SINGULARITY OBSTRUCTIONS TO IMMERSIONS

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1. INTRODUCTION

Let M^m, N^n be connected C^∞ manifolds of dimension m and n, respectively, with $m < n$ and M^m closed (compact, without boundary). Let $f: M^m \to N^n$ be a continuous mapping. Our initial purpose here is to introduce three sets of (cohomological) homotopy invariants for f and show they are obstructions to the process of deforming f to a smooth (C^∞) immersion $g: M^m \to N^n$. By Smale-Hirsch theory [5] any obstruction to such a deformation must involve finding a vector bundle embedding $\phi: TM \to TN$ covering f, and in fact the vanishing of these invariants for an immersion is derived as a consequence of a more general proposition concerning the existence of bundle epimorphisms $\psi: \xi^n \to \tau^p$, ($\xi^n \to M^m$, $\tau^p \to M^m$ real vector bundles over M^m of rank n, p, respectively, $n > p$). Roughly speaking, our approach to this problem involves decomposing the vector bundle $\mathrm{Hom}(\xi, \tau) \to M^m$ into its "singularity subbundles" $S_i(\xi,\tau)$, $0 \leq i \leq p$, and asking when a section

$$\sigma \in \Gamma^\infty(\mathrm{Hom}(\xi, \tau))$$

may be homotoped so as to avoid all the $S_i(\xi,\tau)$, $i > 0$. Using the (Poincare duals of the) fundamental homology classes associated to each $\overline{S_i(\xi,\tau)}$ [13], obstructions to such a deformation are defined.

In Sections 2.1 and 2.2 we give the necessary background information on the homological properties of the $S_i(\xi,\tau)$ and prove the Proposition on bundle epimorphisms [Proposition 2.2.2]. The basic obstruction theorem is then given in (2.3) [Theorem 2.3.2].

In Section 3, we use the obstruction theorem together with a result due to R. Thom and I. Porteous [11] to study immersions of M^m into $\mathbb{C}P^n$, complex projective space of 2n real dimensions. The starting point here is the fact, which follows from the well-known theorem of A. Haefliger ([3], Theorem 1, p. 109), that every $f: M^m \to \mathbb{C}P^m$ deforms to a smooth immersion. The main result of this section [Theorem 3.1.2] then, deals with the question of when this result may be improved and when it is best possible.

Apart from its intrinsic interest, there is a second motivation for considering C^∞ immersions into $\mathbb{C}P^n$. Namely, in [6], A. Holme raised the question of computing the minimal dimension $n(V^m)$ for which V^m, a non-singular projective m-variety (over $\mathbb{C}$) may be embedded holomorphically in $\mathbb{C}P^n$. Since a negative result for C^∞ embeddings into $\mathbb{C}P^n$ is necessarily one for holomorphic embeddings (for M^m complex) the results of (3.1) carry over to give information on this problem. Pursuing

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this question further, we conclude by utilizing the construction of Section 2 in order to produce complex "singularity obstructions" to deforming f: $V^m \to \mathbb{C}P^n$ (V^m any closed, connected complex m-manifold) to a holomorphic immersion provided $n \le 2m-1$ [Theorem 3.2.2, Corollary 3.2.3].

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2. THE SINGULARITY SUBBUNDLES

(2.1) Let M^m be a closed, connected C^∞ m-manifold. For $\xi^n \to M^m$, $\tau^p \to M^m$ real vector bundles over M^m of rank n and p, respectively, with $n > p$, let

$$\mathrm{Hom}(\xi, \tau) \to M^m$$

denote the vector bundle over M^m of rank $n \cdot p$ with

$$\mathrm{Hom}(\xi, \tau)_x = \text{fiber of } \mathrm{Hom}(\xi, \tau) \text{ over } x \in M^m = \{\text{R-linear maps } \phi : \xi_x \to \tau_x\}.$$

Set $S_i(\xi,\tau)$ equal to the submanifold (subbundle) of Hom(ξ,τ) consisting of those elements with rank equal to $p-i$, for $0 \le i \le p$. Similarly, for $\eta^n \to M^m$, $\omega^p \to M^m$ complex vector bundles over M^m, one may form the complex vector bundle

$$\mathrm{Hom}_{\mathbb{C}}(\eta, \omega) \to M^m$$

and then decompose this bundle into its complex singularity subbundles $S_i^{\mathbb{C}}(\eta,\omega)$. (For background on these first-order singularities, see [9], p. 372).

It is a consequence of the work of Borel-Haefliger ([13], p. 23-24; [4], p. 8-02) that for $0 \le i \le p$, the topological closure

$$\overline{S_i(\xi, \tau)} = S_i(\xi, \tau) \cup S_{i+1}(\xi, \tau) \cup \ldots \cup S_p(\xi, \tau),$$

carries a fundamental class in $H_t(\overline{S_i(\xi,\tau)}; \mathbb{Z}_2)$, singular homology with closed supports, where $t = \dim(S_i(\xi, \tau))$. Also, if M^m is orientable, the complex space $\overline{S_i^{\mathbb{C}}(\eta,\omega)}$ possesses such a class ($\mathbb{Z}$ coefficients). Further, according to [12], if the bundle $\mathrm{Hom}(\xi, \tau) \to M^m$ is orientable, $n-p = 2\alpha$, $j = 2r$, then $\overline{S_j(\xi,\tau)}$ has a fundamental class over $\mathbb{Z}$.

Let $[\overline{S_i(\xi,\tau)}]$ denote the image of the fundamental homology class of $\overline{S_i(\xi,\tau)}$ in $H_*(\mathrm{Hom}(\xi, \tau); \mathbb{Z}_2)$ under the inclusion homomorphism. Also, denote by $[\overline{S_i^{\mathbb{C}}(\eta,\omega)}]$ (resp. $[\overline{S_{2r}(\xi,\tau)}]$) the element of $H_*(\mathrm{Hom}_{\mathbb{C}}(\eta, \omega); \mathbb{Z})$ (resp. $H_*(\mathrm{Hom}(\xi,\tau); \mathbb{Z})$) which is the image of the fundamental class of $\overline{S_i^{\mathbb{C}}(\eta,\omega)}$ (resp. $\overline{S_{2r}(\xi,\tau)}$). Finally, define $\mathrm{P.D.}[\overline{S_i(\xi, \tau)}] \in H^*(\mathrm{Hom}(\xi, \tau); \mathbb{Z}_2)$ to be the image of $[\overline{S_i(\xi,\tau)}]$ under Poincare Duality. (Since Hom(ξ,τ) is a paracompact manifold, by [1], p. 20 and [14], Corollary 7, p. 341, one has that $\mathrm{P.D.}[\overline{S_i(\xi, \tau)}]$ is an ordinary singular cohomology class).

Similarly define $\underline{\text{P.D.}[\,S_i^{C}(\eta,\omega)\,]} \in H^*(\mathrm{Hom}_{C}(\eta,\omega);\mathbb{Z})$ and

$$\underline{\text{P.D.}[\overline{S_{2r}(\xi,\tau)}]} \in H^*(\mathrm{Hom}(\xi,\tau);\mathbb{Z})$$

when the corresponding fundamental homology classes exist.

(2.2) Let M^m be, as usual, a closed, connected C^∞ m-manifold and

$$\Pi\colon \mathrm{Hom}(\xi,\tau) \to M^m$$

the Hom bundle of rank $n \cdot p$ as in (2.1). To simplify notation, we make the following definition:

Definition 2.2.1. (1) Let $(\Pi^*)^{-1}\colon H^*(\mathrm{Hom}(\xi,\tau);\mathbb{Z}_2) \to H^*(M^m;\mathbb{Z}_2)$ be the inverse of the cohomology isomorphism induced by the bundle projection Π. Define

$$\underline{b_i(\xi,\tau)} \in H^{i(n-p+i)}(M^m;\mathbb{Z}_2), \qquad 0 \le i \le p,$$

by $b_i(\xi,\tau) = (\Pi^*)^{-1}\text{P.D.}\,[\overline{S_i(\xi,\tau)}]$. (2) Make the further assumption that M^m is orientable and denote by ξ^C, τ^C the complexification of the bundles ξ, τ respectively. Then, for $0 \le i \le p$, define $b_i^C(\xi,\tau) \in H^{2i(n-p+i)}(M^m;\mathbb{Z})$ by

$$b_i^C(\xi,\tau) = (\Pi_C^*)^{-1}\,\text{P.D.}\,[S_i^C(\xi^C,\tau^C)].$$

(Here $\Pi_C\colon \mathrm{Hom}_C(\xi^C,\tau^C) \to M^m$ is the projection). (3) Suppose that both M^m and the bundle $\Pi\colon \mathrm{Hom}(\xi,\tau) \to M^m$ are orientable. Then for $n-p = 2\alpha$, $j = 2r$, define $\underline{r_j(\xi,\tau)} \in H^{j(n-p+j)}(M^m;\mathbb{Z})$ to be $(\Pi^*)^{-1}\text{P.D.}\,[\overline{S_{2r}(\xi,\tau)}]$, $0 \le j \le p$.

The meaning of these cohomological invariants is demonstrated by the following result, which will be the main tool in defining obstructions for smooth immersions in the next section.

PROPOSITION 2.2.2. *Let* M^m, $\mathrm{Hom}(\xi,\tau) \to M^m$ *be as above. Then*

(1) *The classes* $b_i(\xi,\tau)$, $b_i^C(\xi,\tau)$ *and* $r_j(\xi,\tau)$ *are invariants of the isomorphism classes of* ξ^n *and* τ^p; *and*

(2) *Suppose there exists a bundle epimorphism* $\phi\colon \xi^n \to \tau^p$; *i.e., a vector bundle mapping* ϕ *such that for all* $x \in M^m$, $\phi_x\colon \xi_x \to \tau_x$ *is of maximal rank. Then, for* $1 \le i, j \le p$, *the above cohomology classes vanish.*

Remark. This Proposition is a generalization of Theorem 4.2 of [2].

Proof. We will prove the proposition for the classes $b_i(\xi,\tau)$ (the proof for the $r_j(\xi,\tau)$ being identical) and indicate briefly the additional statements needed to extend the proof to the complexified classes $b_i^C(\xi,\tau)$. (1) It suffices to show that if $\psi_1\colon \xi \to \xi'$, $\psi_2\colon \tau \to \tau'$ are vector bundle isomorphisms then there exists an isomorphism $\psi\colon \mathrm{Hom}(\xi,\tau) \to \mathrm{Hom}(\xi',\tau')$ and sections

$$\sigma \in \Gamma^\infty(\mathrm{Hom}(\xi,\tau)), \qquad \sigma' \in \Gamma^\infty(\mathrm{Hom}(\xi',\tau'))$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(\xi, \tau) & \xrightarrow{\psi} & \mathrm{Hom}(\xi', \tau') \\ & \nwarrow \sigma \quad \sigma' \nearrow & \\ & M^m & \end{array}$$

Indeed, if $\sigma: M^m \to \mathrm{Hom}(\xi, \tau)$ is any section one has that $\sigma^* \equiv (\Pi^*)^{-1}$ on cohomology. Then the equality $b_i(\xi, \tau) = b_i(\xi', \tau')$, $0 \le i \le p$, follows immediately noting that ψ of maximal rank on each fiber implies that $\psi(S_i(\xi, \tau)) \equiv S_i(\xi', \tau')$. To verify the existence of the above diagram, however, one need only observe that if

$$\sigma \in \Gamma^\infty(\mathrm{Hom}(\xi, \tau)),$$

$\sigma' = \psi \circ \sigma$ is the desired section of $\mathrm{Hom}(\xi', \tau')$, the isomorphism ψ being determined in the obvious manner by ψ_1 and ψ_2.

The proof of (2) is based on the following lemma:

LEMMA 1. *Suppose that* $\sigma \in \Gamma^\infty(\mathrm{Hom}(\xi, \tau))$ *is transversal to each of the submanifolds* $S_i(\xi, \tau)$. *Set* $S_i(\sigma) = \sigma^{-1}(S_i(\xi, \tau))$. *Then for* $0 \le i \le p$,

(1) $S_i(\sigma)$ *is a regular submanifold of* M^m *of codimension* $i(n-p+i)$.

(2) $$\overline{S_i(\sigma)} = \bigcup_{j=0}^{p-i} S_{i+j}(\sigma) = \sigma^{-1}(\overline{S_i(\xi, \tau)}).$$

(3) $\overline{S_i(\sigma)}$ *possesses a fundamental homology class (over* $\mathbb{Z}_2$*). Further, if* $[\overline{S_i(\sigma)}]$ *denotes the image of this class in* $H_*(M^m; \mathbb{Z}_2)$, *then*

$$\begin{aligned} b_i(\xi, \tau) &= (\Pi^*)^{-1} \mathrm{P.D.}\, [\overline{S_i(\xi, \tau)}] = \sigma^* \mathrm{P.D.}\, [\overline{S_i(\xi, \tau)}] \\ &= \mathrm{P.D.}\, [\overline{S_i(\sigma)}] \in H^{i(n-p+i)}(M^m; \mathbb{Z}_2). \end{aligned}$$

Proof. (1) follows from standard properties of $S_i(\xi, \tau)$ and the transversality of σ. (2) This may be found in ([9], p. 373). (3) By (2) and ([4], p. 8-02), it suffices to show that $\overline{S_i(\sigma)}$ is an ANR. Further, according to a theorem of S.T. Hu ([7], Theorem 7.1, p. 168) this is equivalent to showing that $\overline{S_i(\sigma)}$ is locally contractible, which one accomplishes as follows: By (1), the manifold topology of $S_i(\sigma)$ agrees with the relative topology induced from the inclusion of $S_i(\sigma)$ into M^m. Thus if $x \in \overline{S_i(\sigma)}$ and V is any open set in $\overline{S_i(\sigma)}$ containing x, V is of the form $V' \cap \overline{S_i(\sigma)}$ where V' is open in M^m. Since M^m is locally contractible, there exists an open set U', $x \in U' \subset V'$, with U' contractible to x in V'; *i.e.,* there is a continuous map $G': U' \times I \to V'$ with

$$G'_0 = \text{inclusion} \quad \text{and} \quad G'_1(U') = x.$$

Setting $U = U' \cap \overline{S_i(\sigma)}$ and $G = G' \mid U \times I$ then provides the desired contraction, proving the lemma.

Now, let $\phi: \xi^n \to \tau^p$ be a bundle epimorphism. Define $\sigma_\phi \in \Gamma^\infty(\mathrm{Hom}(\xi, \tau))$ by $\sigma_\phi(x) = \phi_x: \xi_x \to \tau_x$. Then $\mathrm{Image}(\sigma_\phi) \cap S_i(\xi, \tau) = \emptyset$, $1 \le i \le p$, by the definition of the S_i. As $S_0(\xi,\tau)$ is open in $\mathrm{Hom}(\xi,\tau)$, it follows that σ_Φ is transversal to each $S_i(\xi,\tau)$. Hence lemma 1-3 may be applied and one has

$$b_0(\xi,\tau) = \sigma_\Phi^* \mathrm{P.D.}[\overline{S_0(\xi,\tau)}] = \mathrm{P.D.}[\overline{S_0(\sigma_\Phi)}] = \mathrm{P.D.}[M^m]_2 = 1 \in H^0(M;\mathbb{Z}_2).$$

Further, for i>0, $b_i(\xi,\tau) = \sigma_\Phi^* \mathrm{P.D.}[\overline{S_i(\xi,\tau)}] = \mathrm{P.D.}[\overline{S_i(\sigma_\Phi)}] = \mathrm{P.D.}[\emptyset] = 0$, and so the proposition is proved for $b_i(\xi,\tau)$.

To complete the proof for the classes $b_i^c(\xi,\tau)$, let $\overline{\sigma_\Phi} \in \Gamma^\infty \mathrm{Hom}_c(\xi^c,\tau^c))$ be the complexification of σ_Φ. That is, if ξ^c (resp. τ^c) is the complex bundle with total space $\xi\oplus\xi$ (resp.$\tau\oplus\tau$) and complex multiplication $i \cdot (v, w) = (-w, v)$, then $\overline{\sigma_\Phi}$ is given by $\overline{\sigma_\Phi}(v, w) = (\sigma_\Phi(v), \sigma_\Phi(w))$. As it is trivial to show that for $x \in M^m$, real rank(σ_Φ) = complex rank$(\overline{\sigma_\Phi})$, the sets $S_i(\sigma_\Phi)$ and $S_i^c(\overline{\sigma_\Phi})$ will be identical in M^m and the proof for this case will follow exactly as in the real case.

(2.3) Let M^m be a closed, connected C^∞ m-manifold and N^n a connected C^∞ n-manifold, (not necessarily compact), with n > m.

Definition 2.3.1. (1) Let $f{:}M^m \to N^n$ be a continuous mapping. Define

$$\underline{b_i(f)} \in H^{i(n-m+i)}(M^m;\mathbb{Z}_2), \qquad 0 \le i \le m$$

by $b_i(f) = b_i(f^*TN, TM)$. Set $\underline{B(f)} = 1 \oplus b_1(f) \oplus \cdots \oplus b_m(f) \in H^*(M^m;\mathbb{Z}_2)$. (2) Assume that M^m is orientable. Then one defines $\underline{b_i^c(f)}$ to be

$$b_i^c(f^*TN^c, TM^c), \qquad 0 \le i \le m,$$

and $\underline{B^c(f)}$ to be $1 \oplus b_1^c(f) \oplus \cdots \oplus b_m^c(f) \in H^*(M;\mathbb{Z})$. 3) Assume that both M^m and N^n are orientable manifolds. Then for $n-m = 2\alpha$, $j = 2r$, $0 \le j \le m$, set

$$\underline{r_j(f)} = r_j(f^*TN, TM), \quad \text{and} \quad \underline{R(f)} = 1 \oplus \cdots \oplus r_m(f) \in H^*(M^m;\mathbb{Z}).$$

Using Proposition 2.2.2 one may now interpret these classes as obstructions to deforming f to a C^∞ map $g{:}\ M^m \to N^n$ such that the tangent map of g, T(g), avoids all the singularity subbundles of positive codimension.

THEOREM 2.3.2. *Let* $f{:}M^m \to N^n$ *be a continuous mapping. Then* (1) *Each of the total cohomology classes* B(f)), $B^c(f)$ *and* R(f) *is a homotopy invariant of* f. (2) *Suppose that* f *is homotopic to a smooth immersion* g. *Then*

$$B(f) = 1 \in H^0(M^m;\mathbb{Z}_2);$$

and $B^c(f) = R(f) = 1 \in H^0(M^m;\mathbb{Z})$.

Proof. (1) is an immediate consequence of Proposition 2.2.2-1, since

$$f \simeq g \Rightarrow f^*TN \cong g^*TN.$$

To see (2), note that if $g{:}\ M^m \to N^n$ is an immersion, T(g) defines a vector bundle embedding of TM into TN covering g or equivalently an isomorphism of TM onto a subbundle of g*TN. There is thus defined an isomorphism

$$\phi_1 : g^*TN \to TM \oplus \nu^{n-m},$$

where ν^{n-m} is the subbundle of g^*TN orthogonal to $T(g)(TM)$ via a Riemannian metric. Clearly the bundle mapping $\phi: g^*TN \to TM$ given by $\phi_x = (p_1)_x \circ (\phi_1)_x$, where $(p_1)_x$ is the linear projection $TM_x \oplus \nu_x \to TM_x$, is a bundle epimorphism. Hence Proposition 2.2.2-2 may be applied to yield the theorem.

3. APPLICATIONS

(3.1) Let M^m be, as usual, a closed, connected C^∞ m-manifold. In (3.1), as an application of the techniques of Section 2, we restrict our attention to maps $f: M^m \to \mathbb{C}P^n$ and study the problem of finding the minimal dimension n for which every such f deforms to a C^∞ immersion (resp. M^m immerses in $\mathbb{C}P^n$). Our results are summarized in the following theorem:

THEOREM 3.1.1. *Let* M^m *be as above. Then*

(A) *If* M^m *is 2k-connected with* $2 \leq 2k < m/2$, *every map* $f: M^m \to \mathbb{C}P^{m-k}$ *is homotopic to a* C^∞ *embedding.*

(B) *There exists a map* $f: M^m \to \mathbb{C}P^{m-1}$ *not homotopic to an immersion if either of the following two conditions holds:*

(1) $\overline{W_{m-1}}(M^m) \neq 0$; *or*

(2) $\overline{W_{m-1}}(M^m) = 0$ *and there is a class* $u \in H^2(M;\mathbb{Z})$ *such that if* $u' = \rho(u)$,

$$(\rho: H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2) \textit{ reduction} \bmod 2),$$

then $\sum_{k=1}^{[(m-1)/2]} (u')^k \smile \binom{m}{k}_2 \overline{W_{m-2k-1}}(M^m)$ *is non-zero. (Here* $\binom{m}{k}_2$ is the mod 2 *binomial coefficient). Moreover, in case* (1) *if in addition either*

$$H^2(M;\mathbb{Z}) = 0$$

or $m = 2^q$, *then* M^m *does not immerse in* $\mathbb{C}P^{m-1}$. (C) *Suppose that* M^m *is 2-connected,* $m \geq 7$, *and so embeds in* $\mathbb{C}P^{m-1}$ *by* (A). *Then if for some integer* n_0 *with* $4 \leq (m+1)/2 \leq n_0 \leq (3m-4)/4$, *the class*

$$(\overline{W_{2n_0-m+2}}^2 - \overline{W_{2n_0-m+1}} \smile \overline{W_{2n_0-m+3}})(M^m) \neq 0 \in H^{2(2n_0-m+2)}(M^m;\mathbb{Z}_2),$$

M^m *does not immerse in* $\mathbb{C}P^{n_0}$.

Before supplying the proof of Theorem 3.1.1, we present examples to illustrate (B) and (C) above.

Example 3.1.2. (B)-(1). Set $M^m = \mathbb{R}P^m, m = 2^q$. Then by ([8], p. 137), $\overline{W_{m-1}}\,\mathbb{R}P^m) \neq 0$, and so $\mathbb{R}P^{2^q}$ does not immerse in $\mathbb{C}P^{2^q-1}$. (B)-(2). Let M^3 be a closed, connected orientable 3-manifold. Then since $W(M) = 1$, if some $u \in H^2(M;\mathbb{Z})$ satisfies $\rho(u) \neq 0 \in H^2(M;\mathbb{Z}_2)$, there is an element of $[M^3, \mathbb{C}P^2]$ which does not contain an immersion. (Note that since M^3 is parallelizable, by [5], Theorem 5.7, it is immersible in $\mathbb{C}P^2$). This condition is satisfied, for example, by the generator of $H^2(\mathbb{R}P^3;\mathbb{Z})$. (C). To illustrate the final condition, set $M^m = \mathbb{H}P^{2s}$, even-dimensional quaternionic projective space. $\mathbb{H}P^{2s}$ is a 3-connected 8s-manifold. Further, by ([10], Theorem 8, p. 56), $(\overline{W_4}^2 - \overline{W_5} \smile \overline{W_3})(\mathbb{H}P^{2s})$ is the generator of $H^8(\mathbb{H}P^{2s};\mathbb{Z}_2)$ and so $\mathbb{H}P^{2s}$ does not immerse in $\mathbb{C}P^{4s+1}$. When $s = 1$, since by part (A) $\mathbb{H}P^2$ immerses in $\mathbb{C}P^7$, only the case $n = 6$ is open.

Proof of the theorem. We begin with the following lemma:

LEMMA 3.1.3. *Let* M^m *be as above,* n *a positive integer with* $2n > m$, *and* α_n *the canonical generator of* $H^2(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$. *Then there is a 1-1 correspondence between* $H^2(M;\mathbb{Z})$ *and the set of homotopy classes* $[M^m, \mathbb{C}P^n]$ *given by* $u \mapsto [f]$ *with* $f^*(\alpha_n) = u$.

Proof. The lemma is an immediate consequence of (1) $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$ and (2) the cellular approximation theorem.

Notation. For $u \in H^2(M;\mathbb{Z})$ denote by β_u^n the element of $[M^m, \mathbb{C}P^n]$ assigned to it by Lemma 3.1.3.

Proof of (A). As M^m is at least 2-connected, $H^2(M;\mathbb{Z}) = 0$. Thus by the lemma any two maps of M^m into $\mathbb{C}P^n$, $2n > m$, are homotopic, and thus it suffices to exhibit a single embedding $f\colon M^m \subset \mathbb{C}P^{m-k}$. But by [3], the 2k-connectivity of M^m implies that M^m embeds in $\mathbb{R}^{2m-2k}$ and so composing this embedding with any diffeomorphism of $\mathbb{R}^{2m-2k}$ onto a small coordinate chart in $\mathbb{C}P^{m-k}$ yields (A). B)-(1): Let $f\colon M^m \to \mathbb{C}P^{m-1}$ be a continuous map. Then according to Porteous ([11], Proposition 1.3, p. 298)

$$b_1(f) = W_{2m-2-m+1}(f^*T\mathbb{C}P^{m-1} - TM^m) \text{ (the first Thom polynomial)}$$

$$= \sum_{i+j=m-1} f^*(W_i(\mathbb{C}P^{m-1})) \smile \overline{W_m}(M^m) \in H^{m-1}(M;\mathbb{Z}_2).$$

Suppose that $f \in \beta_o^{m-1}$; *i.e.,* f is homotopic to the constant map. Then

$$f^*(W(\mathbb{C}P^{m-1})) = 1 \in H^0(M;\mathbb{Z}_2)$$

and so $b_1(f)$ reduces to $\overline{W_{m-1}}(M^m)$. Hence if $\overline{W_{m-1}}(M^m) \neq 0$, Theorem 2.3.2 implies that β_0^{m-1} does not contain an immersion. Further, if $H^2(M;\mathbb{Z}) = 0$, any map deforms to the constant map and so M^m does not immerse in $\mathbb{C}P^{m-1}$. Finally, if $m = 2^q$, the previously cited formula

$$W_{2i}(\mathbb{C}P^n) = \binom{n+1}{i}_2 (\alpha_n')^i, \qquad \alpha_n' = \rho(\alpha_n),$$

yields $W(\mathbb{C}P^{2^q-1}) = 1$ and so, as before, $\overline{W_{m-1}}(M^m)$ is the obstruction to deforming *any* map $f\colon M^m \to \mathbb{C}P^{m-1}$ to an immersion. (B)-(2): Let $u \in H^2(M;\mathbb{Z})$ and $f \in \beta_u^{m-1}$. Since reduction mod 2 commutes with the homomorphism induced by a continuous map, one has that $u' = f^*(\alpha_n')$. Since it is assumed that $\overline{W_{m-1}}(M^m)$ vanishes, the first singularity obstruction

$$b_1(f) = \overline{W_{m-3}}(M^m) \smile f^*(W_2(\mathbb{C}P^{m-1})) + \overline{W_{m-5}}(M^m) \smile f^*(W_4(\mathbb{C}P^{m-1})) + \cdots$$

which, by the above formula for $W(\mathbb{C}P^n)$, proves (B)-(2). (C): Let $f{:}M^m \to \mathbb{C}P^{n_0}$ be any continuous map. Then for $2(2n_0 - m + 2) \leq m$; *i.e.*, $n_0 \leq (3/4)m - 1$, $b_2(f)$ is defined and represents a possible nontrivial obstruction to deforming f to an immersion. However, since M^m is assumed 2-connected, one has as above that $b_2(f)$ depends only on M^m and, in fact, is given by

$$(\overline{W_{2n_0-m+2}}^2 - \overline{W_{2n_0-m+1}} \smile \overline{W_{2n_0-m+3}})(M^m).$$

Thus (C) follows directly from Theorem 2.3.2 and the proof is complete.

(3.2) A. Holme has shown [6] that if V^m is a non-singular projective variety over $\mathbb{C}$ embedded in $\mathbb{C}P^N$ via i: $V^m \to \mathbb{C}P^N$, then the least integer $n(V^m, i)$ such that V^m can be embedded in $\mathbb{C}P^n$ via a linear projection of $\mathbb{C}P^N \to \mathbb{C}P^n$, may be computed effectively in terms of the degrees of the Chern classes of V^m. As such, it is reasonable to expect $c(V^m)$ to also play a large role in determining $n(V^m)$, the least integer for which V^m embeds holomorphically in $\mathbb{C}P^n$ (by any means). More generally, for V^m, W^n closed, connected complex manifolds and f: $V^m \to W^n$ a continuous mapping, we will realize the Thom polynomials in the Chern classes of $(f^*T_{\mathbb{C}}W - T_{\mathbb{C}}V)$ as obstructions to deforming f to a holomorphic immersion. Specializing to $W^n = \mathbb{C}P^n$ will then give necessary conditions for

$$n(V^m) \leq 2m - 1.$$

Definition 3.2.1. Let $f{:}V^m \to W^n$ be as above and let $T_{\mathbb{C}}V$ (resp. $T_{\mathbb{C}}W$) be the complex tangent bundle of V^m (resp. W^n). By a slight abuse of notation, set

$$\underline{b_i^{\mathbb{C}}(f)} = \det\,(c_{n-m+i-s+t}(f^*T_{\mathbb{C}}W - T_{\mathbb{C}}V))_{s,t=1,\ldots,i}, \qquad \text{for } 0 \leq i \leq m,$$

and set $\underline{B^{\mathbb{C}}(f)} = 1 \oplus b_1^{\mathbb{C}}(f) \oplus \cdots \oplus b_m^{\mathbb{C}}(f) \in H^*(V;\mathbb{Z})$.

The analogue of Theorem 2.3.2 is then

THEOREM 3.2.2. (1) $B^{\mathbb{C}}(f)$ *is a homotopy invariant of* f; *and* (2) *Suppose that* f *is homotopic to a holomorphic immersion* g: $V^m \to W^n$. *Then* $B^{\mathbb{C}}(f) = 1 \in H^0(V;\mathbb{Z})$.

Proof. The techniques used to prove Theorem 2.3.2 carry over to this situation immediately after using [11] to identify $b_i^{\mathbb{C}}(f)$ above with

$$(\Pi^*)^{-1}\,\mathrm{P.D.}[\,\overline{S_i^{\mathbb{C}}(f^*T_{\mathbb{C}}W, T_{\mathbb{C}}V)}\,].$$

In particular, then, for $W^n = \mathbb{C}P^n$ and $b_1^{\mathbb{C}}(f)$ this yields

COROLLARY 3.2.3. *Let* $u \in H^2(V;\mathbb{Z})$ *and let* f: $V^m \to \mathbb{C}P^n$ *be an element of* β_u^n. *Then if* $\sum_{i+j=n-m+1} \binom{n+1}{i} u^i \smile \overline{c_j}(V^m) \neq 0$, f *does not deform to a holomorphic immersion.*

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