ON THE RANK OF AN OPEN MANIFOLD

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Let M denote a differentiable manifold. J. Milnor defined the rank of M to be the maximal number of vector fields $X_1, \cdots, X_k$ on M, everywhere linearly independent, such that their Lie brackets $[X_i, X_j] = 0$ for all i, j. ($X_1, \cdots, X_k$ are called *commuting vector fields* [8].) From Hirsch's immersion theorem, we see that any open parallelizable n-dimensional manifold has rank n. By a *tangent k-field* on a manifold M, we mean k linearly independent vector fields $X_1, \cdots, X_k$ on M. In this short note, we will prove the following result.

THEOREM. *Let M be an n-dimensional open manifold. Then M admits k linearly independent commuting vector fields if and only if M has a tangent k-field which is homotopic to a foliation.*

Proof. Let $f_M: M \to BGL_n$ denote the classifying map for the tangent bundle TM of M. If M has a tangent k-field which is homotopic to a foliation, then f_M has a lifting to $BGL_1 \times \cdots \times BGL_1 \times B\Gamma_{n-k}$; that is, the lifting is homotopic to (constant) $\times \cdots \times$ (constant) $\times\, \phi$, where $\phi: M \to B\Gamma_{n-k}$. ($B\Gamma_{n-k}$ is Haefliger's classifying space for codimension $n - k$ foliations [1].)

Considering the topological groupoid $V\Gamma_p$ consisting of those local diffeomorphisms of R^p such that their first derivatives lie in $SL(p, R)$, we may construct a classifying space $BV\Gamma_p$ for the codimension p volume-preserving foliations [7]. (Here we used the fact that an $SL(n, R)$-structure on an n-dimensional manifold is integrable [3, p. 6].) The argument in [2, p. 148] showed that an open manifold admits a codimension p volume-preserving foliation if and only if the classifying map for its tangent bundle lifts to $BGL_{n-p} \times BV\Gamma_p$ [7].

A *multifoliation* F on M is a collection of foliations $\{F_1, \cdots, F_t\}$, with codimension $F_j = k_j$ and TF_j the tangent bundle of F_j, such that

$$\operatorname{codim}(TF_{i1} \cap \cdots \cap TF_{is}) = k_{i1} + \cdots + k_{is} \quad \text{for any subset } \{i1, \cdots, is\} \subseteq \{1, \cdots, t\}$$

[5, p. 406]. In [6], we called such an F a *multifoliation of type* $(k_1, \cdots, k_t)$, and showed that an open manifold admits a multifoliation of type $(k_1, \cdots, k_t)$ with $k = \sum k_j \leq n$ if and only if the classifying map for its tangent bundle lifts to $B\Gamma_{k_1} \times \cdots \times B\Gamma_{k_t} \times BGL_{n-k}$.

Our classifying map f_M clearly has a lifting to $BV\Gamma_1 \times \cdots \times BV\Gamma_1 \times B\Gamma_{n-k}$. Equivalently, TM can be considered as a normal bundle of a particular codimension n Haefliger Γ-structure [2], where $\Gamma = V\Gamma_1 \times \cdots \times V\Gamma_1 \times \Gamma_{n-k}$. Then the argument in the proof of the result mentioned above ([6]) can be used here to show that M admits a multifoliation $F = \{F_1, \cdots, F_{k+1}\}$ of type $(1, 1, \cdots, 1, n - k)$ such that $F_1, \cdots, F_k$ are codimension 1 volume-preserving foliations with respect to the same volume form on M [7].

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In the language of [4], F is given by a family of open covers $\{U_a\}$ on M such that the Jacobians of the coordinate transformations are of the following form ([6]):

$$\begin{pmatrix} I_k & 0 \\ 0 & * \end{pmatrix}.$$

Let $(x_1, \cdots, x_n)$ denote the corresponding coordinate system on U_a. Hence for each i, with $1 \leq i \leq k$, we may patch up $\partial/\partial x_i$ on each U_a to obtain a globally defined vector field X_i, and $[X_i, X_j] = 0$ for $1 \leq i, j \leq k$. This completes the proof.

Thus, the rank of an open manifold M is the maximum of the numbers k satisfying the condition of the above theorem. Also, the above result does not hold for closed manifolds, as shown in [8].

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