BOUNDEDNESS AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A VOLTERRA EQUATION

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1. INTRODUCTION

We investigate the boundedness and the asymptotic behavior of the solutions of the nonlinear Volterra integral equation

$$x'(t) + \int_{[0,t]} g(x(t - s))\, d\mu(s) = f(t) \qquad (t \in R^+) \tag{1.1}$$

(we write R^+ for the interval $[0, \infty)$). Here, g and f are given real functions, and μ is a given real (Radon) measure. A solution x is required to be locally absolutely continuous and satisfy (1.1) a.e. on R^+.

We suppose throughout that

$$\mu \textit{ is positive definite,} \tag{1.2}$$

$$g \in C(R), \quad \inf_{\xi \in R} G(\xi) > -\infty, \quad \textit{where } G(\xi) = \int_0^{\xi} g(\eta)\, d\eta. \tag{1.3}$$

If, in addition, one has

$$f \in L^1(R^+), \tag{1.4}$$

$$\limsup_{|x| \to \infty} |g(x)|\, (1 + |G(x)|)^{-1} < \infty, \tag{1.5}$$

then it is easy to show that every solution x of (1.1) satisfies

$$\sup_{T \in R^+} G(x(T)) + Q(g \circ x, T, \mu) < \infty, \tag{1.6}$$

where $Q(\phi, T, \mu) = \int_0^T \phi(t) \int_{[0,t]} \phi(t - s)\, d\mu(s)\, dt \quad (\phi \in C[0, T])$.

The proof of this claim proceeds as follows: one multiplies (1.1) by $g(x(t))$ and integrates over $[0, T]$, getting

$$G(x(T)) + Q(g \circ x, T, \mu) = G(x(0)) + \int_0^T g(x(t))\, f(t)\, dt. \tag{1.7}$$

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By (1.2), $Q(g \circ x, T, \mu) \geq 0$. Thus $G(x(T)) \leq G(x(0)) + \int_0^T g(x(t))\, f(t)\, dt$. This together with (1.3) through (1.5) and Gronwall's inequality gives $g \circ x \in L^\infty(R)$, and (1.6) then follows from (1.4), (1.7).

The conclusion (1.6) has very important consequences. Because of (1.2) and (1.3), both terms in (1.6) are bounded from below. Thus, if (1.6) holds, then also

(1.8) $$\sup_{T \in R^+} G(x(T)) < \infty,$$

(1.9) $$\sup_{T \in R^+} Q(g \circ x, T, \mu) < \infty.$$

Of course, if moreover

(1.10) $$\limsup_{\xi \to \pm\infty} G(\xi) = \infty,$$

then it follows from (1.8) that x is bounded; hence, so is $g \circ x$. (Actually, the boundedness of $g \circ x$ follows directly from (1.5) and (1.8) without (1.10), but in the sequel we do not suppose (1.5), and then (1.10) is needed.) The fact that $g \circ x$ is a bounded function satisfying (1.9) can then be used to draw quite detailed conclusions about the asymptotic behavior of $g \circ x$ (see [18]).

We want to find alternative conditions on g, f which imply that every solution x of (1.1) satisfies (1.6). These should not include (1.5), and they should permit a larger asymptotic size of f than (1.4) does. Part of the motivation comes not from (1.1) but from a generalization of (1.1) to a Hilbert space, where (1.5) is quite unrealistic, at least in its most immediate interpretation (see however [21] for a more realistic version). For simplicity, we limit this study to (1.1), but the results that we get are formulated in a way which makes the generalization to an equation in a Hilbert space quite easy.

The basic idea is simple. Instead of dropping $Q(g \circ x, T, \mu)$ in (1.7), we use (1.3) to get

(1.11) $$Q(g \circ x, T, \mu) \leq K + \int_0^T g(x(t))\, f(t)\, dt,$$

where K is a fixed constant. Note that $Q(g \circ x, T, \mu)$ is quadratic in $g \circ x$, whereas $\int_0^T g(x(t))\, f(t)\, dt$ depends linearly on $g \circ x$. Suppose for the moment that μ dominates f in the following sense:

(1.12) *There exists* $\alpha \in R^+$ *such that* $\left| \int_0^T \phi(t)\, f(t)\, dt \right|^2 \leq \alpha Q(\phi, T, \mu)$ *for every* $T \in R^+$, $\phi \in C[0, T]$.

Then it clearly follows from (1.7) and (1.11) that (1.6) holds.

Summarizing the preceding argument, we have

THEOREM 1.1. *Let* (1.2), (1.3) *and* (1.12) *hold. Then every solution* x *of* (1.1) *satisfies* (1.6). *If in addition* (1.10) *holds, then the solutions of* (1.1) *are bounded.*

When the kernel includes a constant term, then an additional perturbation, which need not satisfy (1.12), can be allowed. We discuss this in Section 2.

Condition (1.12) as well as the preceding proof of Theorem 1.1 is partly motivated by a result due to Levin and Nohel [6]. They study (1.1) when $d\mu(t) = a(t)\,dt$ $(t \in R^+)$, and a, f satisfy

$$a \in C[0, \infty),\ (-1)^k a^{(k)}(t) \geq 0\ \ (0 < t < \infty;\ k = 0, 1, 2),\ f \in C[0, \infty) \cap C^1(0, \infty);$$

(1.13) *there exists* $c \in C[0, \infty) \cap C^1(0, \infty)$ *such that* $f^2(t) \leq a(t)c(t)$ *and*

$$(f'(t))^2 \leq a'(t)c'(t) \quad (0 < t < \infty)$$

(they also include another perturbation in (1.1) satisfying (1.4)). The original proof of [6, Theorem 1] does not involve (1.12), but a later proof due to MacCamy and Wong implicitly contains (1.12) (see [11, pp. 28-29] and Section 7 below).

The condition (1.13) has the big disadvantage that it can be applied only when the kernel is a convex function. In addition, it has the unpleasant feature of being quite implicit, and given a, f it is not always easy to construct the function c in (1.13).

When one only has μ positive definite but not convex, then it is much more natural to work with Fourier transform conditions on μ, f, and not with pointwise conditions like (1.13). The following formal computation is made precise in Section 3. Write $Q(\phi, T, \mu)$ in the equivalent Fourier transform form (hats denote Fourier transforms)

$$Q(\phi, T, \mu) = \frac{1}{2\pi} \int_R |\hat{\phi}_T(\omega)|^2\, \Re\hat{\mu}(\omega)\, d\omega,$$

where $\phi_T = \chi_{[0,T]}\phi$, and $\chi_{[0,T]}$ is the characteristic function of the interval $[0, T]$ (here we suppose for simplicity that $\Re\hat{\mu}$ is a function). Also rewrite the integral on the left hand side of (1.12) using Fourier transforms:

$$\int_0^T \phi(t) f(t)\, dt = \frac{1}{2\pi} \int_R \overline{\hat{\phi}_T(\omega)}\, \hat{f}(\omega)\, d\omega. \tag{1.14}$$

Multiply and divide by $\Re\hat{\mu}$ (which is positive because μ is positive definite), and use the Schwarz inequality to get

$$\left|\int_0^T \phi(t) f(t)\, dt\right|^2 \leq (2\pi)^{-2}\left[\int_R |\hat{f}(\omega)|^2 (\Re\hat{\mu}(\omega))^{-1}\, d\omega\right]\left[\int_R |\hat{\phi}_T(\omega)|^2\, \Re\hat{\mu}(\omega)\, d\omega\right].$$

Thus we observe that (1.12) is implied by

$$\hat{f}(\omega)(\Re\hat{\mu}(\omega))^{-1/2} \in L^2(R). \tag{1.15}$$

The main part of this paper is devoted to a study of (1.15).

Some comparisons with earlier results are found in Sections 4 to 6.

2. A KERNEL WITH A CONSTANT TERM

When μ contains a constant term, then one can perturb (1.1) by an additional function of bounded variation:

THEOREM 2.1. *Let* (1.3) *hold. Let* μ, f *be of the form* $d\mu(t) = d\mu_1(t) + \beta\, dt$, $f(t) = f_1(t) + f_2(t)$ $(t \in R^+)$, *where* $\beta > 0$, μ_1 *and* f_1 *satisfy* (1.2), (1.12) *with* μ, f *replaced by* μ_1, f_1, *and* f_2 *is of bounded variation. Then every solution* x *of* (1.1) *satisfies* (1.6). *If moreover* (1.10) *holds, then the solutions of* (1.1) *are bounded.*

The proof of Theorem 2.1 given below is a straightforward modification of an argument in [3].

Proof. We multiply (1.1) by $g(x(t))$ and integrate over $[0, T]$, getting (1.7). Note that

$$Q(g \circ x, T, \mu) = Q(g \circ x, T, \mu_1) + \frac{\beta}{2}\left[\int_0^T g(x(t))\, dt\right]^2 .$$

Substitute this and (1.12) (with μ, f replaced by μ_1, f_1) into (1.7), and use (1.3) and the positive definiteness of μ_1 to obtain

$$\text{(2.1)} \qquad \frac{\beta}{2}\left[\int_0^T g(x(t))\, dt\right]^2 \leq K_1 + \int_0^T g(x(t))\, f_2(t)\, dt ,$$

where $K_1 = G(x(0)) - \inf_{\xi \in R} G(\xi) + \alpha/4$. Redefine f_2 if necessary so that it is continuous from the right, and integrate by parts in the right hand side of (2.1) to get

$$\int_0^T g(x(t))\, f_2(t)\, dt = f_2(T) \int_0^T g(x(s))\, ds - \int_{[0,T]} \int_0^t g(x(s))\, ds\, df_2(t) .$$

Define $\Phi(T) = \sup_{t \in [0,T]} \left|\int_0^t g(x(s))\, ds\right|$. Then clearly $\int_0^T g(x(t))\, f_2(t)\, dt \leq K_2\, \Phi(T)$, where K_2 is the sum of $\sup_{t \in R^+} |f_2(t)|$ and the total variation of f_2 on R^+. Substitute this into (2.1), and use the fact that the right hand side becomes nondecreasing to obtain $\beta(\Phi(T))^2/2 \leq K_1 + K_2\, \Phi(T)$. Thus, $\sup_{T \in R^+} \Phi(T) < \infty$, so also

$$\text{(2.2)} \qquad \sup_{T \in R^+} \left|\int_0^T g(x(t))\, f_2(t)\, dt\right| < \infty.$$

To complete the proof of Theorem 2.1, one substitutes (2.2) into (1.11), and continues as in the proof of Theorem 1.1.

3. A FOURIER TRANSFORM CONDITION

In this section, we explain how (1.15) should be interpreted if $\Re\hat{\mu}$ is not a function and prove that (under an additional technical assumption) (1.15) implies (1.12) (Theorem 3.1 below).

We begin with some notations and conventions which remain valid throughout the paper.

Hats denote Fourier transforms. If u is a tempered distribution, then so is $\hat{u}$, the distribution Fourier transform of u (*cf.* [13]). If a function a and a measure μ induce tempered distributions, then $\hat{a}$ and $\hat{\mu}$ are defined to be the (distribution) Fourier transforms of the corresponding distributions. It is well known that, *e.g.*, when $a \in L^1(R)$, then the distribution $\hat{a}$ is (induced by) a continuous function, which we also denote $\hat{a}$. We normalize the Fourier transform so that in the preceding case, $\hat{a}(\omega) = \int_R e^{-i\omega t} a(t)\, dt$ $(\omega \in R)$.

Another well-known fact is that the Fourier transform maps $L^2(R)$ onto itself.

Unless a specific statement is made to the contrary, functions and measures defined on some interval I are extended to R by zero outside I, and their Fourier transforms are by definition the transforms of the corresponding extensions (this, of course, only makes sense if the extensions induce tempered distributions).

The characteristic function of an interval I is denoted χ_I .

All functions and measures are real-valued, except those that are defined to be the Fourier transforms of other functions and measures (or, to say it in another way, functions and measures on the time domain are real-valued, whereas functions and measures on the frequency domain are complex-valued).

It is known that if μ is positive definite, then the distribution induced by μ is tempered, and $\Re\hat{\mu}$ is (induced by) a positive measure [17, Corollary 1.1]. Moreover, one has

$$Q(\phi, T, \mu) = \frac{1}{2\pi} \int_R |\hat{\phi}_T(\omega)|^2 \, d\{\Re\hat{\mu}\}(\omega),$$

where $\phi_T = \chi_{[0,T]}\phi$ (see [17, Lemma 1.1] and [18, Lemma 4.1]). Thus, in particular,

$$Q(\phi, T, \mu) \geq \frac{1}{2\pi} \int_R |\hat{\phi}_T(\omega)|^2 \lambda(\omega)\, d\omega, \tag{3.1}$$

where λ is (the Radon-Nikodym derivative of) the absolutely continuous part (with respect to the Lebesgue measure) of the measure $\Re\hat{\mu}$.

THEOREM 3.1. *Let* μ *be positive definite. Let* f *be the inverse Fourier transform of a function* $\hat{f} \in L^1(R) + L^2(R)$, *and suppose that*

$$\int_R |\hat{f}(\omega)|^2 (\lambda(\omega))^{-1}\, d\omega < \infty, \tag{3.2}$$

where λ *is the absolutely continuous part of* $\Re\hat{\mu}$. *Then* μ *dominates* f *in the sense of* (1.12).

Henceforth, when we say that μ dominates f, then we mean that μ, f satisfy (1.12).

It follows from the hypothesis on f that $f \in L^2(R) + L^\infty(R)$, and that the given function $\hat{f}$ actually is the (distribution) Fourier transform of f. An equivalent condition is to ask f to induce a tempered distribution (so that the Fourier transform $\hat{f}$ exists) together with the earlier $\hat{f} \in L^1(R) + L^2(R)$. Note that f need not vanish on $(-\infty, 0)$.

Proof of Theorem 3.1. By (3.1) and (3.2), the only step in the formal proof of Theorem 3.1 given in Section 1 which is still questionable is (1.14).

By the hypothesis, one can write $\hat{f}$ in the form $\hat{f} = r_1 + r_2$, where $r_1 \in L^1(R)$, $r_2 \in L^2(R)$. Let f_1 and f_2 be the inverse Fourier transforms of r_1 and r_2, respectively. Then $f = f_1 + f_2$, and to verify (1.14) it clearly suffices to show that

$$\int_R \overline{\phi_T(t)}\, f_i(t)\, dt = \frac{1}{2\pi} \int_R \overline{\hat{\phi}_T(\omega)}\, r_i(\omega)\, d\omega \quad (i = 1, 2) \tag{3.3}$$

($\phi_T = \overline{\phi_T}$, because ϕ is real). When i = 1, then (3.3) is an immediate consequence of Fubini's theorem (note that $r_1, \phi_T \in L^1(R)$, and $f_1, \hat{\phi}_T \in L^\infty(R)$). When i = 2, then (3.3) is a special case of Parseval's identity. Thus (1.14) holds, and the proof is complete.

4. A "STRONGLY" POSITIVE DEFINITE KERNEL

As a first application of Theorem 3.1, we study the case when μ is "strongly" positive definite in the following sense (let n be a nonnegative integer, and recall that $\Re\hat{\mu}$ is a positive measure if μ is positive definite):

(4.1) *There exists* $\varepsilon > 0$ *such that the measure* $d(\Re\hat{\mu})(\omega) - \varepsilon(1+\omega^2)^{-n}\, d\omega$ $(\omega \in R)$ *is positive.*

Taking n = 1 in (4.1), one gets MacCamy's and Wong's [11] strong positivity.

PROPOSITION 4.1. *Let* μ *be a positive definite measure satisfying* (4.1) *for some* $n \in \{0, 1, 2, \cdots\}$, *and let*

$$f,\ f^{(n)} \in L^2(R^+), \tag{4.2}$$

where $f^{(n)}$ *is the* n:*th* (*distribution*) *derivative of* f *on* $(0, \infty)$. *Then* μ *dominates* f.

The two most interesting cases are n = 0 and n = 1. We discuss these after the proof of Proposition 4.1.

Proof of Proposition 4.1. The first step in the proof consists of extending f to R in an appropriate way. If n = 0, then we use our standard extension f(t) = 0 (t < 0), and get (4.3) below. If $n \geq 1$, then it follows from (4.2) that f (redefined on a set of measure zero) is n - 1 times continuously differentiable on R^+. In particular, the right-derivatives $f^{(i)}(0)$ (i = 0, ⋯, n - 1) exist. Let η be an arbitrary C^∞-function with compact support such that $\eta^{(i)}(0) = f^{(i)}(0)$ (i = 0, ⋯, n - 1), and define $f(t) = \eta(t)$ (t < 0). Then it follows from the construction that

$$f,\ f^{(n)} \in L^2(R), \tag{4.3}$$

where $f^{(n)}$ is the n:th (distribution) derivative of f on R (and not only on $(0, \infty)$).

One can now use (4.3) and standard properties of the Fourier transform to conclude that $\int_R |\hat{f}(\omega)|^2\, d\omega < \infty$, and $\int_R |\omega^n \hat{f}(\omega)|^2\, d\omega < \infty$. Clearly then

$$\int_R (1 + \omega^2)^n\, |\hat{f}(\omega)|^2\, d\omega < \infty. \tag{4.4}$$

Let λ be the absolutely continuous part of $\Re\,\hat{\mu}$. Then, by (4.1), $\lambda(\omega) \geq \varepsilon(1 + \omega^2)^{-n}$ a.e. on R, and together with (4.4) this yields (3.2). Thus Theorem 3.1 applies, and the proof of Proposition 4.1 is complete.

By Theorem 1.1 and Proposition 4.1, if μ, f satisfy the hypothesis of Proposition 4.1, g satisfies (1.3), and x is a solution of (1.1), then (1.6) holds. If moreover (1.10) is true, then

$$x \in L^\infty(R^+). \tag{4.5}$$

In the special case $n = 0$, the estimates that we obtain are very similar to those used, *e.g.*, by Barbu in [2, Section 4]. The assumption (4.2) simply becomes $f \in L^2(R^+)$. Condition (4.1) with $n = 0$ implies $Q(\phi, T, \mu) \geq \varepsilon \int_0^T |\phi(t)|^2\, dt$ (this follows from (3.1) and Parseval's identity). Thus (1.9) yields

$$g \circ x \in L^2(R^+). \tag{4.6}$$

Now, suppose in addition that

$$\inf_{0<|\xi|\leq y} g(\xi)/\xi > 0 \quad \textit{for each } y \in R^+. \tag{4.7}$$

Then it follows from (4.5) and (4.6) that

$$x \in L^2(R^+). \tag{4.8}$$

If the kernel μ has a finite total variation, then one can in addition get a pointwise convergence to zero of x(t) and g(x(t)), because in this case (1.1), (4.2), (4.6) and a standard theorem on convolutions imply $x' \in L^2(R^+)$. In particular, x is uniformly continuous. Combining this with (1.3), (4.7) (which imply $g(0) = 0$) and (4.8), one has

$$x(t) \to 0, \quad g(x(t)) \to 0 \quad (t \to \infty). \tag{4.9}$$

Remark 4.2. Using the continuity of g, one can easily show that (4.7) is equivalent to

$$\xi g(\xi) > 0 \quad (\xi \neq 0), \qquad \liminf_{\xi \to 0} g(\xi)/\xi > 0.$$

The case $n = 1$ in Proposition 4.1 enables us to improve a result due to MacCamy [10, Theorem I(i)]. For simplicity we only discuss Theorem A below, which is a scalar analogue of [10, Theorem I(i)], and leave the obvious generalization to MacCamy's vector-valued equation to the reader.

THEOREM A. (i) *Let* (1.3) *hold. Let* $d\mu(t) = a(t)\,dt$ $(t \in R^+)$, *where*

$$a \in C^2(R^+), \qquad t^3 a^{(k)}(t) \in L^1(R^+) \quad (k = 0, 1, 2), \tag{4.10}$$

and suppose that (4.1), (4.2) *hold with* $n = 1$. *Then every solution* x *of* (1.1) *satisfies*

$$x' \in L^2 \cap L^\infty(R^+). \tag{4.11}$$

(ii) *If in addition* (1.10), (4.7) *hold, then so do* (4.5), (4.8) *and* (4.9).

(The statement of Theorem A differs substantially from the statement of [10, Theorem I(i)]. MacCamy writes his assumption on a in a different but equivalent way. The hypotheses on f and g in Theorem A are weaker than the corresponding ones in [10]. However, a careful examination of [10, Section 4] shows that MacCamy's proof applies under the hypothesis of Theorem A. The conclusions (4.5), (4.8), and (4.11) are given not in [10, Theorem I(i)] but in [10, Lemma 4.1]. In the Hilbert space version of Theorem A one should drop the claim $g(x(t)) \to 0$ $(t \to \infty)$.)

Proposition 4.1 enables us to weaken the assumption on a in Theorem A:

THEOREM 4.3. *Theorem* A *is true if* (4.10) *is replaced by*

$$a \in L^1(R^+), \qquad a' \in L^1 \cap BV(R^+). \tag{4.12}$$

Here, BV stands for the set of functions of bounded variation, and a' is the (distribution) derivative of a on $(0, \infty)$.

Proof of Theorem 4.3. (i) By Proposition 4.1 and Theorem 1.1, every solution x of (1.1) satisfies (1.6), hence also (1.8), (1.9). It follows from (1.9) and two inequalities due to Staffans (see [17, Lemma 6.1] and [19, Lemma 1 and Theorem 2(ii)]) that $a * g \circ x \in L^2 \cap L^\infty(R^+)$, where

$$a * g \circ x(t) = \int_0^t g(x(t-s))\,a(s)\,ds \quad (t \in R^+).$$

By the hypothesis, $f \in L^2 \cap L^\infty(R^+)$ also ($f' \in L^2(R^+)$ implies that f is (a.e. equal to) a uniformly continuous function, and this together with $f \in L^2(R^+)$ yields $f \in L^\infty(R^+)$). Thus by (1.1), $x' \in L^2 \cap L^\infty(R^+)$, and this completes the proof of (i).

(ii) The first claim (4.5) clearly follows from (1.8), (1.10).

If necessary, redefine f on a set of measure zero so that it becomes absolutely continuous, and (1.1) holds everywhere on R^+. Write (1.1) in the form

$$x'(t) + \int_0^t a(t-s)\,g(x(s))\,ds = f(t),$$

and differentiate:

$$x''(t) + a(0)\,g(x(t)) + v(t) = f'(t), \tag{4.13}$$

where $v(t) = \int_0^t a'(t-s)\,g(x(s))\,ds$ $(t \in R^+)$. We claim that

$$v \in L^2(R^+). \tag{4.14}$$

Define $\phi_T = \chi_{[0,T]}\, g \circ x$, and compute

$$\int_0^T |v(t)|^2\,dt = \int_0^T \left| \int_0^t a'(t-s)\,\phi_T(s)\,ds \right|^2 dt$$

$$\leq \int_{R^+} \left| \int_0^t a'(t-s)\,\phi_T(s)\,ds \right|^2 dt = \frac{1}{2\pi} \int_R |\hat{\phi}_T(\omega)\,(a')^{\wedge}(\omega)|^2\,d\omega .$$

It follows from (4.12) that $|(a')^{\wedge}(\omega)|^2 \leq \gamma(1+\omega^2)^{-1}$ $(\omega \in R)$ for some positive constant γ. Combining this with (3.1) and (4.1) (with $n = 1$), we get

$$\int_0^T |v(t)|^2\,dt \leq \frac{\gamma}{\varepsilon}\, Q(g \circ x, T, \mu),$$

and (4.14) is a consequence of (1.9).

Multiply (4.13) by $x(t)$, and integrate over $[0, T]$. This yields

$$a(0) \int_0^T x(t)\,g(x(t))\,dt = x(0)\,x'(0) - x(T)\,x'(T) + \int_0^T (x'(t))^2\,dt + \int_0^T x(t)\,(f'(t) - v(t))\,dt .$$

Use (4.2), (4.5), (4.11), (4.14) and the Schwarz inequality to conclude that

$$a(0) \int_0^T x(t)\,g(x(t))\,dt \leq K_1 + K_2 \left[\int_0^T |x(s)|^2\,ds \right]^{1/2}, \tag{4.15}$$

where K_1 and K_2 are constants independent of T. By (4.1) and the fact that $a(0) = \left[\int_R \Re\,\hat{a}(\omega)\,d\omega \right]/\pi$ (*cf.* [17, line (6.12)]), $a(0) > 0$. Together with (4.5), (4.7), and (4.15), this implies (4.8).

The final claim (4.9) follows from (4.8), the uniform continuity of x, the continuity of g, and the fact that $g(0) = 0$.

5. A FINITE NUMBER OF ZEROS IN $\Re\hat{\mu}$

In this section, we generalize Proposition 4.1 by allowing a finite number of zeros of finite order in $\Re\hat{\mu}$. More precisely, we define

$$\eta(\omega) = \omega^2(1+\omega^2)^{-1} \quad (\omega \in R), \tag{5.1}$$

and suppose that there exist a finite distinct set of points $\{\omega_1, \cdots, \omega_m\}$, a corresponding set of positive integers $\{q_1, \cdots, q_m\}$, and constants $\varepsilon > 0$, $n \in \{0, 1, 2, \cdots\}$, such that

(5.2) *the measure* $d(\Re\hat{\mu})(\omega) - \gamma(\omega)\,d\omega$ $(\omega \in R)$ *is positive, where*

$$\gamma(\omega) = \varepsilon(1+\omega^2)^{-n} \prod_{j=1}^{m} (\eta(\omega - \omega_j))^{q_j}.$$

Note that the zeros of γ are exactly $\{\omega_1, \cdots, \omega_m\}$, and that the order of each zero ω_j is $2q_j$.

PROPOSITION 5.1. *Let* (1.3), (4.2) *and* (5.2) *hold. Moreover, suppose that for each* $j = 1, \cdots, m$, *there exist a function* $h_j \in L^2(R^+)$ *and constants* $\alpha_{j,p}$ $(p = 0, \cdots, q_j - 1)$ *such that*

$$\int_0^t (t-s)^{q_j - 1} e^{-i\omega_j s} f(s)\,ds = h_j(t) + \sum_{p=0}^{q_j - 1} \alpha_{j,p} t^p \quad (t \in R^+). \tag{5.3}$$

Then μ *dominates* f.

Proposition 5.1 contains Proposition 4.1.

Remark 5.2. A sufficient condition for (5.3) to hold is

$$(1 + t^q) f(t) \in L^2(R^+), \tag{5.4}$$

where $q = \max\{q_j\}$. This one can show, *e.g.*, by expanding $(t-s)^{q_j - 1}$, writing $\int_0^t = \int_0^\infty - \int_t^\infty$, and using Hardy's inequality (see [22, p. 272]) to verify that

$$t^p \int_t^\infty s^{(q_j - p - 1)} e^{-i\omega_j s} f(s)\,ds \in L^2(R^+) \quad (0 \le p \le q_j - 1).$$

However, (5.4) is by no means necessary for (5.3) to hold. For instance, every nonnegative and nonincreasing $f \in L^2(R^+)$ satisfies (5.3) whenever $\omega_j \neq 0$ and $q_j = 1$ $(j = 1, \cdots, m)$ (integrate by parts to show that $\left|\int_t^\infty e^{-i\omega_j s} f(s)\,ds\right| \le 2\,|\omega_j|^{-1} f(t)$).

Proof of Proposition 5.1. We begin by extending f to R as in the proof of Proposition 4.1. In particular, (4.3) holds, and the support of f is bounded to the left.

It follows from (5.3) that for some new constants $\beta_{j,p}$,

$$\int_{-\infty}^t (t-s)^{q_j - 1} e^{-i\omega_j s} f(s)\,ds = h_j(t) + \sum_{p=0}^{q_j - 1} \beta_{j,p} t^p \quad (t \in R^+) \tag{5.5}$$

(write $\int_{-\infty}^{t} = \int_{-\infty}^{0} + \int_{0}^{t}$, and expand $(t - s)^{q_j - 1}$ in the first integral). Moreover, by adding a suitably chosen C^{∞}-function with compact support contained in $(-\infty, 0)$ to f, we can without loss of generality suppose that all the constants $\beta_{j,p}$ vanish (this function can, *e.g.*, be chosen as a linear combination of smooth approximations of the functions $\chi_{[-2,-1]}(s)s^p e^{i\omega_j s}$ $(j = 1, \cdots, m;\ p = 0, \cdots, q_j - 1)$). Extend each h_j to R so that (5.5) holds also for negative t:

$$\int_{-\infty}^{t} (t - s)^{q_j - 1} e^{-i\omega_j s} f(s)\, ds = h_j(t) \quad (t \in R). \tag{5.6}$$

Then $h_j \in L^2(R)$ $(j = 1, \cdots, m)$.

Differentiate (5.6) q_j times in the distribution sense, and then take the Fourier transform of each side to get

$$(q_j - 1)!\, \hat{f}(\omega + \omega_j) = (i\omega)^{q_j} \hat{h}_j(\omega)$$

a.e. on R. Together with $\hat{h}_j \in L^2(R)$, this implies $(\omega - \omega_j)^{-q_j} \hat{f}(\omega) \in L^2(R)$. As this is true for each $j = 1, \cdots, m$, and (4.3), (5.1) and (5.2) hold, we find that (3.2) is satisfied. Thus Theorem 3.1 applies, and the proof of Proposition 5.1 is complete.

Taking $n = 0$, $m = 1$, $\omega_1 = 0$, $q_1 = 1$ in Proposition 5.1, we get a result which overlaps Londen's [9, Corollary 1]. Londen studies a vector-valued equation, but reformulating his result for the scalar equation (1.1) we get (here LAC stands for the set of locally absolutely continuous functions)

THEOREM B (Londen). (i) *Let* $a \in LAC(R^+)$,

$$a(t) \geq 0 \quad (t \in R^+), \qquad a'(t) \leq 0 \ \textit{a.e. on}\ R^+, \quad \textit{and} \quad a(0) > 0. \tag{5.7}$$

Let $h \in LAC(R^+)$, $h' \in L^2_{loc}(R^+)$, *and*

$$\sum_{n=0}^{\infty} \left[\int_{n}^{n+1} |h'(s)|^2\, ds \right]^{1/2} < \infty. \tag{5.8}$$

Let g *satisfy* (1.3). *Then every solution* x *of*

$$x(t) + \int_{0}^{t} a(t - s)\, g(x(s))\, ds = h(t) \quad (t \in R^+) \tag{5.9}$$

satisfies

$$\sup_{t \in R^+} \int_{t}^{t+1} |g(x(s))|^2\, ds < \infty, \tag{5.10}$$

and

$$x' \in L^2(R^+). \tag{5.11}$$

(ii) *Let the hypothesis of* (i) *hold, except that* (5.8) *is weakened to* $h' \in L^2(R^+)$. *In addition, suppose that* $a(\infty) > 0$. *Then every solution* x *of* (5.9) *satisfies* (4.6) *and* (5.11).

(For simplicity we take g to be continuous, which is not required by Londen. Part (i) is a reformulation of [9, Corollary 1], and part (ii) a reformulation of [9, Corollary 2].)

To see the connection between Proposition 5.1 and Theorem B, one must first differentiate (5.9):

$$x'(t) + a(0)g(x(t)) + \int_0^t a'(t-s)g(x(s))\,ds = h'(t). \tag{5.12}$$

This differentiated equation is equivalent to (1.1) if one defines $f = h'$, $d\mu(t) = a(0)\,d\delta_0(t) + a'(t)\,dt$ ($t \in R^+$), where δ_0 is the Dirac measure at zero. Compute

$$\Re\hat{\mu}(\omega) = a(0) + \int_0^\infty \cos(\omega t)\,a'(t)\,dt = a(\infty) - \int_0^\infty (1-\cos(\omega t))\,a'(t)\,dt. \tag{5.13}$$

Now one should distinguish between two cases: either $a(\infty) > 0$, or $a(\infty) = 0$. If $a(\infty) > 0$, then, by (5.13), $\Re\hat{\mu}(\omega) \geq a(\infty) > 0$ ($\omega \in R$) and Proposition 4.1 applies with $n = 0$, $\varepsilon = a(\infty)$. The discussion of the case $n = 0$ in Section 4 yields exactly part (ii) of Theorem B. In the case $a(\infty) = 0$, one has $\Re\hat{\mu}(0) = 0$, and thus (4.1) cannot possibly hold. However, we claim that (5.2) is satisfied with $n = 0$, $m = 1$, $\omega_1 = 0$, $q_1 = 1$. It follows from (5.13) and the Riemann-Lebesgue lemma that

$$\lim_{|\omega|\to\infty} \Re\hat{\mu}(\omega) = a(0) > 0.$$

Moreover, since $1 - \cos(\omega t) > 0$ a.e. in R^+ whenever $\omega \neq 0$, we have $\Re\hat{\mu}(\omega) > 0$ ($\omega \neq 0$). Thus, in order to verify (5.2), it suffices to show that

$$\liminf_{\omega\to 0} \omega^{-2}\Re\hat{\mu}(\omega) > 0. \tag{5.14}$$

Since $a(0) > 0$ but $a(\infty) = 0$, one can find some $T < \infty$ such that $a(0) > a(T)$. Then for $\omega \neq 0$,

$$\begin{aligned} \omega^{-2}\Re\hat{\mu}(\omega) &= -\omega^{-2}\int_0^\infty (1-\cos(\omega t))\,a'(t)\,dt \\ &\geq -\omega^{-2}\int_0^T (1-\cos(\omega t))\,a'(t)\,dt = \int_0^T \frac{\sin(\omega t)}{\omega}(a(t)-a(T))\,dt. \end{aligned}$$

By the Lebesgue dominated convergence theorem,

$$\lim_{\omega\to 0}\int_0^T \frac{\sin(\omega t)}{\omega}(a(t)-a(T))\,dt = \int_0^T t(a(t)-a(T))\,dt > 0.$$

This verifies (5.14), and shows that (5.2) holds with $n = 0$, $m = 1$, $\omega_1 = 0$, $q_1 = 1$.

Proposition 5.1 and Theorem 1.1 yield the following modification of Theorem B(i):

THEOREM 5.3. *Theorem* B(i) *is true if* (5.8) *is replaced by*

$$h' \in L^2(R^+), \quad \lim_{t \to \infty} h(t) = h(\infty) \textit{ exists}, \quad \textit{and} \quad h - h(\infty) \in L^2(R^+). \tag{5.15}$$

Proof of Theorem 5.3. By the preceding argument, one can apply Proposition 5.1 and Theorem 1.1 to the differentiated version (5.12) of (5.9). This means that every solution x of (5.9) satisfies (1.6), hence also (1.9) (where μ is defined as in the lines following (5.12)). Now we use Londen's second identity [9, line (4.2)]

$$Q(\phi, T, \mu) = -\frac{1}{2}\int_0^T \int_0^t a'(s)(\phi(t) - \phi(t-s))^2\, ds\, dt$$

$$+\frac{1}{2}\int_0^T a(T-t)(\phi(t))^2\, dt + \frac{1}{2}\int_0^T a(t)(\phi(t))^2\, dt,$$

(1.9) and (5.7) to conclude that

$$\sup_{T \in R^+} \int_{T-\delta}^T |g(x(t))|^2\, dt < \infty, \tag{5.16}$$

where $\delta > 0$ is chosen so that $a(t) > 0$ in $[0, \delta]$. But (5.16) is equivalent to (5.10), so we have shown that (5.10) holds.

The claim (5.11) follows from (1.9), (5.12), (5.15) and the inequality

$$\int_0^T [a(0)\, g(x(t)) + \int_0^t a'(t-s)\, g(x(s))\, ds]^2\, dt \le 4a(0)\, Q(g \circ x, T, \mu)$$

(see [19, Lemma 1 and Theorem 2(i)]). This completes the proof of Theorem 5.3.

Neither of Theorems B(i) and 5.3 implies the other, because (5.8) and (5.15) are overlapping; *e.g.*, $h(t) = (1+t)^{-1/2}$ satisfies (5.8) but not (5.15), and

$$h(t) = (1+t)^{-1}\cos(t)$$

satisfies (5.15) but not (5.8).

Theorems B and 5.3 overlap Levin's [5, Theorem 1]. In [5], the hypotheses are (here m is the Lebesgue measure)

$$a \textit{ is nonnegative and nonincreasing on } R^+, \textit{ and } a(0) < \infty, \tag{5.17}$$

$$h \in C \cap BV(R^+), \tag{5.18}$$

$$g \in C(R), \quad m\{\xi \in R: \xi g(\xi) < 0\} < \infty. \tag{5.19}$$

Clearly (5.17) is weaker than (5.7) combined with $a \in LAC(R^+)$, (5.18) is weaker than (5.8) and overlaps (5.15), and (5.19) overlaps (1.3). The conclusion of [5, Theorem 1] is that the solutions of (5.9) are *a priori* bounded.

6. A CONVEX, NONINTEGRABLE KERNEL

We next turn to the case when the function f in (1.1) is comparatively large at infinity. Note that in Sections 4 and 5 we have throughout used conditions on f which imply $f \in L^2(R^+)$. Thus, the only one of our previous results which applies to (1.1), *e.g.*, when $f(t) = (1+t)^{-1/2}$, is Theorem 2.1 (with $f = f_2$). In this particular case, Theorem 2.1 roughly says that if the kernel μ contains a constant term, then every solution x of (1.1) satisfies (1.6).

A similar result holds when the kernel not necessarily contains a constant term, but is (or contains) a convex, nonintegrable function. The fact that (1.13) implies (1.12) provides us with one such example (see Section 7, and the illustration in [6, p. 433]). Here we develop a somewhat different result, which is based on Theorem 3.1:

PROPOSITION 6.1. *Let* $d\mu(t) = a(t)\,dt$ $(t \in R^+)$, *where*

$$\text{(6.1)} \qquad \begin{aligned} &a \textit{ is nonnegative, nonincreasing and convex on } (0, \infty),\\ &a \in L^1(0,1), \textit{ and } a(\infty) = 0. \end{aligned}$$

Let $f \in BV(R^+)$, $f(\infty) = 0$, *and let* $V(t)$ *be the total variation of* f *on* $[t, \infty)$. *Moreover, suppose that*

$$\text{(6.2)} \qquad \int_0^1 \left[\int_0^{1/\omega} V(t)\,dt\right]^2 \left[\int_{(0,1/\omega]} t^2\,da'(t)\right]^{-1} d\omega < \infty .$$

Then μ *dominates* f.

At the end of this section we give an example which illustrates Proposition 6.1.

We begin the proof by stating and proving two lemmas:

LEMMA 6.2. *Let* a *satisfy* (6.1), *and define*

$$\text{(6.3)} \qquad \lambda(\omega) = \lim_{T\to\infty} \int_0^T \cos(\omega t)\,a(t)\,dt \qquad (\omega \neq 0)$$

(*this limit exists because* a *is monotone and tends to zero*). *Then*

$$\text{(6.4)} \qquad \lambda(\omega) \geq \frac{11}{24}\left[\int_{(0,1/\omega]} t^2\,da'(t)\right] \qquad (\omega \neq 0).$$

Proof of Lemma 6.2. It follows from (6.1) that

$$\text{(6.5)} \qquad ta(t) \to 0, \quad t^2 a'(t) \to 0 \qquad (t \to 0+)$$

(*cf.* [4, Lemma 1]). Fix $\omega \in R$, $\omega \neq 0$. Integrating by parts twice in (6.3) (this is possible because of (6.5)), one can show that

$$\text{(6.6)} \qquad \lambda(\omega) = \omega^{-2} \int_{(0,\infty)} (1 - \cos(\omega t))\,da'(t).$$

Use the Taylor expansion of $\cos(\omega t)$ to check that $1 - \cos(\omega t) \geq 11(\omega t)^2/24$ $(0 \leq t \leq 1/|\omega|)$. Clearly $1 - \cos(\omega t) \geq 0$ $(1/|\omega| < t < \infty)$. Substituting this into (6.6) and using (6.1), one gets (6.4).

LEMMA 6.3. *Let* $f \in BV(R^+)$, $f(\infty) = 0$, *and define*

$$\tilde{f}(\omega) = \lim_{T\to\infty} \int_0^T e^{-i\omega t} f(t)\, dt \qquad (\omega \neq 0) \tag{6.7}$$

(*this limit exists because* f *is the sum of two monotone functions tending to zero*). *Then*

$$|\tilde{f}(\omega)| \leq 2 \int_0^{1/|\omega|} V(t)\, dt \qquad (\omega \neq 0), \tag{6.8}$$

where $V(t)$ *is the total variation of* f *on* $[t, \infty)$.

Different versions of (6.8) have been used by Hannsgen [4] and by Shea and Wainger [15].

Proof of Lemma 6.3. Integrate by parts to show that

$$\tilde{f}(\omega) = \frac{1}{i\omega} \int_{R^+} (e^{-i\omega t} - 1)\, df(t) \qquad (\omega \neq 0). \tag{6.9}$$

It is easy to see that $|e^{-i\omega t} - 1| \leq 2h(|\omega t|)$, where

$$h(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 1, & t > 1. \end{cases} \tag{6.10}$$

Substituting this into (6.9) and integrating by parts, one gets (6.8).

Proof of Proposition 6.1. The proof is based on Theorem 3.1. The first fact which we need is that the function λ in Theorem 3.1 is the same as the function λ in (6.3). This follows from (6.4) (which yields $\lambda \geq 0$) and [20, Lemma 1.1] (integrate by parts to show that

$$\lambda(\omega) = \Re \left\{ \int_{R^+} e^{-i\omega t}(1 - h(t))\, a(t)\, dt + \frac{1}{i\omega} \int_{R^+} e^{-i\omega t} (h(t)\, a(t))'\, dt \right\},$$

where h is the function defined in (6.10)). In particular, the function λ in (6.3) is locally integrable.

Next we have to show that the distribution Fourier transform $\hat{f}$ of f belongs to $L^1(R) + L^2(R)$. For this we use (6.2) and Lemmas 6.2 and 6.3. Define

$$r(\omega) = 2 \int_0^{1/|\omega|} V(t)\, dt \qquad (\omega \in R). \tag{6.11}$$

We first claim that

$$r \in L^1_{loc}(R). \tag{6.12}$$

Clearly

$$r(\omega) \leq 2V(0)\,|\omega|^{-1} \qquad (|\omega| \geq 1), \tag{6.13}$$

so (6.12) follows if one can prove that $r \in L^1(-1, 1)$. Using the Schwarz inequality one gets (first multiply and divide by $\lambda^{1/2}$)

$$\left[\int_{-1}^{1} r(\omega)\,d\omega\right]^2 \leq \left[\int_{-1}^{1} (r(\omega))^2 (\lambda(\omega))^{-1}\,d\omega\right]\left[\int_{-1}^{1} \lambda(\omega)\,d\omega\right].$$

But $\int_{-1}^{1} \lambda(\omega)\,d\omega < \infty$, and by (6.2), (6.4) and (6.11) also $\int_{-1}^{1} (r(\omega))^2 (\lambda(\omega))^{-1}\,d\omega < \infty$. This shows that $r \in L^1(-1, 1)$, and verifies (6.12).

Define $f_n = \chi[0,n]\,f$. Then $f_n \to f$ in the space of tempered distributions as $n \to \infty$, so also $\hat{f}_n \to \hat{f}$ in the same space. On the other hand, we claim that $\hat{f}_n \to \tilde{f}$ in the distribution sense. Let $V_n(t)$ be the total variation of f_n on $[t, \infty)$. Then $V_n(t) \leq V(t)$ $(t \in R^+)$ (because $f(\infty) = 0$). Now use Lemma 6.3 and (6.11) to conclude that each function $\hat{f}_n$ satisfies $|\hat{f}_n(\omega)| \leq r(\omega)$ $(\omega \in R)$. Together with (6.7) (which shows that $\hat{f}_n$ converges pointwise to $\tilde{f}$), (6.12), and the Lebesgue dominated convergence theorem, this implies that $\hat{f}_n \to \tilde{f}$ in $L^1_{loc}(R)$, hence also in the distribution sense.

By the uniqueness of a distribution limit, $\hat{f} = \tilde{f}$. From (6.8), and (6.11) through (6.13), we then get

$$\hat{f} = \tilde{f} \in L^1(R) + L^2(R). \tag{6.14}$$

It follows from (6.2), (6.4), (6.8) and (6.14) that Theorem 3.1 applies, and the proof of Proposition 6.1 is complete.

To illustrate Proposition 6.1, we investigate what (6.2) means in the particular case when a is of the form $a(t) = (1+t)^{-\alpha}$, where $0 < \alpha < 1$. For simplicity we also take f nonincreasing. Then (6.2) becomes $\int_0^1 \omega^{1-\alpha}\left[\int_0^{1/\omega} f(t)\,dt\right]^2 d\omega < \infty$ (here we have used the fact that

$$\lim_{\omega \to 0+} \omega^{1-\alpha} \int_0^{1/\omega} \frac{t^2\,dt}{(1+t)^{2+\alpha}} = \frac{1}{1-\alpha} \neq 0).$$

In particular, we notice that if $f(t) = O(t^{-\beta})$, where $\beta > \alpha/2$, then (6.2) holds.

It is interesting to observe that we get the same relation between the exponents α and β as Levin and Nohel [6, p. 433] do by using (1.13).

7. ON LEVIN'S AND NOHEL'S CONDITION

The following result was announced in Section 1:

LEMMA 7.1. *Let* (1.13) *hold, and define* $d\mu(t) = a(t)\,dt$. *Then* μ *dominates* f.

Proof of Lemma 7.1. Fix $T \in R^+$, $\phi \in C[0, T]$. Integrating by parts, one gets

$$\int_0^T \phi(t) f(t)\, dt = f(T)\Phi(T) - \int_0^T f'(t)\Phi(t)\, dt,$$

where $\Phi(t) = \int_0^t \phi(s)\, ds$. By (1.13) and the Schwarz inequality (note that c is non-negative and nonincreasing),

$$|f(T)\Phi(T)|^2 \leq c(0)\, a(T)\, [\Phi(T)]^2,$$

$$\left|\int_0^T f'(t)\Phi(t)\, dt\right|^2 \leq -c(0) \int_0^T a'(t)\, [\Phi(t)]^2\, dt;$$

so altogether

$$\left|\int_0^T \phi(t) f(t)\, dt\right|^2 \leq 2\left[\, |f(T)\Phi(T)|^2 + \left|\int_0^T f'(t)\Phi(t)\, dt\right|^2 \right]$$

$$\leq 4c(0)\left[\frac{1}{2} a(T)\, [\Phi(T)]^2 - \frac{1}{2}\int_0^T a'(t)\, [\Phi(t)]^2\, dt\right].$$

The estimate (1.12) now follows from Londen's first identity [8, line (2.4)]

$$\int_0^T \phi(t) \int_0^t \phi(t - s)\, a(s)\, ds\, dt = \frac{1}{2} a(T)\, [\Phi(T)]^2 - \frac{1}{2}\int_0^T a'(t)\, [\Phi(t)]^2\, dt$$

$$- \frac{1}{2}\int_0^T a'(t)\, [\Phi(T) - \Phi(T - t)]^2\, dt + \frac{1}{2}\int_0^T \int_0^t a''(s)\, [\Phi(t) - \Phi(t - s)]^2\, ds\, dt,$$

and the proof of Lemma 7.1 is complete.

8. ADDED IN PROOF

Theorem 3.1 can be sharpened. By weakening the Fourier transform condition on f used in Theorem 3.1, one can make it equivalent to a strengthened version of (1.12).

We denote by $L^2(R; \operatorname{Re} \hat{\mu})$ the set of Radon measures on R which can be written in the form $d\nu(\omega) = h(\omega) d\{\operatorname{Re} \hat{\mu}\}(\omega)$, with h square-integrable with respect to $\operatorname{Re} \hat{\mu}$ (h is the Radon-Nikodym derivative of ν with respect to $\operatorname{Re} \hat{\mu}$). It follows from (3.2) together with $\hat{f} \in L^1_{loc}(R)$ that

(8.1) $$\hat{f} \in L^2(R; \operatorname{Re} \hat{\mu})$$

(because $\hat{f} \in L^2(R; \lambda) \subset L^2(R; \operatorname{Re} \hat{\mu})$).

Replace (1.12) by

there exists $\alpha \in R^+$ *such that*

(8.2) $$\left| \int_R \psi(t)f(t)\,dt \right|^2 \leq \alpha \int_R \psi(t) \int_R \psi(t-s)\,d\mu(s)\,dt\,,$$

for every $\psi \in L^\infty(R)$ *with compact support.*

Putting $\psi = \chi_{[0,T]}\phi$, we observe that (8.2) is stronger than (1.12). The crucial difference between (1.12) and (8.2) is not the fact that ψ is less smooth than $\chi_{[0,T]}\phi$, but the fact that in (8.2) we integrate over negative values of t as well as positive values.

THEOREM 8.1. *Let* μ *be positive definite, and let* $f \in L^1_{loc}(R)$ *induce a tempered distribution. Then* (8.1) *and* (8.2) *are equivalent.*

That (8.1) implies (8.2) is shown in approximately the same way as in Theorem 3.1 (first take ψ to be a C^∞-function so that the analogue of (1.14) does not cause any difficulties, and then complete the proof using a density argument).

The proof of the converse part is omitted.

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