# THE ESSENTIAL SPECTRUM OF A HANKEL OPERATOR WITH PIECEWISE CONTINUOUS SYMBOL

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A Hankel operator S on a complex Hilbert space with complete orthonormal basis $\{e_n; n = 0, 1, 2, \ldots\}$ is one whose representing matrix has the form

$$S_{ij} = c_{i+j}, \qquad i, j = 0, 1, 2, \ldots .$$

A classical theorem of Nehari [6] shows that a sequence $(c_n)_{n=0}^{\infty}$ defines a bounded Hankel operator if and only if it is the sequence of positive Fourier coefficients of an essentially bounded measurable function $\phi$ on the unit circle. Hartman subsequently showed that S is compact if and only if $\phi$ can be chosen to be continuous (see [4] or [1]).

In this note we determine the essential spectrum of S when $\phi$ is a function possessing left and right limits at every point on the circle.

*Notation.* Let $L^2$ be the Hilbert space of square integrable functions on the unit circle T with the usual orthonormal basis $\{z^n; n = 0, \pm 1, \pm 2, \ldots\}$. The unitary operator J on $L^2$ is defined by $Jz^n = z^{-n}$ and we shall let P denote the orthogonal projection of $L^2$ onto the Hardy subspace $H^2$ spanned by $\{z^n; n = 0, 1, 2, \ldots\}$.

For an essentially bounded measurable function $\phi$ in $L^\infty$, the Toeplitz operator $T_\phi$, on $H^2$, is defined by $T_\phi = PM_\phi \mid H^2$ where $M_\phi$ is the usual multiplication operator on $L^2$. We call $\phi$ the symbol of the Toeplitz operator $T_\phi$. The Hankel operator on $H^2$, with symbol $\phi$ in $L^\infty$, is defined by $S_\phi = PJM_\phi \mid H^2$.

Let PC denote the collection of functions on T which possess left and right limits at each point. For $\phi$ in PC and $\alpha$ in T we shall write

$$\phi_\alpha = \frac{1}{2} \lim_{t \to 0+} \{\phi(\alpha e^{it}) - \phi(\alpha e^{-it})\}$$

and call $\phi_\alpha$ the jump of $\phi$ at $\alpha$.

Let T′ denote the non-real points of T and, for $\gamma, \nu \in \mathbb{C}$, let $[\gamma, \nu]$ denote the line segment joining $\gamma$ and $\nu$. We shall prove the following:

THEOREM 1. Let $\phi$ be a function in PC. Then

$$\sigma_e(S_\phi) = [0, i\phi_1] \cup [0, i\phi_{-1}] \cup \bigcup_{\alpha \in T'} [-(-\phi_\alpha \phi_{\bar\alpha})^{1/2}, +(-\phi_\alpha \phi_{\bar\alpha})^{1/2}].$$

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In particular notice that a jump at $\alpha$ only contributes to the essential spectrum if it is accompanied by a jump at $\bar{\alpha}$. The key results we shall use are the following two theorems of Gohberg and Krupnik [3] (see also [2] p.20) concerning Toeplitz operators with PC symbols.

THEOREM 2. *Let $\phi$ and $\psi$ be piecewise continuous functions. Then*

$$T_{\phi\psi} - T_\phi T_\psi$$

*is compact if and only if $\phi$ and $\psi$ do not have common points of discontinuity.*

For $\phi$ in PC define $\hat{\phi}$ on $T \times [0,1]$ by $\hat{\phi}(e^{it}, s) = s\phi(e^{it-}) + (1-s)\phi(e^{it+})$.

THEOREM 3. *If* $\{\phi_{ij}\}_{i=1}^{n}{}_{j=1}^{m}$ *are functions in* PC *then* $\sum_i \prod_j T_{\phi_{ij}}$ *is Fredholm if and only if* $\hat{\phi} = \sum_i \prod_j \hat{\phi}_{ij} \neq 0$.

These two theorems can be related to Hankel operators by means of the following formula which has proved useful in other situations (*e.g.,* [9] and [7]). For $\phi$ in $L^\infty$ let $\tilde{\phi}(z) = \phi(\bar{z})$.

LEMMA 4. *For $\phi$ and $\psi$ in $L^\infty$ we have* $S_\phi S_\psi = T_{\tilde{\phi}\psi} - T_{\tilde{\phi}z} T_{\psi\bar{z}}$.

*Proof.*
$$\begin{aligned} S_\phi S_\psi &= PJM_\phi PJM_\psi \,|\, H^2 = PM_{\tilde{\phi}z}(M_{\bar{z}} JPJM_z) M_{\bar{z}\psi} \,|\, H^2 \\ &= PM_{\tilde{\phi}z}(I-P) M_{\bar{z}\psi} \,|\, H^2 = T_{\tilde{\phi}\psi} - T_{\tilde{\phi}z} T_{\psi\bar{z}}. \end{aligned}$$

We shall prove Theorem 1 through a series of lemmas which deal with simple symbol functions in PC.

Let $\psi(e^{it}) = i(t-\pi)e^{it}$, $0 \le t < 2\pi$. Then $\psi \in PC$ and has a single jump at 1 where $\psi_1 = -i\pi$. A simple computation shows that

$$(S_\psi z^n, z^m) = (n+m+1)^{-1}, \qquad n, m = 0, 1, 2, \ldots,$$

and so $S_\psi$ is the Hankel operator defined by Hilbert's matrix. Magnus [5] has shown that the essential spectrum of this operator is the interval $[0, \pi]$ (*i.e.,* $[0, i\psi_1]$). Alternatively by Lemma 4, $S_\psi^2 = T_{(t-\pi)^2} - T_{(t-\pi)}{}^2$. Theorem 3 can be applied to show that $\sigma_e(S_\psi^2) = [0, \pi^2]$ and, since $S_\psi$ is positive and $\|S_\psi\| \le \|\psi\| \le \pi$, we have proved Magnus's result.

LEMMA 5. *Let $\phi$ be continuous apart from a jump at $\alpha$. Then*

(i) *If* $\alpha = 1$ *then* $\sigma_e(S_\phi) = [0, i\phi_1]$

(ii) *If* $\alpha = -1$ *then* $\sigma_e(S_\phi) = [0, i\phi_{-1}]$.

*Proof.* (i) We may write $\phi = \lambda\psi + \theta$ where $\lambda \in \mathbb{C}$ and $\theta$ is a continuous function. Since, by Hartman's theorem, $S_\theta$ is compact, we have

$$\sigma_e(S_\phi) = \sigma_e(\lambda S_\psi) = [0, i\lambda\psi_1] = [0, i\phi_1].$$

(ii) Let V be the unitary operator on $L^2$ defined by $(Vf)(e^{it}) = f(-e^{it})$. Then V commutes with J and P and $V^*S_\phi V = S_{\phi'}$ where $\phi'(e^{it}) = \phi(-e^{it})$. By (i)

$$\sigma_e(S_\phi) = \sigma_e(S_{\phi'}) = [0, i\phi'_1] = [0, i\phi_{-1}].$$

LEMMA 6. *Let $\phi$ be continuous apart from a jump at $\alpha$ and a jump at $\bar{\alpha}$ where $\alpha$ is a non-real point of* T. *Then $\sigma_e(S_\phi)$ is the line segment*

$$[-(-\phi_\alpha\phi_{\bar{\alpha}})^{1/2}, +(-\phi_\alpha\phi_{\bar{\alpha}})^{1/2}].$$

*Proof.* Let $\phi_\alpha = \lambda_1$ and $\phi_{\bar{\alpha}} = \lambda_2$. Since we may add a continuous function to the symbol without altering the essential spectrum we may suppose that $\phi(\alpha-) = 0$, $\phi(\alpha+) = 2\lambda_1$, $\phi(\bar{\alpha}-) = 0$, and $\phi(\bar{\alpha}+) = 2\lambda_2$. By Lemma 4,

$$S_\phi^2 = T_{\tilde{\phi}\phi} - T_{\tilde{\phi}z}T_{\phi\bar{z}}$$

so that by Theorem 1 $\sigma_e(S_\phi^2)$ is the range of $\widehat{\tilde{\phi}\phi} - \widehat{\tilde{\phi}z}\,\widehat{\phi\bar{z}}$ on $T \times [0,1]$. Since $\phi$ is continuous except possibly at $\alpha$ and $\bar{\alpha}$ it suffices to consider the range of this function on $\alpha \times [0,1]$ and $\bar{\alpha} \times [0,1]$. Now $\tilde{\phi}\phi$ is continuous near $\alpha$ and $\bar{\alpha}$ and vanishes at $\alpha$ and $\bar{\alpha}$, thus $\sigma_e(S_\phi^2)$ is given by the range of $-\hat{\tilde{\phi}}\,\hat{\phi}$ on $\alpha \times [0,1]$ and $\bar{\alpha} \times [0,1]$. Since $\hat{\tilde{\phi}}\,\hat{\phi}(\alpha, s) = \hat{\tilde{\phi}}\,\hat{\phi}(\bar{\alpha}, s) = 4\lambda_1\lambda_2 s(1-s)$ we see that $\sigma_e(S_\phi^2) = [0, -\lambda_1\lambda_2]$.

We now show that $\sigma_e(S_\phi) = -\sigma_e(S_\phi)$ which will complete the proof of the lemma.

Let $\theta$ be a function on T which is continuous apart from a proper jump at $\alpha$. It follows that there exist complex numbers $\nu_1$, $\nu_2$ and a continuous function $\phi'$ on T such that $\phi = \nu_1\theta + \nu_2\bar{\tilde{\theta}} + \phi'$. Thus, since $S_\theta^* = S_{\bar{\tilde{\theta}}}$ it will be sufficient to show that $\sigma_e(S) = -\sigma_e(S)$ where $S = \nu_1 S_\theta + \nu_2 S_\theta^*$. Since $S_\theta^2 = T_{\tilde{\theta}\theta} - T_{\tilde{\theta}z}T_{\theta\bar{z}}$, it follows from Theorem 2 that $S_\theta^2$ is compact. Let $\pi$ be the homomorphism of $B(H^2)$ into the Calkin algebra A, and let $\Phi$ be a faithful representation of A as a C*-algebra of operators on a Hilbert space. Then $(\Phi\circ\pi(S_\theta))^2 = 0$. By a result of Radjavi and Rosenthal ([8] Theorem 1) $(\Phi\circ\pi)(S_\theta)$ has the form $\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$ Thus $(\Phi\circ\pi)(S)$ has the form $\begin{pmatrix} 0 & \nu_1 C \\ \nu_2 C^* & 0 \end{pmatrix}$ and so $\sigma(\Phi\circ\pi(S)) = -\sigma(\Phi\circ\pi(S))$ which implies

$$\sigma(\pi(S)) = -\sigma(\pi(S)),$$

completing the proof.

LEMMA 7. *Let $(a_n)_{n=1}^{\infty}$ be elements of a complex unital commutative Banach algebra* A *such that $a = \sum_{n=1}^{\infty} a_n$ converges in norm and*

$$a_n a_m = a_m a_n = 0 \qquad \textit{for } n \neq m.$$

*Then* $\sigma(a) \cup \{0\} = \bigcup_{n=1}^{\infty} \sigma(a_n)$.

*Proof.* Let M be the set of multiplicative linear functionals $\phi$ on A, so that $\sigma(b) = \{\phi(b); \phi \in M\}$ for b in A. Since $\phi(a) = \sum_{n=1}^{\infty} \phi(a_n)$ and $\phi(a_n) \neq 0$ implies $\phi(a_m) = 0$ for $m \neq n$, the result follows.

We now put together the pieces.

*Proof of Theorem 1.* We first show that the theorem is true if $\phi$ is a piecewise continuous function. In this case we can write $\phi = \phi' + \phi'' + \sum_{i=1}^{n} \phi^{(\alpha_i)}$ where $\phi'$ (resp. $\phi''$) is continuous apart from possibly a jump at 1 (resp. $-1$) and $\phi^{(\alpha_i)}$ is continuous apart from possible jumps at $\alpha_i$ and $\bar{\alpha}_i$. Since, by Theorem 2 and Lemma 4 any pair of operators A, B from $\{\pi(S_{\phi'}), \pi(S_{\phi''}), \pi(S_{\phi(\alpha_i)}); i = 1, \ldots, n\}$ satisfies $AB = 0$ the theorem follows from lemmas 5, 6 and 7.

Suppose now that $\phi \in PC$. We first show that there exists a sequence of piecewise continuous functions $\phi^{(n)}$, $n=1,2,\ldots$, such that

(i) $\phi^{(n)}$ and $\tilde{\phi}^{(m)}$ have no common discontinuities for $n \neq m$.

(ii) $|\psi_\alpha^{(n)}| \leq 2^{-n}$ for $\alpha \in T$, where $\psi^{(n)} = \phi - \sum_{i=1}^{n} \phi^{(i)}$.

Let $\Lambda'_n = \{\alpha: |\phi_\alpha| \geq 2^{-n}\}$. Since $\phi \in PC$ it follows that $\Lambda'_n$ is finite. Let

$$\Lambda''_n = \{\bar{\alpha}; \alpha \in \Lambda'_n\} \cup \Lambda'_n$$

and let $\Lambda_1 = \Lambda''_1$, $\Lambda_{n+1} = \Lambda''_{n+1} \setminus \Lambda''_n$, $n = 1, 2, \ldots$. Now choose $\phi^{(n)}$ to be any piecewise continuous function such that $\phi^{(n)}$ is continuous on $T \setminus \Lambda_n$ and

$$\phi_\alpha^{(n)} = \phi_\alpha \qquad \text{for } \alpha \in \Lambda_n.$$

Then the $\phi^{(n)}$ satisfy (i) and (ii).

By theorem 2 and Lemma 4, (i) shows that $\pi(S_{\phi(n)})\pi(S_{\phi(m)}) = 0$ for $n \neq m$. Also the second condition (ii) shows that $\|\pi(S_{\psi(n)})\| \leq 2.2^{-n}$. This can be seen by first approximating $\psi^{(n)}$ by a piecewise continuous function $\theta$ so that $\|\psi^{(n)} - \theta\| \leq \varepsilon$. Since $|\theta_\alpha| \leq 2^{-n} + \frac{1}{2}\varepsilon$ for $\alpha \in T$, there exists a continuous function $\theta'$ such that $\|\theta - \theta'\| \leq 2.2^{-n} + \varepsilon$. Thus

$$\|\pi(S_{\psi(n)})\| = \|\pi(S_{\psi(n)-\theta'})\| \leq \|\psi^{(n)} - \theta'\| \leq 2(2^{-n} + \varepsilon).$$

So $\pi(S_\phi) = \sum_{n=1}^{\infty} \pi(S_{\phi(n)})$ and the theorem follows from Lemma 7 and the first part of this proof.

*Remarks.* Although zero is always a point in the essential spectrum of a Hankel operator it is not always true that the essential spectrum is 'connected to the origin' as in Theorem 1. To see the first half of this statement suppose that A is a left inverse for the Hankel operator S modulo the compacts, so that $AS = I + K$ for some compact operator K. Since $SU^n = U^{*n}S$, where U is the shift on $H^2$, we have $(I + K)U^n = AU^{*n}S$, and so, for $x \in H^2$, $U^n x = AU^{*n}Sx - KU^n x$. However, $U^n x \to 0$ weakly and so $KU^n x \to 0$ in norm. Since $U^{*n}Sx \to 0$ in norm also, we have a contradiction when $x \neq 0$. The second half of our initial remark follows by considering the Hankel operator $S_{z\bar{\phi}}$ where $\phi$ is the inner function

$$\phi(z) = \exp\left(\frac{1+z}{z-1}\right).$$

It can be shown that $S_{z\bar{\phi}}$ is a self-adjoint partial isometry (a partial symmetry?) and $\sigma(S_{z\bar{\phi}}) = \sigma_e(S_{z\bar{\phi}}) = \{-1\} \cup \{0\} \cup \{1\}$.

Just how arbitrary can the spectrum or essential spectrum of a Hankel operator be? In particular, is any compact subset of the complex plane which contains the origin the spectrum of a Hankel operator?

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