# NORMAL ANALYTIC FUNCTIONS AND A THEOREM OF DOOB

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Let f be an analytic function in the open unit disk D, and let $\Lambda$ denote the collection of all one-to-one conformal mappings of D onto D. Then f is called *normal* if the family of functions $\{f \circ \lambda\}_{\lambda \in \Lambda}$ is normal in D in the sense of Montel.

Let X be a chord of the unit circle C having one endpoint at $\zeta = 1$, and let $\alpha$ be a finite complex value. For each $\varepsilon > 0$, let

$$\delta_\varepsilon = \liminf_{r \to 0^+} \frac{\mu[\{z: |f(z) - \alpha| < \varepsilon\} \cap X_r]}{r},$$

where $\mu[\cdot]$ denotes linear Lebesgue measure and $X_r = X \cap \{z: |z - 1| < r\}$. We say (see [3, p. 163]) that $\alpha$ is a *metric cluster value of the first kind of* f *on* X *at* 1 if

$$\lim_{\varepsilon \to 0} \varepsilon^{(\delta_\varepsilon)^d} = 0 \quad \text{for some } d > 0. \tag{1}$$

The upper limit of the exponents d for which (1) holds is called the *order* of the cluster value.

We shall prove the following theorem, which has already been established by J. L. Doob in the particular case that f is a bounded analytic function [3, Theorem 4].

THEOREM. *Let* f *be a normal analytic function in* D, *and let the finite complex value* $\alpha$ *be a metric cluster value of the first kind of* f *on some chord* X *at* 1. *If the order of* $\alpha$ *is greater than* 2, *then* f *has angular limit* $\alpha$ *at* 1.

*Proof.* To begin, suppose that f is bounded on X. Corresponding to each $\varepsilon_0 > 0$, let X' be a chord of C having one endpoint at $\zeta = 1$ and forming the angle $\pi/2(1 + \varepsilon_0)$ with X. By a theorem of F. Bagemihl [1, Theorem 4], there exists a positive number r such that f is bounded in the sector S determined by the chords X and X' and an arc of the circle $|z - 1| = r$. Since f has the metric cluster value $\alpha$ of the first kind of order greater than 2 on X at 1, we can now use the exact argument given by Doob [3, proof of Theorem 4] to conclude that f has the limit $\alpha$ as $z \to 1$ along certain paths in S ending at 1. Then, according to O. Lehto and K. I. Virtanen [4, Theorem 2], it follows that f possesses the angular limit $\alpha$ at 1.

To complete the proof of our theorem, we need only show that condition (1) implies that f is bounded on X. Suppose not. Then there exists a sequence $\{z_n\} \subset X$ with $z_n \to 1$ and $f(z_n) \to \infty$.

Let $\tau$ be an arbitrary positive number. For each pair of points z and z' in D, let $\rho(z, z')$ denote the non-Euclidean distance between z and z', and for each point z' in D, let $D(z', \tau) = \{z \in D: \rho(z, z') < \tau\}$. An argument of F. Bagemihl and W. Seidel [2, proof of Theorem 1] shows that our sequence $\{z_n\}$ has the property that

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(2) $$f(z) \to \infty \quad \text{as } z \to 1 \text{ through } \bigcup_{n=1}^{\infty} D(z_n, \tau).$$

For each n, let $z_n'$ be the point on X between $z_n$ and 1 for which $\rho(z_n, z_n') = \tau$. It can be shown by straightforward calculations that

(3) $$\lim_{n \to \infty} \frac{|z_n - z_n'|}{|1 - z_n|} = 1 - \ell_\tau,$$

where $\ell_\tau = q - \sqrt{q^2 - 1}$, $q = 1 + 2\cos^2\theta \sinh^2\tau$, and $\theta$ is the angle between X and the radius to 1.

Now suppose that $\varepsilon > 0$. By (2) we can associate with each $\tau > 0$ an integer $N = N(\tau)$ such that $D(z_n, \tau) \subset \{z: |f(z) - \alpha| \geq \varepsilon\}$ for each $n > N$; hence, it follows from (3) that

$$\delta_\varepsilon \leq 1 - (1 - \ell_\tau) = \ell_\tau \qquad \text{for each } \tau > 0.$$

Since $\lim_{\tau \to \infty} \ell_\tau = 0$, we conclude that $\delta_\varepsilon = 0$. This precludes condition (1), and the proof is complete.

*Remark.* In the proof of our theorem, we used the hypothesis concerning the order of the cluster value $\alpha$ only to employ an argument of Doob in [3]. As indicated by Doob [3, p. 165], it is easy to see from his argument that the order of the cluster value $\alpha$ on X could have been made dependent on the angle $\theta$ between X and the radius to 1. If $\theta$ is nearly $\pi/2$, the order can be taken near 1.

## REFERENCES

1. F. Bagemihl, *Some boundary properties of normal functions bounded on nontangential arcs*. Arch. Math. 14 (1963), 399-406.

2. F. Bagemihl and W. Seidel, *Sequential and continuous limits of meromorphic functions*. Ann. Acad. Sci. Fenn. Ser. AI, no. 280 (1960), 17 pp.

3. J. L. Doob, *The boundary values of analytic functions*. Trans. Amer. Math. Soc. 34 (1932), 153-170.

4. O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions*. Acta Math. 97 (1957), 47-65.

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