MATRIX ALGEBRAS OVER O_n

William L. Paschke and Norberto Salinas

This paper is concerned with the extension theory of the C*-algebras O_n studied by J. Cuntz in [7] and their tensor products with the algebra M_k of complex $k \times k$ matrices. We show by computing various Ext groups that the O_n's are pairwise non-isomorphic (a result which has also been obtained independently by M. Pimsner and S. Popa [8]), and that O_n and $O_n \otimes M_k$ are non-isomorphic if k and $n-1$ are not relatively prime. We also prove that O_n is isomorphic to $O_n \otimes M_k$ if k divides n or is congruent to 1 mod$(n-1)$.

We briefly indicate our notation and summarize essential prerequisites from extension theory for C*-algebras. Throughout, H is complex, infinite-dimensional separable Hilbert space. We write L(H) and Q(H) for, respectively, the algebra of all bounded operators on H and the Calkin algebra (the quotient of L(H) by the compacts), and let π: L(H) $\to$ Q(H) denote the quotient map. To avoid unnecessary clumsiness of expression, we once and for all make fixed identifications of $H \otimes \mathbb{C}^n$ (the direct sum of n copies of H) with H for $n = 2, 3, \ldots$, and thereby identify $L(H \otimes \mathbb{C}^n)$ with L(H) and $Q(H \otimes \mathbb{C}^n)$ with Q(H). We also identify $L(H \otimes \mathbb{C}^n)$ and $Q(H \otimes \mathbb{C}^n)$ with $L(H) \otimes M_n$ and $Q(H) \otimes M_n$, respectively, in the natural way. For a separable unital C*-algebra A, we write E(A) for the set of all unital *-monomorphisms (*extensions*) of A into Q(H). We say that extensions τ and σ are *strongly* (respectively, *weakly*) *equivalent* if there is a unitary $U \in L(H)$ (respectively, unitary $u \in Q(H)$) such that $\tau(\cdot) = \pi(U)\sigma(\cdot)\pi(U^*)$ (resp. $u\sigma(\cdot)u^*$). For $\tau \in E(A)$, $[\tau]$ denotes the strong equivalence class of τ. We write $\mathrm{Ext}^s(A)$ for $\{[\tau] : \tau \in E(A)\}$ and let $\mathrm{Ext}^w(A)$ denote the set of weak equivalence classes in E(A). Given $\tau, \sigma \in E(A)$, we define $\tau \oplus \sigma \in E(A)$ (via our identification of Q(H) with $Q(H) \otimes M_2$) by

$$(\tau \oplus \sigma)(a) = \begin{pmatrix} \tau(a) & 0 \\ 0 & \sigma(a) \end{pmatrix}.$$

The operations thereby induced on $\mathrm{Ext}^s(A)$ and $\mathrm{Ext}^w(A)$ make them into abelian semigroups. An extension τ is called *trivial* if it lifts to a unital *-representation of A on H. D. Voiculescu showed in [12] (see also [2]) that all trivial extensions of A are strongly equivalent and that the resulting strong equivalence class serves as the zero element of $\mathrm{Ext}^s(A)$. Correspondingly, the weak equivalence class of any trivial extension is the zero element of $\mathrm{Ext}^w(A)$. It is not the case in general that $\mathrm{Ext}^s(A)$ (and hence $\mathrm{Ext}^w(A)$) is a group; see [1] for an example of a non-invertible extension. When $\mathrm{Ext}^s(A)$ is a group, though, $\mathrm{Ext}^w(A)$ can be naturally identified with the quotient of $\mathrm{Ext}^s(A)$ by the subgroup consisting of those $[\tau]$ for which τ is weakly equivalent to a trivial extension.

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For further information on extension theory for non-commutative C*-algebras, we refer the reader to [2], [3], [4], and [10].

1. $\mathrm{Ext}^s(O_n)$ AND $\mathrm{Ext}^w(O_n)$

In this section we give a concise account of the extension theory for the C*-algebras O_n studied by J. Cuntz in [7]. Our results here have been obtained recently (and independently) by M. Pimsner and S. Popa [8], and also by L. Brown (private communication) but our treatment has direct bearing on the material in the next section.

As in [7], O_n (for $n = 2, 3, \ldots$) is the C*-algebra generated by isometries $S_1, S_2, \ldots, S_n$ of H with orthogonal ranges whose direct sum is H. For fixed n, all choices of n isometries subject to these requirements give rise to isomorphic C*-algebras [7]. Let $\tau\colon O_n \to Q(H)$ be a unital $*$-monomorphism. Our immediate goal is to associate to τ an integer $m(\tau)$ that will measure the obstruction to lifting τ to a $*$-representation on H. Let v_τ be the $n \times n$ matrix in $Q(H \otimes \mathbb{C}^n)$ $(= Q(H) \otimes M_n)$ with zeros in the second through n^{th} rows and with first row

$$\tau(S_1) \quad \tau(S_2) \quad \ldots \quad \tau(S_n)$$

Since $S_i^* S_j = \delta_{ij} I$ and $S_1 S_1^* + \ldots + S_n S_n^* = I$, we see that v_τ is an isometry in the Calkin algebra with $v_\tau v_\tau^* = \pi(P_1)$, where P_1 is the projection of $H \otimes \mathbb{C}^n$ $(= H \oplus \ldots \oplus H)$ onto the first direct summand.

Our definition of $m(\tau)$ requires a lemma which is frequently cited as a consequence of the proof for 2.5 of [5]. This result has been used to compute obstructions in other situations also, *e.g.* J. Thayer's treatment (essentially a computation of Ext^s) of extensions of UHF algebras in [11]. We indicate the proof of the lemma here for completeness.

LEMMA 1.1. *Let* P *and* Q *be projections in* L(H) *and* v *a partial isometry in* Q(H) *such that* $vv^* = \pi(P)$ *and* $v^*v = \pi(Q)$. *There is a partial isometry* V *in* L(H) *such that*

(a) $\pi(V) = v$; *and*

(b) $VV^* \leqq P$ *and* $V^*V \leqq Q$.

Moreover, the integer $\dim(Q - V^*V) - \dim(P - VV^*)$ *is uniquely determined by these conditions.*

Proof. Let $T \in L(H)$ be such that $\pi(T) = v$. If we let V be the partial isometry in the polar decomposition of PTQ, then V clearly satisfies (b). Noting that

$$\pi(|PTQ|) = |\pi(PTQ)| = |v| = \pi(Q),$$

we have $v = \pi(V)\pi(Q) = \pi(V)$ (because $V^*V \leqq Q$), so (a) holds as well. For the last assertion, regard V as a map from QH to PH and let W be a unitary transformation from PH onto QH. The partial isometry $\tilde{V}$ in $L(PH \oplus QH)$ with matrix

$$\begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix}$$

is easily seen to be Fredholm with index $\dim(Q - V^*V) - \dim(P - VV^*)$.

We apply Lemma 1.1 (with $v = v_\tau$, $P = P_1$ as above, and $Q = I$) to get a partial isometry V_τ of $H \otimes \mathbb{C}^n$ with $\pi(V_\tau) = v_\tau$, $V_\tau V_\tau^* \leqq P_1$ and $V_\tau^* V_\tau$ a projection of finite co-dimension.

Definition 1.2. We set

$$m(\tau) = \dim(I - V_\tau^* V_\tau) - \dim(P_1 - V_\tau V_\tau^*).$$

By the lemma, this definition of $m(\tau)$ is unambiguous. Suppose that $\tau' \in E(O_n)$ is strongly equivalent to τ, so there is a unitary U on H such that

$$\tau'(\cdot) = \pi(U^*)\tau(\cdot)\pi(U).$$

Let $\tilde{U} \in L(H \otimes \mathbb{C}^n)$ be the direct sum of n copies of U, so $\tilde{U}$ commutes with P_1 and $v_{\tau'} = \pi(\tilde{U}^*)v_\tau\pi(\tilde{U})$. It is clear that we may take $V_{\tau'} = \tilde{U}^* V_\tau \tilde{U}$ and hence $m(\tau') = m(\tau)$, so m is constant on strong equivalence classes. For any $\sigma \in E(O_n)$, we have $m(\tau \oplus \sigma) = m(\tau) + m(\sigma)$ because after making a natural identification of $(H \oplus H) \otimes \mathbb{C}^n$ with $(H \otimes \mathbb{C}^n) \oplus (H \otimes \mathbb{C}^n)$, we may take $V_{\tau\oplus\sigma} = V_\tau \oplus V_\sigma$ and compute

$$\begin{aligned} m(\tau \oplus \sigma) &= \dim(I \oplus I - V_\tau^* V_\tau \oplus V_\sigma^* V_\sigma) \\ &\quad - \dim(P_1 \oplus P_1 - V_\tau V_\tau^* \oplus V_\sigma V_\sigma^*) \\ &= m(\tau) + m(\sigma). \end{aligned}$$

These remarks show that m induces an additive map (which we will also call m) from $\mathrm{Ext}^s(O_n)$ to $\mathbb{Z}$.

LEMMA 1.3. $m(\tau) = 0$ *if and only if* τ *is trivial.*

Proof. If τ is trivial, then τ is strongly equivalent to the $*$-monomorphism τ_0 that sends each S_j to $\pi(S_j)$ (1.4 and 1.5 of [12], 1.12 of [7]). There is an obvious choice of V_{τ_0} such that $V_{\tau_0}^* V_{\tau_0} = I$ and $V_{\tau_0} V_{\tau_0}^* = P_1$, so $m(\tau) = m(\tau_0) = 0$.

Conversely, suppose that $m(\tau) = 0$. Since $\dim(I - V_\tau^* V_\tau) = \dim(P_1 - V_\tau V_\tau^*)$, we may add to V_τ a (finite-rank) partial isometry with initial space $(I - V_\tau^* V_\tau)H$ and final space $(P_1 - V_\tau V_\tau^*)H$ and assume that $V_\tau^* V_\tau = I$ and $V_\tau V_\tau^* = P_1$. Regard V_τ as an $n \times n$ operator matrix. The second through n^{th} rows of V_τ must be 0 because the second through n^{th} diagonal entries of $V_\tau V_\tau^*$ are 0. Let T_j ($j = 1,2,\ldots,n$) be the j^{th} entry in the first row of V_τ. We have $T_j^* T_j = I$ for each j (because $V_\tau^* V_\tau = I$) and $T_1 T_1^* + \ldots + T_n T_n^* = I$ (because the first diagonal entry of $V_\tau V_\tau^*$ is I). By 1.12 of [7], there is a $*$-isomorphism $\theta\colon O_n \to C^*(T_1,\ldots,T_n)$ such that $\theta(S_j) = T_j$ for each j. Since $\pi \circ \theta(S_j) = \pi(T_j) = \tau(S_j)$, τ is trivial.

LEMMA 1.4. *The range of* m *is* $\mathbb{Z}$.

Proof. It will suffice to find $\sigma,\tau \in E(O_n)$ such that $m(\sigma) = 1$ and $m(\tau) = -1$. Let $R_1,\ldots,R_n \in L(H)$ be isometries with orthogonal ranges such that

$$R_1R_1^* + \dots + R_nR_n^* = I - Q,$$

where Q is a one dimensional projection. Further, choose $T_1, \dots, T_n \in L(H)$ with orthogonal ranges such that $T_1^*T_1 = I - Q$, the other T_j's are isometries, and

$$T_1T_1^* + \dots + T_nT_n^* = I.$$

By 1.12 of [7], there exist $\sigma, \tau \in E(O_n)$ with $\sigma(S_j) = \pi(T_j)$ and $\tau(S_j) = \pi(R_j)$ $(j = 1, \dots, n)$. Obvious choices of V_σ and V_τ yield $V_\sigma^*V_\sigma = (I - Q) \oplus I \oplus \dots \oplus I$, $V_\sigma V_\sigma^* = P_1$, $V_\tau^*V_\tau = I$, $V_\tau V_\tau^* = (I - Q) \oplus 0 \oplus \dots \oplus 0$, so $m(\sigma) = 1$ and $m(\tau) = -1$.

Lemmas 1.3 and 1.4 imply that $\mathrm{Ext}^s(O_n)$ is a group (because if $\tau \in E(O_n)$, we can find $\tau' \in E(O_n)$ with $m(\tau') = -m(\tau)$, so $m(\tau \oplus \tau') = 0$, so $\tau \oplus \tau'$ is trivial) and hence that m is an isomorphism of $\mathrm{Ext}^s(O_n)$ with $\mathbb{Z}$. We record this as

THEOREM 1.5. $\mathrm{Ext}^s(O_n)$ *is a group isomorphic to* $\mathbb{Z}$ *for* $n = 2, 3, \dots$.

That $\mathrm{Ext}^s(O_n)$ is a group follows also from the fact that O_n is nuclear (see 2.3 of [7]) and Theorem 8 of [2].

To compute $\mathrm{Ext}^w(O_n)$, let U_+ be the standard unilateral shift and let $\tau_1 : O_n \to Q(H)$ be the $*$-monomorphism that maps each S_j to $\pi(U_+S_jU_+^*)$; the subgroup of $\mathrm{Ext}^s(O_n)$ generated by $[\tau_1]$ consists precisely of the weakly trivial classes. If we choose for V_{τ_1} the $n \times n$ operator matrix with zeros in the second through n^{th} rows and first row

$$U_+S_1U_+^* \quad \dots \quad U_+S_nU_+^*,$$

then $V_{\tau_1}V_{\tau_1}^*$ has $U_+U_+^*$ in the (1,1)-position and zeros elsewhere, while $V_{\tau_1}^*V_{\tau_1}$ is diagonal with $U_+U_+^*$ in each diagonal position. Hence $m(\tau_1) = n - 1$. This proves

THEOREM 1.6. $\mathrm{Ext}^w(O_n)$ *is isomorphic to* $\mathbb{Z}_{n-1}$ *for* $n = 2, 3, \dots$.

It is of course immediate from this that O_n and O_m are nonisomorphic if $m \neq n$.

2. ISOMORPHISM AND NON-ISOMORPHISM OF $O_n \otimes M_k$ WITH O_n

For the time being, we fix the integers $n \geqq 2$ and $k \geqq 2$. For $r = 0, 1, \dots, k - 1$, let $U_{k,r} \in L(H \otimes \mathbb{C}^k)$ be the direct sum of r copies of U_+ (the standard unilateral shift) and $k - r$ copies of I_H. Notice that the last direct summand of $U_{k,r}$ is always I_H. It is well known that every $\rho \in E(M_k)$ is strongly equivalent to one of the unital $*$-monomorphisms $\rho_r : M_k \to Q(H) \otimes M_k$ $(r = 0, 1, \dots, k - 1)$ defined by

$$\rho_r(T) = \pi(U_{k,r}(I_H \otimes T)U_{k,r}^*).$$

If we let $d(\rho)$ be the integer $r \in \{0, 1, \dots, k - 1\}$ such that ρ is strongly equivalent to ρ_r, then d induces an isomorphism of $\mathrm{Ext}^s(M_k)$ with $\mathbb{Z}_k$.

Our next lemma is a refined version of 3.15 of [4] and 3.4 of [10].

LEMMA 2.1. *Let* A *be a separable unital* C^**-algebra and let* $\tau \in E(A \otimes M_k)$. *If* ρ *is the restriction of* τ *to* $1 \otimes M_k$ *and* $r = d(\rho)$, *then there is a* $\sigma \in E(A)$ *such that* τ *is strongly equivalent to the unital* $*$*-monomorphism*

$$\pi(U_{k,r})(\sigma \otimes id_k)(\cdot)\,\pi(U^*_{k,r}): A \otimes M_k \to Q(H) \otimes M_k.$$

Proof. We may assume that τ maps $A \otimes M_k$ to $Q(H) \otimes M_k$ and that $\tau(1 \otimes T) = \rho_r(T)$ for $T \in M_k$. Let $\{e_{ij}\}_{i,j=1}^{k}$ be the standard matrix units for M_k. We have $\tau(1 \otimes e_{kk}) = 1 \otimes e_{kk} \in Q(H) \otimes M_k$ and hence for any $a \in A$ we have $\tau(a \otimes e_{kk}) = (1 \otimes e_{kk})\tau(a \otimes 1_k)(1 \otimes e_{kk})$, where 1_k is the identity matrix in M_k. Regarded as a $k \times k$ matrix in $Q(H) \otimes M_k$, $\tau(a \otimes e_{kk})$ must therefore have the form $\sigma(a) \otimes e_{kk}$ for some $\sigma(a) \in Q(H)$. The map $\sigma: A \to Q(H)$ so obtained is clearly a unital $*$-monomorphism. For $j = 1, 2, \ldots, k$, we have

$$\begin{aligned}\tau(a \otimes e_{jj}) &= \tau(1 \otimes e_{jk})(\sigma(a) \otimes e_{kk})\tau(1 \otimes e_{kj}) \\ &= \pi(U_{k,r})(1 \otimes e_{jk})(\sigma(a) \otimes e_{kk})(1 \otimes e_{kj})\,\pi(U^*_{k,r}) \\ &= \pi(U_{k,r})(\sigma(a) \otimes e_{jj})\,\pi(U^*_{k,r}).\end{aligned}$$

(In the second equality we have used the fact that $\sigma(a) \otimes e_{kk}$ commutes with $\pi(U_{k,r})$.) Summing on j, we obtain $\tau(a \otimes 1_k) = \pi(U_{k,r})(\sigma(a) \otimes 1_k)\,\pi(U^*_{k,r})$, and finally for $T \in M_k$, $\tau(a \otimes T) = \tau(a \otimes 1_k)\,\rho_r(T) = \pi(U_{k,r})(\sigma(a) \otimes T)\,\pi(U^*_{k,r})$, as required.

We can use this simple fact to describe $Ext^s(A \otimes M_k)$ in terms of k, $Ext^s(A)$, and the weakly trivial extensions of A when $Ext^s(A)$ has no elements of order k. For $\tau \in E(A \otimes M_k)$, define $i_*\tau \in E(A)$ (respectively, $j_*\tau \in E(M_k)$) to be the restriction of τ to $A \otimes 1_k$ (respectively, $1 \otimes M_k$). We then obtain a homomorphism $\gamma: Ext^s(A \otimes M_k) \to Ext^s(A) \times \mathbb{Z}_k$ defined by $\gamma([\tau]) = ([i_*\tau], d(j_*\tau))$. Further, for $\sigma \in E(A)$ we define $\tilde{\sigma}(a) = \pi(U_+)\,\sigma(a)\,\pi(U^*_+)$. One checks easily that $\tilde{\sigma}$ is strongly equivalent to any $*$-monomorphism obtained from σ by conjugating with a unitary in the Calkin algebra of index 1. Let $\sigma_0 \in E(A)$ be trivial and set $\sigma_1 = \tilde{\sigma}_0$. Since σ is strongly equivalent to $\sigma_0 \oplus \sigma$ and $(\sigma_0 \oplus \sigma)^{\sim}$ is strongly equivalent to

$$\pi(U_{2,1})(\sigma_0 \oplus \sigma)(\cdot)\,\pi(U^*_{2,1}) = \sigma_1 \oplus \sigma,$$

we have

$$(*) \qquad [\tilde{\sigma}] = [\sigma] + [\sigma_1].$$

PROPOSITION 2.2. *Let* A *be a separable unital* C*-*algebra such that* $Ext^s(A)$ *is a group with no elements of order* k. *Then* $Ext^s(A \otimes M_k)$ *is a group isomorphic to*

$$\{(k[\sigma] + r[\sigma_1], \langle r \rangle) \in Ext^s(A) \times \mathbb{Z}_k : \sigma \in E(A), r \in \mathbb{Z}\}$$

(*where* $r \to \langle r \rangle$ *is the quotient map of* $\mathbb{Z}$ *onto* $\mathbb{Z}_k$).

Proof. Let G be the subgroup of $Ext^s(A) \times \mathbb{Z}_k$ described in the statement of the proposition. Take $\sigma \in E(A)$, $r \in \{0, 1, \ldots, k-1\}$ and consider

$$\tau = \pi(U_{k,r})(\sigma \otimes id_k)(\cdot)\,\pi(U^*_{k,r}): A \otimes M_k \to Q(H) \otimes M_k.$$

We have $j_*\tau = \rho_r$ and $[i_*\tau] = r[\tilde{\sigma}] + (k-r)[\sigma] = k[\sigma] + r[\sigma_1]$ by (*). It now follows from Lemma 2.1 that the range of γ is G. Now suppose that

$\tau \in E(A \otimes M_k)$ is such that $\gamma([\tau]) = (0,0)$; *i.e.*, that the restrictions of τ to $A \otimes 1_k$ and $1 \otimes M_k$ are trivial. By Lemma 2.1, we may assume that $\tau = \sigma \otimes id_k$ for some $\sigma \in E(A)$. But then $[i_*\tau] = k[\sigma] = 0$, so σ must be trivial by our assumption on $Ext^s(A)$, and we conclude $[\tau] = 0$. As in the proof of Theorem 1.5 this is enough to show that $Ext^s(A \otimes M_k)$ is a group and γ is an isomorphism.

We specialize to the case $A = O_n$.

THEOREM 2.3. $Ext^s(O_n \otimes M_k)$ *is isomorphic to* $\mathbb{Z} \times \mathbb{Z}_g$, *where* g *is the greatest common divisor of* k *and* $n-1$.

Proof. Using the isomorphism m of $Ext^s(O_n)$ with $\mathbb{Z}$ obtained in Section 1 and recalling that $m([\sigma_1]) = n-1$, we see from the previous proposition that $Ext^s(O_n \otimes M_k)$ is isomorphic to

$$G = \{(sk + r(n-1), \langle r \rangle) \in \mathbb{Z} \times \mathbb{Z}_k : r, s \in \mathbb{Z}\}.$$

It is immediate that $(k,0)$ and $(n-1,1)$ belong to G. Let $x = (n-1)/g$ and $y = k/g$. Since x and y are relatively prime, we can find $s_0, r_0 \in \mathbb{Z}$ such that $s_0 y + r_0 x = 1$ and hence $s_0 k + r_0(n-1) = g$. We have $s_0(k,0) + r_0(n-1,1) = (g, \langle r_0 \rangle) \in G$. Further, $y(n-1,1) - x(k,0) = (kx - kx, \langle y \rangle) = (0, \langle y \rangle) \in G$. We claim that G is generated by $(g, \langle r_0 \rangle)$ and $(0, \langle y \rangle)$. Every element of G has the form $((sy + rx)g, \langle r \rangle)$ for some $r, s \in \mathbb{Z}$. We write

$$((sy + rx)g, \langle r \rangle) = (sy + rx)(g, \langle r_0 \rangle) - r_0 s(0, \langle y \rangle) + (0, \langle r - rr_0 x \rangle).$$

Since $r - rr_0 x = rs_0 y$, this shows that $G = \mathbb{Z}(g, \langle r_0 \rangle) + \mathbb{Z}(0, \langle y \rangle)$, as claimed. This sum is clearly direct, and since $\langle y \rangle$ has order g in $\mathbb{Z}_k$ and $(g, \langle r_0 \rangle)$ has infinite order, the theorem is proved.

COROLLARY 2.4. *If* k *and* $n-1$ *are not relatively prime, then* $O_n \otimes M_k$ *and* O_n *are non-isomorphic.*

Proof. In this case, $Ext^s(O_n \otimes M_k)$ contains a nonzero element of finite order and so cannot be isomorphic to $Ext^s(O_n)$.

Question. Conversely, are $O_n \otimes M_k$ and O_n isomorphic whenever k and $n-1$ are relatively prime?

We provide a partial answer to this question by showing below that $O_n \otimes M_k$ is isomorphic to O_n at least in the following two cases:

(i) when k divides n; and

(ii) when $k \equiv 1 \pmod{n-1}$.

PROPOSITION 2.5. *If* k *divides* n, *then* O_n *and* $O_n \otimes M_k$ *are isomorphic.*

Proof. Our argument is modeled after M. D. Choi's proof in [6] that $O_2 \otimes M_2$ is isomorphic to O_2. Let $j = n/k$. For $r = 0,1,\dots,j-1$ and $s = 1,2,\dots,k$, let $T_{rk+s} \in O_n \otimes M_k$ be the $k \times k$ matrix whose s^{th} row is

$$S_{rk+1} \quad S_{rk+2} \quad \cdots \quad S_{(r+1)k}$$

and all of whose other rows are zero. One checks easily that each T_i is an isometry, and that $T_1T_1^* + \dots + T_nT_n^* = I$, so by 1.12 of [7] the C*-subalgebra A of $O_n \otimes M_k$ generated by the T_i's is isomorphic to O_n. Further, a straightforward computation shows that for s,t $\in \{1,2,\dots,k\}$, the matrix $\sum_{r=0}^{j-1} T_{rk+s}T_{rk+t}^*$ has I in the (s,t)-position and zeros in all other positions, so A contains all of the standard matrix units for $1 \otimes M_k$. Since all of the S's occur as entries of the T's, this means that $A = O_n \otimes M_k$.

In [6], Choi considers unitaries u and v on H described as follows. Break H into two isomorphic direct summands; let u be an order-two unitary permuting these. Now break the second direct summand further into two isomorphic direct summands; let v be an order-three unitary that cyclically permutes the resulting three direct summands of H. The proof of our next lemma requires a generalization of this construction in which H is initially broken into k pieces to define a unitary u of order k, with the last of these pieces being broken further into n pieces to define a unitary v of order k + n − 1.

LEMMA 2.6. *For any* n, k $\geqq$ 2, $O_n \otimes M_k$ *is isomorphic to* $O_n \otimes M_{k+n-1}$.

Proof. Let $H = H_1 \oplus \dots \oplus H_{k-1} \oplus K_1 \oplus \dots \oplus K_n$, where each of the k + n − 1 direct summands is isomorphic to H, and let $H_k = K_1 \oplus \dots \oplus K_n$, which we regard as a subspace of H. Let e be the projection of H on H_1 and let $u \in L(H)$ be a unitary of order k that permutes the H_j's one notch to the left, that is $u^k = I$, $uH_{k-1} = H_{k-2}, \dots, uH_2 = H_1$, and $uH_1 = H_k$. Let v be a unitary of order k + n − 1 that permutes the original direct summands one notch to the left, that is $v^{k+n-1} = I$, $vK_n = K_{n-1}, \dots, vK_2 = K_1$, $vK_1 = H_{k-1}$, $vH_{k-1} = H_{k-2}, \dots, vH_2 = H_1$, and $vH_1 = K_n$. We further require that $v|_{H_2\oplus\dots\oplus H_{k-1}} = u|_{H_2\oplus\dots\oplus H_{k-1}}$. Consider the C*-algebra C*(e,u,v) generated by e,u and v. We will prove the lemma by showing first that C*(e,u,v) is isomorphic to $O_n \otimes M_k$ and then that C*(e,u,v) is isomorphic to $O_n \otimes M_{k+n-1}$.

Identify $H_2, H_3, \dots, H_k$ with H_1 in such a way that u is represented with respect to the decomposition $H = H_1 \oplus \dots \oplus H_k$ by a k × k matrix of O's and I's. The projection e is of course the k × k matrix with I in the (1,1)-position and O's elsewhere. Notice that e and u together generate all of the scalar matrices because u is a complete permutation matrix. The k × k matrix that represents v in this setting has the form

$$\begin{pmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & & \vdots & \vdots \\ 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & B \\ A & 0 & 0 & \cdots & 0 & C \end{pmatrix}$$

We observe the following:

(i) $A^*A = I$, $B^*B + C^*C = I$, $BB^* = I$, $AA^* + CC^* = I$ (because v is unitary);

(ii) $C^n = 0$ (because the compression of v to H_k annihilates K_1 and moves K_j to K_{j-1} for $j = 2, \ldots, n$);

(iii) $BC^jA = 0$ for $j = 0, 1, \ldots, n-2$ (because $v^{j+1}H_1 = K_{n-j}$ is orthogonal to K_1 for such j); and

(iv) $BC^{n-1}A = I$.

(To see that (iv) holds, take $\xi \in H_1$ and set $\bar{\xi} = \xi \oplus 0 \oplus \ldots \oplus 0 \in H$. We have $v\bar{\xi} = 0 \oplus \ldots \oplus 0 \oplus A\xi$ and successive applications of v yield

$$v^n\bar{\xi} = 0 \oplus \ldots \oplus 0 \oplus C^{n-1}A\xi,$$
$$v^{n+1}\bar{\xi} = 0 \oplus \ldots \oplus 0 \oplus BC^{n-1}A\xi \oplus 0,$$

and finally

$$\bar{\xi} = v^{n+k-1}\bar{\xi} = BC^{n-1}A\xi \oplus 0 \oplus \ldots \oplus 0.)$$

Define operators T_j $(j = 1, \ldots, n)$ by $T_j = C^{j-1}A$. We claim that $C^*(T_1, \ldots, T_n)$ is isomorphic to O_n. We have $T_1^* T_1 = A^*A = I$ by (i). For $j = 1, \ldots, n-1$,

$$T_{j+1}^* T_{j+1} = T_j^* C^* C T_j = T_j^*(I - B^*B) T_j$$

by (i). By (iii), we have $BT_j = 0$ for such j and hence $T_{j+1}^* T_{j+1} = T_j^* T_j$, so by induction the T's are all isometries. Further, (i) shows that

$$T_1 T_1^* + \ldots + T_n T_n^* = \sum_{j=0}^{n-1} C^j(I - CC^*)(C^j)^*,$$

and since $C^n = 0$ (ii), this sum collapses to I. It now follows from 1.12 of [7] that $C^*(T_1, \ldots, T_n)$ is isomorphic to O_n. We next claim that

$$C^*(T_1, \ldots, T_n) = C^*(A,B,C).$$

Certainly $A = T_1 \in C^*(T_1, \ldots, T_n)$. That $B \in C^*(T_1, \ldots, T_n)$ follows from (iv), (i) and (ii): $B^* = B^*BC^{n-1}A = (I - C^*C)C^{n-1}A = C^{n-1}A = T_n$. We have $C \in C^*(T_1, \ldots, T_n)$ because $CT_n = 0$ by (iii) and thus

$$\sum_{j=1}^{n-1} T_{j+1} T_j^* = C \sum_{j=1}^{n-1} T_j T_j^* = C(I - T_n T_n^*) = C.$$

Since $C^*(e,u,v)$ contains all scalar matrices, we see that $C^*(e,u,v)$ is isomorphic to $C^*(A,B,C) \otimes M_k$ and thus to $O_n \otimes M_k$.

We now consider the decomposition $H = H_1 \oplus \ldots \oplus H_{k-1} \oplus K_1 \oplus \ldots \oplus K_n$, where all summands are identified with H_1 in such a way that the $(n + k - 1) \times (n + k - 1)$ matrix which represents v is a complete permutation matrix (with all entries 0 or I). The matrix for e is the obvious one; e and v generate all scalar

$$(n + k - 1) \times (n + k - 1)$$

matrices. The matrix for u has the form

$$\begin{bmatrix} X & \begin{matrix} 0 \\ V_1 \dots V_n \end{matrix} \\ \begin{matrix} W_1 & \\ \vdots & 0 \\ W_n & \end{matrix} & 0 \end{bmatrix}$$

where X is $(k-1) \times (k-1)$ with I's on the superdiagonal and 0's elsewhere. Since u is unitary, the V_j's are isometries and $V_1 V_1^* + \dots + V_n V_n^* = I$ (so $C^*(V_1, \dots, V_n)$ is isomorphic to O_n). For $\xi \in H_1$, let $\bar{\xi} = \xi \oplus 0 \dots \oplus 0$. Successive applications of u to $\bar{\xi}$ yield $\bar{\xi} = u^k \bar{\xi} = (V_1 W_1 \xi + \dots + V_n W_n \xi) \oplus 0 \oplus \dots \oplus 0$, so

$$V_1 W_1 + \dots + V_n W_n = I.$$

Since the V's have orthogonal ranges, we can apply V_j^* on the left to get $V_j^* V_j W_j = W_j = V_j^*$ for $j = 1, \dots, n$. We conclude that $C^*(e,u,v)$ is isomorphic to $O_n \otimes M_{n+k-1}$. This proves the lemma.

Since $O_n \otimes M_n$ is isomorphic to O_n (by Proposition 2.5), we have as an immediate consequence of the lemma

PROPOSITION 2.7. *$O_n \otimes M_k$ is isomorphic to O_n whenever $k \equiv 1 \pmod{n-1}$.*

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Department of Mathematics
University of Kansas
Lawrence, Kansas 66045