lambda|\,d\lambda$$

for P, Q homogeneous polynomials such that $\deg P+\deg Q=4$. By (5) of lemma 2 and the fact that $\Delta_Q\mathscr{R}_{\psi_r}(\lambda,m,\alpha)=0$ except for just one value m_0 of m, one has

$$\begin{aligned}
J &= \int_{-\infty}^{+\infty}\sum_{\alpha}\|\Delta_P M(\lambda)\,W_\alpha^{m_0}(\lambda)\|_{HS}^2\,|\Delta_Q\hat{\psi}_r(\lambda,m_0,\alpha)|^2|\lambda|\,d\lambda \\
&\leq \sum_{j=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\|\Delta_P M(\lambda)\,\hat{\Pi}_{2^j}(\lambda)\|_{HS}^2\,C_Q r^2\,2^{j(4-\deg Q)}f_Q(r2^{2j}) \\
&\leq Cr\sum_{j=-\infty}^{+\infty}r2^{j(4-\deg Q)}f_Q(r2^{2j})\,2^{j(2-\deg P)}\leq C\|f_Q\|_1 r.
\end{aligned}$$

This proves (4) and hence the multiplier theorem.

Remarks. From the proof of the theorem it is easily seen that one has to verify condition (2′) only for those operators Δ_P which actually act on $M(\lambda)$ in the Leibniz formula.

Both the tools we use in the proof of the theorem for the three-dimensional Heisenberg group H, i.e. the theory of singular integrals on homogeneous spaces and the machinery of the Fourier tranform, are available in the more general case of the $2n + 1$-dimensional Heisenberg group H^n. However the increased number and complexity of the difference-differential operators arising as Fourier transforms of the multiplication by homogeneous monomials on H^n, would make an extension of the proof to the general case prohibitively long and involved.

The following corollary is a straightforward consequence of the multiplier theorem and simplifies considerably its application to multipliers associated with the sub Laplacian $\mathscr{L}_0$.

COROLLARY 1. *Let f be a function of class C^4 in $\mathbf{C}\setminus\{0\}$. Assume that for every differential monomial $\partial^\gamma = \partial_x^{\gamma_1}\partial_y^{\gamma_2}$, with $|\gamma| = \gamma_1 + \gamma_2 \leq 4$ one has $|\partial^\gamma f(z)| < C|z|^{-|\gamma|}$. Then the multiplier $M(\lambda) = \sum_\alpha f((2\alpha + z)|\lambda|)\, W_\alpha^0(\lambda)$, $z \in \mathbf{C}$ is bounded on $L^p(H)$, $1 < p < \infty$, and is of weak type* (1,1).

4. APPLICATIONS

Among the multipliers that in the classical case fall under the domain of applicability of Hörmander's theorem are the imaginary powers $(-\Delta)^{it}$, $(I - \Delta)^{it}$, $t \in \mathbf{R}$, of the Laplacian and the second derivatives of the fundamental solution of the Laplace equation. In this section we show that our theorem may be used to prove analogous results for a family of hypoelliptic operators on H. Some of the multipliers we consider here were studied previously by other authors. However we discuss them because they serve the purpose of illustrating which multipliers fall under the scope of the theorem.

For $z \in \mathbf{C}$ consider the left invariant differential operator $\mathscr{L}_z = \mathscr{L}_o + izT$. The operator $\mathscr{L}_z$ was studied by Folland and Stein [5], who showed among other things that for z admissible, *i.e.*, $z \neq \pm 1, \pm 3, \ldots, \mathscr{L}_z$ is hypoelliptic and has a fundamental solution E_z. From the result of Folland [3] it follows that $\mathscr{L}_o$ generates a symmetric diffusion semigroup on $L^p(H)$, $1 \leq p \leq \infty$. Let $\mathscr{L}_0 = \int_0^\infty \lambda\, dE(\lambda)$ be the spectral resolution of $\mathscr{L}_0$. Then, according to the Littlewood-Paley-Stein theory [12], if

$$f(\lambda) = \int_0^\infty e^{-\lambda s}\phi(s)\, ds \tag{7}$$

for some $\phi \in L^\infty(0,\infty)$, the operator $f(\mathscr{L}_0) = \int_0^\infty f(\lambda)\, dE(\lambda)$ is bounded on $L^p(H)$, $1 < p < \infty$.

An easy computation shows that the operator

$$\hat{\mathscr{L}}_z(\lambda) = \sum_\alpha (2\alpha + 1 - iz)|\lambda|\, W_\alpha^0(\lambda)$$

is the Fourier transform of $\mathscr{L}_z$. Hence the Fourier transform of $f(\mathscr{L}_z)$ is

$$f(\mathscr{L}_z(\lambda)) = \sum_\alpha f((2\alpha + 1 - iz)|\lambda|)\, W_\alpha^0(\lambda),$$

and since (7) implies $|f^{(n)}(r)| < C_n r^{-n}$ for every $n > 0$, we may consider Corollary 1 as a sharpening of the result in [4, Lemma 3.13] in the case of the three-dimensional Heisenberg group. In particular one obtains that the fractional operators $(\mathscr{L}_z)^{it}$, $(I + \mathscr{L}_z)^{it}$, $t \in \mathbf{R}$, are bounded on $L^p(H)$, $1 < p < \infty$ and of weak type (1,1).

Consider next the second derivatives $Z\bar{Z}E_z$, $\bar{Z}ZE_z$, TE_z of the fundamental solution E_z of $\mathscr{L}_z$. Their Fourier transforms are

$$(Z\bar{Z}E_z)\hat{}\,(\lambda) = \sum_\alpha -\left(\frac{2\alpha}{2\alpha + 1 - iz}\right) W_\alpha^0(\lambda)$$

$$(\bar{Z}ZE_z)\hat{}\,(\lambda) = \sum_\alpha -\left(\frac{2(\alpha + 1)}{2\alpha + 1 - iz}\right) W_\alpha^0(\lambda)$$

$$(Te_z)\hat{}\,(\lambda) = \sum_\alpha \operatorname{sign}(\lambda) \frac{1}{2\alpha + 1 - iz} W_\alpha^0(\lambda)$$

Thus it is a straightforward consequence of the multiplier theorem that $Z\bar{Z}E_z$, $\bar{Z}ZE_z$, TE_z define bounded convolution operators on $L^p(H)$, $1 < p < \infty$ which are also weak type (1,1). This result, for all admissible z, has been obtained by Folland and Stein [5], without using the Fourier transform.

REFERENCES

1. R. R. Coifman and G. Weiss, *Multipliers transformations of functions on SU(2) and* $\sum_2$. Rev. Un. Mat. Argentina, 25, (1970), 145–166.

2. ———, *Analyse harmonique non-commutative sur certains espaces homogènes.* Lecture Notes in Math., vol. 242, Springer-Verlag, Berlin and New York, (1971).

3. G. B. Folland, *A fundamental solution for a subelliptic operator.* Bull. Amer. Math. Soc., 79, (1973), 373–376.

4. ———, *Subelliptic estimates and function spaces on nilpotent Lie groups.* Ark. Mat., 13, (1975), 161–207.

5. G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group.* Comm. Pure Appl. Math., 27, (1974), 429–522.

6. D. Geller, *Fourier analysis on the Heisenberg group.* Proc. Natl. Acad. Sci. USA, 74, (1977), 1328–1331.

7. C. S. Herz, *The theory of p-spaces with an application to convolution operators.* Trans. Amer. Math. Soc., 154, (1971), 69–82.

8. L. Hörmander, *Estimates for translation invariant operators in L^p spaces.* Acta Math. 104, (1960), 93–140.

9. A. W. Knapp and E. M. Stein, *Intertwining operators for semisimple groups.* Ann. of Math., 93, (1971), 489-578.

10. A. Korányi and S. Vági, *Singular integrals on homogeneous spaces and some problems of classical analysis.* Ann. Scuola Norm. Sup. Pisa, 25, (1971), 575-648.

11. R. L. Rubin, *Multipliers on the rigid motions of the plane and their relations to multipliers on direct products.* Proc. Amer. Math. Soc., 59, No. 1, (1976), 89-98.

12. E. M. Stein, *Topics in harmonic analysis.* Ann. of Math. Studies, 63, Princeton University Press, Princeton 1970.

Instituto di Matematica
Università di Milano
via C. Saldini 50,
20133, Milano, Italy.
and
Instituto di Matematica
Università di Genova
via L. B. Alberti 4,
16132 Genova, Italy