UNIQUENESS FOR SETS OF FUNCTIONS GENERATED BY MATRICES

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1. INTRODUCTION

In this paper we define admissible matrices. Using these matrices we define sets of admissible functions. That each of these sets of admissible functions is a complete orthonormal system for $L^1[[0, 1), \mathbb{C}]$ and that any admissible Fourier series of these functions (hereafter called Fourier series) converges almost everywhere are almost immediate from the definitions. The major result of this paper supplies sufficient conditions for uniqueness of admissible series. Some results on Walsh-like series found in [1], [3], [4], [5], and [8], among others, are similiar to our results.

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2. BASIC DEFINITIONS AND PROPERTIES

Definition 1. Let $A = (a_{ij})$ be an $n \times n$ matrix with complex entries and let $\bar{A} = (\bar{a}_{ij})$, where $\bar{a}_{ij}$ is the complex conjugate of a_{ij}. The nonsingular matrix A will be called an *admissible matrix* if $A\bar{A}^T$ is a diagonal matrix and $a_{1j} = 1$ for $1 \leq j \leq n$.

Note that a matrix composed of the characters of the cyclic group of order n arranged properly satisfies this condition.

Definition 2. Let $A = (a_{ij})$ be an $n \times n$ admissible matrix normalized so that $A\bar{A}^T = nI$. For $1 \leq i \leq n$ and $x \in [0, 1)$, let $g_i(x) = a_{ik}$, where $(k-1)/n \leq x < k/n$ and $1 \leq k \leq n$. The functions $(g_i)_{i=1}^n$ defined in this way will be called the *admissible functions* derived from the matrix A. (In the future we will just refer to these functions as admissible functions.)

The family of complex-valued functions $(g_i)_{i=1}^n$ defined on $[0, 1)$ can be extended periodically to any interval (a, b). Note that $A\bar{A}^T = nI$ implies that the g_i are orthonormal.

Definition 3. Let $A = (a_{ij})$ be an $n \times n$ complex matrix, and let $B = (b_{ij})$ be an $m \times m$ complex matrix. $A * B$ will be used to denote the product of A by B, which is defined as follows. Let B_i be the ith row of B, and let $\otimes$ be the Kronecker product of two matrices; then

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$$A * B = \begin{bmatrix} A \otimes B_1 \\ \cdot \\ \cdot \\ \cdot \\ A \otimes B_m \end{bmatrix} .$$

Although similar to the tensor product, the $*$ product of admissible matrices does not alter the order of the admissible functions as would the tensor product.

Since the $*$ product is not associative, we shall assume, unless otherwise noted, that $A * B * C$ means $(A * B) * C$.

Note that if A is $n \times n$ and B is $m \times m$, then $A * B$ is $nm \times nm$. Further, if both matrices have nonzero determinant, it follows from the similar property of the tensor product that

$$|\det(A * B)| = |\det(B * A)| = |[\det(A)]^m[\det(B)]^n| \neq 0 .$$

It is easy to see that if A and B are admissible matrices, then $A * B$ is also an admissible matrix.

Definition 4. Let $H = (H_i)_{i=1}^{\infty}$ be a sequence of admissible matrices, let

$$K_n = H_1 * H_2 * \dots * H_n ,$$

and let s(n) be the dimension of K_n . Let $(g_i)_{i=1}^{\infty}$ be the sequence of admissible functions derived from $(K_n)_{n=1}^{\infty}$. Now let F_n be the set of end-points of the intervals of constancy of $(g_i)_{i=1}^{s(n)}$ (the functions derived from K_n). We define the *F-set* of H to be $F = \bigcup_{n=1}^{\infty} F_n$.

Note that $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$, and that F is a countable set.

The definition of an F-set may be extended in a natural way to any interval (a, b) which contains [0, 1). We define F(a, b), the F-set of (a, b), by saying that $x \in F(a, b)$ if and only if $x \pmod 1 \in F$. Since the context of future results makes it clear whether we are on [0, 1) or (a, b), we will use F to denote F(a, b).

Let $\alpha_n(x) = \dfrac{k}{s(n)}$ and $\beta_n(x) = \dfrac{k+1}{s(n)}$, where k is an integer and

$$\frac{k}{s(n)} \leq x < \frac{k+1}{s(n)} .$$

Then we observe that Definition 2 gives us

$$\sum_{i=1}^{s(n)} \overline{g(t)}\, g(x) = \begin{cases} s(n) & \text{if } \alpha_n(t) = \alpha_n(x) \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\sum_{i=1}^{s(n)} \int_0^1 f(t)\,\overline{g_i(t)}\,g(x)dt \to f(x)$ almost everywhere and thus the g_i form a complete orthonormal system. Observe that if $H_i = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ for all i, one gets the Walsh functions as discussed by Paley in [6].

MAIN RESULT

This section of the paper deals with uniqueness of admissible series, hereafter called series. For any $x \in (a, b)$, define $\alpha'_n(x)$ by $\alpha'_n(x) = \alpha_n(x)$ if $x \notin F$ and

$$\alpha'_n(x) = \alpha_n(x) - 1/s(n) \qquad \text{if } x \in F\,.$$

Then by substituting s(n) for 2^n in the theorem of [5], one gets the following.

LEMMA 1. *Let* G *be a real-valued function defined on* $(a, b) \cap F$ *which satisfies the following conditions:*

(i) $\liminf_{n\to\infty} [G(x) - G(\alpha'_n(x))] \le 0\,, \qquad x \in F;$

(ii) $\liminf_{n\to\infty} [G(\beta_n(x)) - G(\alpha_n(x))] \le 0\,, \qquad x \in (a, b);$

(iii) *there exists a countable set* E *such that*

$$\liminf_{n\to\infty} [G(\beta_n(x)) - G(\alpha_n(x))] \cdot s(n) \le 0\,, \qquad x \in (a, b) - E.$$

Then G *is monotone decreasing on* F.

The following notation and observations will be needed. Let $(g_i)_{i=1}^{\infty}$ be a sequence of admissible functions, and let s(n) be as defined in Definition 4. Define

$$h_i(x) \doteq \int_0^x g_i(t)\,dt.$$

Thus $h_i(k/s(n)) \equiv 0$ for all $i > s(n)$. Let $R(x) = \sum_{i=1}^{\infty} a_i g_i(x)$ be a series, and define $R(s(n), x) = \sum_{i=1}^{s(n)} a_i g_i(x)$. Define $L(R, x) = \lim_{n\to\infty} \sum_{i=1}^{s(n)} a_i h_i(x)$. We shall write L(x) in place of L(R, x) when R is fixed. We now state a lemma which we will need for our theorem.

LEMMA 2. *If* $R(x) = \sum_{i=1}^{\infty} a_i g_i(x)$ *is a series that has the property that*

$$(1/s(n)) \sum_{i=1}^{s(n)} |a_i g_i(t)| \to 0$$

uniformly in t *as* $n \to \infty$, *then for each* x *and every* $n > 0$, *both* $L(\alpha_n(x))$ *and* $L(\beta_n(x))$ *exist and are finite.*

Further:

(a) $L(\beta_n(x)) - L(\alpha_n(x)) = (1/s(n))R(s(n), x)$;

(b) $L(\beta_n(x)) - L(\alpha_n(x)) \to 0$ uniformly in x as $n \to \infty$.

Also, if $L(x_0)$ *exists, then:*

(c) $L(\alpha_n'(x_0)) \to L(x_0)$;

(d) $L(\beta_n(x_0)) \to L(x_0)$.

The proof of the above lemma is the same as for Lemmas 1 and 2 in [2, page 552 and 553], with 2^n replaced by s(n).

THEOREM. *Let* $(g_i)_{i=1}^{\infty}$ *be a sequence of real-valued admissible functions. Let* $R(x) = \sum_{i=1}^{\infty} a_i g_i(x)$ *be a series, and let* E *be a countable subset of* [0, 1]. *Suppose that*

(1) $(1/s(n)) \sum_{i=1}^{s(n)} |a_i g_i(t)| \to 0$ *uniformly in* t *as* $n \to \infty$, *and*

(2) if $x \in [0, 1] - E$, then

$$\liminf_{n\to\infty} R(s(n), x) \le q(x) \le \limsup_{n\to\infty} R(s(n), x),$$

where q *is some function in* $L^1[0, 1]$.

Then R *is the Fourier series of* q.

Proof. Define $q(x) = 0$ for all $x \in E$ and let $Q(x) = \int_0^x q(t)\, dt$.

Let (a, b) be an interval which contains [0, 1]. Extend q and $(g_i)_{i=1}^{\infty}$ (and consequently R(x), R(s(n), x), and L(x)) periodically to (a, b). Next, using the Vitali-Carathéodory theorem as in [2, page 557] and Lemma 2, it is easily seen that $L(x) = Q(x)$ for $x \in F$.

Now let $\sum_{i=1}^{\infty} b_i g_i(x)$ be the Fourier series of q(x), so that (by Lemma 2)

$$\begin{aligned}\sum_{i=1}^{s(n)} b_i g_i(\alpha_j(x)) &= s(n) \int_{\alpha_j(x)}^{\beta_j(x)} q(t)\, dt = s(n)\, [Q(\beta_j(x)) - Q(\alpha_j(x))] \\ &= s(n)\, [L(\beta_j(x)) - L(\alpha_j(x))] = \sum_{i=1}^{s(n)} a_i g_i(\alpha_j(x)).\end{aligned}$$

Thus, by completeness, we have $a_i = b_i$ for $1 \le i \le s(n)$. This argument is valid for all n, so that $a_i = b_i$ for all i. Therefore, R is the Fourier series of q, and the proof is complete.

COMMENTS

1) The conditions on the theorem can apparently be weakened to correspond to those in [2] by considering the underlying groups, but the authors felt that this would tend to confuse the notation even more.

2) It is possible to alter the definition of the matrix product in such a way as to get Haar-like functions.

3) We have no idea whether complete systems of continuous functions can be created in this way, although the trigonometric functions apparently can be generated in a manner reminiscent of how the Walsh functions are generated by the Rademacher functions.

It is possible that a result similar to Lemma 1 can be proved if the F-set is replaced by any countable dense subset of (a, b) .

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