EXTREME POINTS OF THE UNIT BALL OF THE BLOCH SPACE $\mathscr{B}_0$

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1. INTRODUCTION

Let Δ denote the open unit disc in the complex plane $\mathbb{C}$, and let Γ denote the boundary of Δ. If f is a function holomorphic in Δ, define M(f) by

$$M(f) = \sup\{|f'(z)|(1-|z|^2) : z \in \Delta\}.$$

The *Bloch space* $\mathscr{B}$ consists of those holomorphic functions f for which M(f) is finite. The norm $\|f\| = |f(0)| + M(f)$ makes $\mathscr{B}$ a Banach space. The set of f in $\mathscr{B}$ for which $\lim_{|z|\to 1} |f'(z)|(1-|z|^2) = 0$ is a closed subspace of $\mathscr{B}$, denoted by $\mathscr{B}_0$. There are several characterizations of the functions in the Bloch space, and we refer the reader to [1], [2], [4], and [5]. The dual space of $\mathscr{B}_0$ is linearly homeomorphic with a Banach space I of functions holomorphic on Δ [1]. In fact,

$$I = \left\{ g : \int_0^1 \int_0^{2\pi} |g'(re^{i\theta})|\, r\, dr\, d\theta < \infty \right\}.$$

Further, the second dual of $\mathscr{B}_0$ is isometrically isomorphic to $\mathscr{B}$. Alaoglu's Theorem and the Krein-Milman Theorem then imply that the unit ball of $\mathscr{B}$ has extreme points. We show that the unit ball of $\mathscr{B}_0$ also has extreme points. The principal result of this paper is a characterization of the extreme points of the unit ball of $\mathscr{B}_0$.

We list here a theorem which plays a fundamental role in later proofs.

THEOREM A. *Let* G(x, y) *be a convergent real power series such that* G(0, 0) = 0 *and* $G(0,y) = \sum_{n=s}^{\infty} b_n y^n$, *where* $s \geq 1$ *and* $b_s \neq 0$. *Then there are power series* $\Omega(x,y)$, $A_i(x)$ $(i = 0, 1, \ldots, s-1)$ *such that*

$$G(x,y) = (y^s + A_{s-1}(x)\, y^{s-1} + \ldots + A_0(x))\, \Omega(x,y),$$

and $\Omega(0,0) \neq 0$.

Theorem A is a special case of the real analytic version of the Weierstrass Preparation Theorem (*cf., e.g.,* [7, p. 145]). A C^∞ version of this result (the Malgrange-Mather Theorem) can be found in [6, p. 94].

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2. CHARACTERIZATION OF THE EXTREME POINTS

We begin by restricting our attention to functions $f \in \mathscr{B}$ normalized by $f(0) = 0$. Thus let $\tilde{\mathscr{B}} = \{f \in \mathscr{B} : f(0) = 0\}$, and let $\tilde{\mathscr{B}}_0 = \mathscr{B}_0 \cap \tilde{\mathscr{B}}$. Then for $f \in \tilde{\mathscr{B}}$, $\|f\| = M(f)$. We first determine the extreme points of the unit ball of $\tilde{\mathscr{B}}_0$, (denoted ball $\tilde{\mathscr{B}}_0$), and then we will discuss the extreme points of ball $\mathscr{B}_0$.

For $f \in$ ball $\tilde{\mathscr{B}}$, let $L_f = \{z \in \Delta : |f'(z)|(1 - |z|^2) = 1\}$.

THEOREM 1. *Let* f *be in ball* $\tilde{\mathscr{B}}$. *If there is an* $R < 1$ *so that* $L_f \cap \{z : |z| \leq R\}$ *is an infinite set, then* f *is an extreme point of ball* $\tilde{\mathscr{B}}$.

Suppose that $g_1, g_2 \in$ ball $\tilde{\mathscr{B}}$ and $f = \frac{1}{2}(g_1 + g_2)$. If $z \in L_f \cap \{z : |z| \leq R\}$, then

$$|f'(z)| = (1 - |z|^2)^{-1} \quad \text{and} \quad |g_i'(z)| \leq (1 - |z_k|^2)^{-1} \qquad \text{for } i = 1, 2.$$

Thus $|f'(z)| \geq \frac{1}{2}(|g_1'(z)| + |g_2'(z)|)$. But $f'(z) = \frac{1}{2}(g_1'(z) + g_2'(z))$, so that f', g_1', and g_2' agree on $L_f \cap \{z : |z| \leq R\}$. Thus $f = g_1 = g_2$, and f is an extreme point of ball $\tilde{\mathscr{B}}$.

COROLLARY 1. *If* f *is in ball* $\tilde{\mathscr{B}}_0$ *and* L_f *is an infinite set, then* f *is an extreme point of ball* $\tilde{\mathscr{B}}_0$.

A routine computation shows that the function $f(z) = \frac{1}{2}\log(1 + z)(1 - z)^{-1}$ is an extreme point of ball $\tilde{\mathscr{B}}$. In fact, L_f is the interval $(-1, 1)$ on the real axis. Note that f is not in $\tilde{\mathscr{B}}_0$. In Section 3 we discuss the functions

$$f_n(z) = z^n / \|z^n\|, \qquad n = 2, 3, \ldots .$$

These functions are extreme points in ball $\tilde{\mathscr{B}}_0$.

The following theorem is the converse of Corollary 1.

THEOREM 2. *Let* f *be in ball* $\tilde{\mathscr{B}}_0$. *If* L_f *is finite, then* f *is not an extreme point of ball* $\tilde{\mathscr{B}}_0$.

The proof of Theorem 2 is based on the following lemmas.

LEMMA 0. *If* $f : [-\delta, \delta] \to \mathbb{R}$ *is real analytic at* $x = 0$ *and if* f *has an isolated local minimum at* $x = 0$, f *has Taylor series expansion of form*

$$f(x) = f(0) + \sum_{k=2j}^{\infty} a_k x^k ,$$

where $j \geq 1$ *and* $a_{2j} > 0$.

LEMMA 1. *Let* G *be a real valued function on* Δ *of the form*

$$G(x, y) = y^2 + A_1(x)y + A_0(x) , \tag{1}$$

where A_0 *and* A_1 *are real analytic functions on* $|x| < 1$. *Suppose that* $G(0,0) = 0$ *and that there is a* $\delta > 0$ *so that* $G(x,y) > 0$ *for* $0 < x^2 + y^2 < \delta$. *Then there is an integer* $n > 0$ *and a* δ' *with* $0 < \delta' \leq \delta$ *so that* $0 < x^2 + y^2 < \delta'$ *implies* $(x^2 + y^2)^n < G(x,y)$.

Proof. Since $G(0,0) = 0$, we have $A_0(0) = 0$. Also, since $G(0,y) = y^2 + A_1(0)y$, we must have $A_1(0) = 0$. The function A_1 has power series expansion of the form $A_1(x) = ax^j + \sum_{k=j+1}^{\infty} a_k x^k$, where $j \geq 1$ and $a \neq 0$. We write G in the form

$$G(x,y) = \left(y + \frac{A_1(x)}{2}\right)^2 + \left(A_0(x) - \frac{A^2{}_1(x)}{4}\right). \tag{2}$$

Observe that for $0 < x^2 + \frac{A_1^2(x)}{4} < \delta$, $G\left(x, \frac{-A_1(x)}{2}\right) = A_0(x) - \frac{A_1^2(x)}{4} > 0$. An application of Lemma 0 gives $A_0(x) - \frac{A_1^2(x)}{4} = bx^{2n} + \sum_{k=2n+1}^{\infty} b_k x^k$, where $n \geq 1$ and $b > 0$. Thus there is a δ' with $0 < \delta' \leq \delta$ so that if $0 < x^2 < \delta'$, we have

$$\frac{b}{2} x^{2n} \leq A_0(x) - \frac{A_1^2(x)}{4}, \quad \text{and} \tag{3}$$

$$A_1^2(x) \leq 4a^2x^2. \tag{4}$$

We first consider the set

$$A = \{(x,y) : y^2 \leq 4a^2x^2 \text{ and } 0 < x^2 < \delta'\}.$$

Using (2) and (3), we obtain the inequality

$$G(x,y) \geq A_0(x) - \frac{A_1^2(x)}{4} \geq \frac{b}{2} x^{2n} \geq K_1(x^2 + y^2)^n \qquad \text{for } (x,y) \in A,$$

where $K_1 = \frac{1}{2} b(2 + 4a^2)^{-n}$. Now consider the set

$$\mathscr{B} = \{(x,y) : y^2 \geq 4a^2x^2, y \neq 0, \text{ and } x^2 < \delta'\}.$$

Using (2) and (4), we obtain

$$G(x,y) \geq \left(g + \frac{A_1(x)}{2}\right)^2 \geq \frac{y^2}{4} \geq K_2(x^2 + y^2)^n \qquad \text{for } (x,y) \in \mathscr{B},$$

where $K_2 = \frac{1}{4}(2 + (4a^2)^{-1})^{-1}$. Hence there is a constant K such that

$$K(x^2+y^2)^n < G(x,y) \qquad \text{if } 0 < x^2+y^2 < \delta'.$$

Further, we may suppose that $K = 1$ by replacing n by $n+1$ and choosing a smaller δ'.

LEMMA 2. *Suppose that* f *is in ball* $\tilde{\mathscr{B}}$, $|f'(0)| = 1$, *and there is a* $\delta > 0$ *such that* $|f'(z)|(1-|z|^2) < 1$ *for* $0 < |z| < \delta$. *Then there is a positive integer* n *and a* δ' *with* $0 < \delta' \le \delta$ *such that* $(|f'(z)| + |z|^n)(1-|z|^2) < 1$ *for* $0 < |z| < \delta'$.

Proof. Without loss of generality, we may assume that $f'(0) = 1$, so that $f'(z) = 1 + a_1 z + \sum_{k=2}^{\infty} a_k z^k$. If $a_1 \neq 0$, we can choose θ so that for r sufficiently small, we have $|f'(re^{i\theta})| \ge 1 + \frac{|a_1|}{2} r$. Thus $|f'(re^{i\theta})|(1-r^2) > 1$ for small r, a contradiction since $||f|| = 1$. Hence $a_1 = 0$. If $|a_2| > 1$, we can again choose θ so that for r sufficiently small we have $|f'(re^{i\theta})| \ge 1 + \frac{1+|a_2|}{2} r^2$, which again contradicts $||f|| = 1$. Thus $|a_2| \le 1$.

By a rotation of the variable z, we may assume that $0 \le a_2 \le 1$. If $a_2 < 1$, we choose $n = 3$. There is a $\delta' > 0$ so that

$$|z|^3 + \left|\sum_{k=3}^{\infty} a_k z^k\right| < (1-a_2)|z|^2 \qquad \text{if } |z| < \delta'.$$

Thus for $|z| < \delta'$, we have

$$|f'(z)| + |z|^3 < 1 + a_2|z|^2 + \left|\sum_{k=3}^{\infty} a_k z^k\right| + |z|^3 < 1 + |z|^2 < (1-|z|^2)^{-1}.$$

Thus the only case remaining is $a_2 = 1$. In this case we write $z = x + iy$ and obtain the expansion

$$|f'(x+iy)|^2 = 1 + 2(x^2-y^2) + O_3(x,y), \tag{5}$$

where O_3 is a convergent power series having only terms of order 3 or higher. We know that for $0 < |z| < \delta$, we have,

$$|f'(x+iy)|^2 < (1-(x^2+y^2))^{-2} = \sum_{k=0}^{\infty} (k+1)(x^2+y^2)^k. \tag{6}$$

Let $G(x,y) = (1-(x^2+y^2))^{-2} - |f'(x+iy)|^2$. Then G is a real analytic function with $G(0,0) = 0$ and $G(x,y) > 0$ if $0 < x^2+y^2 < \delta^2$. It is easy to see from (5) and (6) that G has the form $G(x,y) = 4y^2 + O_3(x,y)$, where as before O_3 is a convergent power series with terms of order 3 or higher. We now appeal to Theorem A. We may write G as a pseudopolynomial;

$$G(x,y) = (y^2 + A_1(x)y + A_0(x))\,\Omega(x,y),$$

where A_0, A_1, and Ω are real analytic, and $\Omega(0,0) \neq 0$. (Actually $\Omega(0,0) = 4$.) By Lemma 1, there is an $n > 0$ and a $\delta_1' > 0$ so that if $0 < x^2 + y^2 < \delta_1'$, then $(x^2 + y^2)^n < (y^2 + A_1(x)y + A_0(x))$. Since $\Omega(0,0) = 4$, a possibly smaller choice of δ_1' yields that if $0 < x^2 + y^2 < \delta_1'$, then $(x^2 + y^2)^n < G(x,y)$.

Let $K = \sup\{2|f'(z)| + |z|^{n+1} : |z| < \delta\}$. It is clear that we can choose $\delta' > 0$ so that if $0 < x^2 + y^2 < (\delta')^2$, then $K(x^2 + y^2)^{n+1} < G(x,y)$. Thus if $0 < |z| < \delta'$, then

$$\begin{aligned}(|f'(z)| + |z|^{n+1})^2 &= |f'(z)|^2 + 2|f'(z)||z|^{n+1} + |z|^{2n+2} \\ &\leq |f'(z)|^2 + K|z|^{n+1} < (1 - |z|^2)^{-2}.\end{aligned}$$

The lemma is now proven.

Let f be in ball $\mathscr{B}_0$. An immediate consequence of Lemma 2 is that if $L_f = \{0\}$, then f is not an extreme point. Lemma 3 is a corollary of Lemma 2, and Lemma 3 will imply that if $L_f = \{z_0\}$ for $z_0 \in \Delta$, then f is not an extreme point.

LEMMA 3. *Suppose that* f *is in ball* $\tilde{\mathscr{B}}_0$, *and that for some* $z_0 \in \Delta$ *we have* $|f'(z_0)|(1 - |z_0|^2) = 1$, *and there is a* $\delta > 0$ *so that*

$$|f'(z)|(1 - |z|^2) < 1 \qquad \text{for } 0 < |z - z_0| < \delta.$$

Then there is a positive integer n *and a* δ' *with* $0 < \delta' \leq \delta$ *such that*

$$\left(|f'(z)| + \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|^n\right)(1 - |z|^2) < 1 \qquad \text{for } 0 < |z| < \delta'.$$

Proof. Let ϕ be the holomorphic automorphism of Δ given by

$$z = \phi(w) = (w + z_0)(1 + \bar{z}_0 w)^{-1}.$$

Choose $g \in \tilde{\mathscr{B}}_0$ with $g'(w) = f'(\phi(w))\phi'(w)$. Then g satisfies the conditions of Lemma 2.

Let $K = \sup\{|\phi'(w)| : |w| < 1\}$. The proof of Lemma 2 shows that there is a $\delta_1' > 0$ and an integer n such that

$$(|g'(w)| + K|w|^n)(1 - |w|^2) < 1 \qquad \text{for } 0 < |w| < \delta_1'.$$

By the Schwarz-Pick Lemma, $1 - |z|^2 = |\phi'(w)|(1 - |w|^2)$. Choose δ' so that $|z - z_0| < \delta'$ implies $|w| < \delta_1'$. It follows easily that for $0 < |z - z_0| < \delta'$, we have

$$\left(|f'(z)| + \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|^n\right)(1 - |z|^2) < 1.$$

We now prove Theorem 2. Let f be in ball $\tilde{\mathscr{B}}_0$ and let $L_f = \{z_1, \ldots, z_n\}$. Let $b_\delta(z_j)$ denote the closed ball with radius δ and center z_j, and choose δ small enough that the balls $\{b_\delta(z_j)\}_{j=1}^k$ are disjoint and are contained in Δ. A multiple application of Lemma 3 yields a positive number $\delta' \le \delta$ and an integer n such that for $i = 1,2,\dots,k$, $0 < |z - z_i| < \delta'$ implies $|f'(z)| + \left|\frac{z - z_i}{1 - \bar{z}_i z}\right|^n < (1 - |z|^2)^{-1}$. Let $M = \sup\left\{|f'(z)|(1 - |z|^2) : z \in \Delta \setminus \bigcup_{j=1}^{k} b_\delta(z_j)\right\}$. Choose $g \in \tilde{\mathscr{B}}_0$ so that

$$g'(z) = (1 - M)\prod_{j=1}^{k}\left(\frac{z - z_i}{1 - \bar{z}_i z}\right)^n.$$

Then clearly for $z \in b_{\delta'}(z_j)$, we have

$$\begin{aligned}|f'(z) \pm g'(z)|(1 - |z|^2) &\le (|f'(z)| + |g'(z)|)(1 - |z|^2)\\ &\le \left(|f'(z)| + \left|\frac{z - z_j}{1 - \bar{z}_j z}\right|^n\right)(1 - |z|^2) < 1.\end{aligned}$$

Also, if $z \in \Delta \setminus \bigcup_{j=1}^{k} b_{\delta'}(z_j)$ we have

$$\begin{aligned}|f'(z) \pm g'(z)|(1 - |z|^2) &\le |f'(z)|(1 - |z|^2) + |g'(z)|(1 - |z|^2)\\ &< M + (1 - M) = 1.\end{aligned}$$

Thus $f + g$ and $f - g$ are in ball $\tilde{\mathscr{B}}_0$, and $f = \frac{1}{2}(f + g) + \frac{1}{2}(f - g)$ is not an extreme point of ball $\tilde{\mathscr{B}}_0$.

Thus far we have considered extreme points of the unit ball of the normalized Bloch space $\tilde{\mathscr{B}}_0$. We can determine the extreme points of ball $\mathscr{B}_0$ from the following proposition.

PROPOSITION 1. *Let* X *and* Y *be Banach spaces with norms* $\|\ \|_1$ *and* $\|\ \|_2$ *respectively. Let* $N : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *be a function so that* $(x, y) \to N(|x|, |y|)$ *is a norm on* $\mathbb{R}^2$. *Define a norm* $\|\ \|$ *on* $X \oplus Y$ *by*

$$\|x \oplus y\| = N(\|x\|, \|y\|) \qquad \textit{for } x \in X,\ y \in Y.$$

Then $x \oplus y$ *is an extreme point for ball* $(X \oplus Y)$ *if and only if,*

(i) x *is an extreme point for the ball of radius* $\|x\|$ *of* X,

(ii) y *is an extreme point of the ball of radius* $\|y\|$ *of* Y, *and*

(iii) $(\|x\|, \|y\|)$ *is an extreme point of the unit ball of* $\mathbb{R}^2$ *with norm* N.

The proof of the proposition is routine (and probably known), and we omit it. We have the following immediate corollary.

COROLLARY 2. *Let* f *be in ball* $\mathscr{B}_0$. *Then* f *is an extreme point of ball* $\mathscr{B}_0$ *if and only if either*

(i) f *is a constant function of modulus* 1,

(ii) $f(0) = 0$ *and f is an extreme point of ball* $\tilde{\mathscr{B}}_0$.

Proof. $\mathscr{B}_0 = \mathbb{C} < \tilde{\mathscr{B}}_0$, where $\mathbb{C}$ here denotes the constant functions. For $f \in$ ball $\mathscr{B}_0$, the norm of f is given by $\|f\| = |f(0)| + M(f)$, so that norm N of Proposition 1 is the L^1-norm of $\mathbb{R}^2$. Thus $(|f(0)|, M(f))$ is an extreme point for ball $\mathbb{R}^2$ in L^1-norm if and only if $f(0) = 0$ and $\|f\| = 1$ or $|f(0)| = 1$ and $M(f) = 0$. The Corollary now follows.

Remarks. 1. Consider the following equivalent norms on $\mathscr{B}_0$. For $1 \le p \le \infty$ and $f \in \mathscr{B}_0$. Let $\|f\|_p = (|f(0)|^p + (M(f))^p)^{1/p}$ for $1 < p < \infty$. Proposition 1 produces an essentially different class of extreme points from the extreme points in Corollary 2 for the $p = 1$ case.

2. An analog of Corollary 2 is valid which characterizes the extreme points of ball $\mathscr{B}$ in terms of the extreme points of ball $\tilde{\mathscr{B}}$.

3. FURTHER RESULTS, EXAMPLES, AND QUESTIONS

We begin this section by studying the sets of the form

$$L_f = \{z : |f'(z)|(1 - |z|^2) = 1\}$$

for $f \in$ ball $\tilde{\mathscr{B}}_0$.

THEOREM 3. *If* f *is an extreme point of ball* $\tilde{\mathscr{B}}_0$, *then there are simple closed pairwise disjoint analytic curves* $\gamma_1, \gamma_2, \ldots, \gamma_k$ *with* $k \ge 1$ *and points* $W_1, \ldots, W_j$ *with* $j \ge 0$ *so that* $L_f = \left(\bigcup_{i=1}^{k} \gamma_i\right) \cup \{W_1, \ldots, W_j\}$. *Thus in particular* L_f *is uncountable.*

Proof. Suppose z_0 is an accumulation point of L_f. As in the proof of Lemma 3, we may replace f' by $(f' \circ \phi)\phi'$ and hence assume that $z_0 = 0$. Thus we have $\{z_n\} \subset L_f$ with $z_n \to 0$. As in the proof of Lemma 2, we can assume that the Taylor series for f' has form $f'(z) = 1 + a_2 z^2 + \ldots$, where $0 \le a_2 \le 1$. If $0 \le a_2 < 1$, then as in Lemma 2, $|f'(z)|(1 - |z|^2)$ has an isolated local minimum at $z = 0$. Thus $a_2 = 1$. As before we form $G(x,y) = (1 - (x^2 + y^2))^{-2} - |f'(x + iy)|^2$ and apply Theorem A to get

$$G(x,y) = (y^2 + A_1(x)y + A_0(x))\,\Omega(x,y), \qquad \text{where } \Omega(0,0) > 0.$$

Write $z_n = x_n + iy_n$. For n sufficiently large, $H(x_n, y_n) \equiv y_n^2 + A_1(x_n)y_n + A_0(x_n) = 0$, and $H(x,y) \ge 0$ if $x^2 + y^2$ is small. Thus $A_0(x_n) - \dfrac{A_1^2(x_n)}{4} = 0$ for large n. Since $x_n \to 0$, and A_0 and A_1 are real analytic we find that either $A_0(x) - \dfrac{A_1^2(x)}{4} \equiv 0$ for $|x|$ sufficiently small or $x_n = 0$ for all large n. In the first case we have

$$\left\{(x,y) : x^2 + y^2 < \delta \quad \text{and} \quad y = -\frac{A_1(x)}{2}\right\} \subset L_f.$$

In the second case we have $H(0,y_n) = y_n^2 = 0$ for large h, contradicting the fact that $z_n \neq 0$. The preceding argument shows that in a neighborhood of an accumulation point of L_f, we know that L_f is an analytic arc. This fact and the fact that L_f is a compact subset of Δ imply the Theorem.

We do not know which sets described in the Theorem can arise as an L_f for some f in ball $\bar{\mathscr{B}}_0$. However, we have the following result.

THEOREM 4. *For each* $w \in \Delta$, *there is a countable family of circles* $\Gamma_{w,n}$ *in* Δ *with the following properties.* (a) *Given* $\Gamma_{w,n}$, *there is a function* $f_{w,n}$ *in ball* $\bar{\mathscr{B}}_0$ *so that* $L_{f_{w,n}} = \Gamma_{w,n}$. *Moreover,* $f_{w,n}$ *is unique up to a multiplicative constant of modulus* 1. (b) *Conversely, if* f *is in ball* $\bar{\mathscr{B}}_0$ *and* L_f *contains a circle* Γ, *then* $\Gamma = \Gamma_{w,n}$ *for some* (w, n).

Proof. We first consider part (b) of the theorem. Suppose that f is in ball $\bar{\mathscr{B}}_0$ and that $\{z : |z| = r_0\} \subset L_f$. If we set $g(z) = (1 - r_0^2) f'(r_0 z)$, then g is in the disc algebra with modulus identically equal to one in Γ. Hence there is a finite Blaschke product

$$B(z) = \lambda z^k \prod_{j=1}^{J} \left(\frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \right), \qquad |\lambda| = 1, \alpha_j \neq 0,$$

such that $f'(z) = \dfrac{1}{1 - r_0^2} B\left(\dfrac{z}{r_0}\right)$. The maximum of $|f'(z)|(1 - |z|^2)$ occurs at $|z| = r_0$. Hence

$$\frac{d}{dr} \left[\frac{1 - r^2}{1 - r_0^2} \frac{r^k}{r_0{}^k} \prod_{j=1}^{J} \left| \frac{z/r_0 - \alpha_j}{1 - \bar{\alpha}_j z/r_0} \right| \right] \Bigg|_{r=r_0} = 0, \qquad z = re^{i\theta}. \tag{7}$$

Let $P(r, \theta - \phi)$ be the Poisson kernel evaluated at $z = re^{i\theta}$ and observe that

$$\frac{d}{dr} \left| \frac{z/r_0 - \alpha_j}{1 - \bar{\alpha}_j z/r_0} \right|_{r=r_0} = \frac{1}{r_0} P(|\alpha_j|, \theta - \theta_j), \tag{8}$$

where $\alpha_j = |\alpha_j| e^{i\theta_j}$. If we carry out the differentiation in (7) and use (8), we see that

$$\frac{k - (k+2) r_0^2}{1 - r_0^2} = \sum_{j=1}^{J} P(|\alpha_j|, \theta - \theta_j). \tag{9}$$

Suppose that $J \neq 0$ and form the polynomial $q(z) = \prod_{j=1}^{J} (z - \alpha_j)$. Then

$$0 = \sum_{j=1}^{J} q(\alpha_j) = \int_{\Gamma} q(e^{i\theta}) \sum_{j=1}^{J} P(|\alpha_j|, \theta - \theta_j)\, d\theta$$
$$= \int_{\Gamma} q(e^{i\theta}) \frac{k - (k+2) r_0^2}{(1 - r_0^2)} d\theta = q(0) \frac{k - (k+2) r_0^2}{1 - r_0^2}$$

so that $k - (k+2)r_0^2 = 0$. (9) now says that $\sum_{j=1}^{J} P(|\alpha_j|, \theta - \theta_j) = 0$, which is clearly false. Thus $J = 0$ and $B(z) = \lambda z^k$. A straightforward calculation shows that $\max\{|w|^n(1-|w|^2) : |w| < 1\} = \left(\frac{n}{n+2}\right)^{n/2}\left(\frac{2}{n+2}\right)$, and that this maximum occurs on $|w| = \left(\frac{n}{n+2}\right)^{1/2}$. Thus $r_0 = \left(\frac{k}{k+2}\right)^{1/2}$, and

$$\Gamma_{0,k} = \left\{z : |z| = \left(\frac{k}{k+2}\right)^{1/2}\right\}.$$

Further $f_{0,k}$ satisfies

$$f'_{0,k}(z) = \frac{\lambda z^k}{r_k^k(1-r_k^2)}, \qquad \text{where } r_k = \left(\frac{k}{k+2}\right)^{1/2} \quad \text{and } |\lambda| = 1.$$

Now suppose that f is in ball $\bar{\mathscr{B}}_0$ and that Γ is a circle centered at z_0 contained in L_f. Let ϕ be a holomorphic automorphism of Δ which maps Γ to a circle centered at the origin, and let $\psi = \phi^{-1}$. Choose h in $\bar{\mathscr{B}}_0$ with $h'(w) = f'(\psi(w))\psi'(w)$. From the first part of the proof, we conclude that

$$h'(w) = \frac{\lambda w^k}{r_k^k(1-r_k^2)} \qquad \text{where } |\lambda| = 1 \text{ and } k > 0.$$

It follows that $f'(z) = f'(\psi(w)) = \dfrac{\lambda w^k}{\psi'(w)} = \lambda(\phi(z))^k\phi'(z)$, and that

$$\Gamma = \Gamma_{z_0,k} = \phi^{-1}(\Gamma'), \qquad \text{where } \Gamma' = \Gamma_{0,k} = \left\{w : |w| \le \left(\frac{k}{k+2}\right)^{1/2}\right\}.$$

Thus the circles $\Gamma_{w,n}$ in the statement of Theorem 3 are just images of the circles $\Gamma_{0,n}$ under automorphisms of Δ.

In some Banach spaces, the notion of "strong extreme point" has been of interest. For example the strong extreme points of ball H^∞ are the inner functions [3].

Definition. Let X be a Banach space and let $x \in X$ with $||x|| = 1$. Then X is a strong extreme point for ball X if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\max\{||x+y||, ||x-y||\} \le 1 + \delta$ implies $||y|| < \varepsilon$.

PROPOSITION. *The unit ball of $\bar{\mathscr{B}}_0$ has no strong extreme points.*

Proof. Let f be in $\bar{\mathscr{B}}_0$ with $||f|| = 1$. Given $\delta > 0$, there is an $r \in (0,1)$ such that $|f'(z)|(1-|z|^2) < \delta$ if $r \le z < 1$. Let $g_k(z) = z^k/||z^k||$. Then $g_k \in$ ball $\bar{\mathscr{B}}_0$ and a routine calculation shows that $\lim_{k\to\infty} \sup_{|z|\le r} |g'_k(z)| = 0$. Thus there is a k_0 so that $|g'_{k_0}(z)|(1-|z|^2) < \delta$ for $|z| \le r$.

We conclude that $||f \pm g_{k_0}|| \le ||f|| + \delta$, and $||g_{k_0}|| = 1$.

We end our paper with some questions.

Question 1. What is the closed convex hull of the extreme points of ball $\tilde{\mathscr{B}}_0$?

Question 2. Are there extreme points f of ball $\tilde{\mathscr{B}}_0$ such that L_f is not a circle? A weaker question is whether L_f must be connected.

Question 3. All of the extreme points of ball $\tilde{\mathscr{B}}_0$ are extreme points of ball $\tilde{\mathscr{B}}$. However, ball $\tilde{\mathscr{B}}$ has other extreme points. What are the extreme points of ball $\tilde{\mathscr{B}}$? Can ball $\tilde{\mathscr{B}}$ have an extreme point f such that L_f is empty?

REFERENCES

1. J. M. Anderson, J. Clunie, and Ch. Pommerenke, *On Bloch functions and normal functions.* J. Reine Angew. Math. 270 (1974), 12–37.
2. J. M. Anderson and L. A. Rubel, *Hypernormal meromorphic functions,* preprint.
3. J. A. Cima and J. Thomson, *On extreme points in* H^p. Duke Math. J. 40 (1973), 529–532.
4. R. R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables,* preprint.
5. P. L. Duren, B. W. Romberg, and A. L. Shields, *Linear functionals on* H^p *with* $0 < p < 1$. J. Reine Angew. Math. 238 (1969), 32–60.
6. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities.* Graduate Texts in Mathematics, Vol. 14. Springer-Verlag, New York-Heidelberg, 1973.
7. O. Zariski and P. Samuel, *Commutative Algebra,* Vol. II. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960.

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