THE mod p SMITH INDEX AND A GENERALIZED BORSUK-ULAM THEOREM

Ewing L. Lusk

1. INTRODUCTION

In [3], R. Fenn proved a generalization of the Borsuk-Ulam Theorem, using the index theory for spaces with free $\mathbb{Z}_2$-actions developed by C.-T. Yang in [5]. Here we extend the index theory, using a slightly different approach, to spaces with free $\mathbb{Z}_p$-actions for arbitrary primes p, and we use it to generalize Fenn's theorem to $\mathbb{Z}_p$-homology spheres with free $\mathbb{Z}_p$-actions. In particular, let S^m be a $\mathbb{Z}_p$-homology m-sphere, and let $T: S^m \to S^m$ be a piecewise linear homeomorphism of period p. Let $f: S^m \to S^m$ be a map whose degree d is not divisible by p, and let $g: S^m \to R^n$ be an arbitrary map.

THEOREM 1. *If* $m > (n-1)(p-1)$, *then there is a pair* $\{x, y\}$ *of distinct points in* S^m *such that* $f(x) = Tf(y)$ *and* $g(x) = g(y)$.

THEOREM 2. *If* $m > n(p-1) - 1$, *then there is a set* $\{x_0, \cdots, x_{p-1}\}$ *of* p *distinct points in* S^m *such that*

$$f(x_0) = Tf(x_1) = \cdots = T^{p-1} f(x_{p-1}) \quad \textit{and} \quad g(x_0) = g(x_1) = \cdots = g(x_{p-1}).$$

In the case where $p = 2$, each of these two theorems reduces to Fenn's theorem. If f is the identity map, then the requirement that T be PL may be dropped. In this case Theorem 1 is a slightly stronger version of a theorem in [1], and Theorem 2 appears in [4]. Theorem 2 can be used to show as in [3] that if M is manifold covered by R^n, then for each map

$$F: S^m \to (S^m/\mathbb{Z}_p) \times M \qquad (m > n(p-1) - 1)$$

whose degree is prime to p, there is a point $y \in (S^m/\mathbb{Z}_p) \times M$ such that $F^{-1}(y)$ contains at least p points.

There are several parts to the argument. The first is the development of the index theory for free $\mathbb{Z}_p$-actions, given in Section 2. In Section 3 we find an upper bound for the cohomology of a certain configuration-like space and hence for its index. The proofs of the theorems occur in Section 4.

2. THE INDEX HOMOMORPHISM

Let X be a simplicial complex, and suppose that T generates a free, simplicial $\mathbb{Z}_p$-action on X, where p is prime. Let C(X) denote the simplicial chain complex of X with coefficients in $\mathbb{Z}_p$. It can be shown that if z is an invariant cycle in C(X), then $z = c + T_\# c + \cdots + T_\#^{p-1} c$, for some $c \in C(X)$. The homology groups of the

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chain complex $(1 + T_\# + \cdots + T_\#^{p-1})\, C(X)$ are isomorphic to the homology groups of $X/\mathbb{Z}_p$. The requirements that X be a complex and T be simplicial can be eliminated by the technique of Čech theory.

In the case $p = 2$, Yang [5] denotes these groups by $H_*(X; T)$ and defines for each n an index homomorphism $\nu\colon H_n(X; T) \to \mathbb{Z}_2$. Its basic properties are that the index of a class is preserved under the homomorphism induced by an equivariant map, and that the fundamental class of an m-sphere has index m. Motivated by the approach in [2], we give here a different definition, which is simpler than Yang's inductive approach, generalizes his definition to primes other than two, and makes particularly simple the proofs of the various properties. In what follows, all homology and cohomology is taken with $\mathbb{Z}_p$ coefficients.

We recall that a free $\mathbb{Z}_p$-action on a space X induces a principal covering $X \to X/\mathbb{Z}_p$, and that there is a principal covering $E\mathbb{Z}_p \to B\mathbb{Z}_p$ such that equivalence classes of coverings are in one-to-one correspondence with homotopy classes of maps from $X/\mathbb{Z}_p$ to $B\mathbb{Z}_p$. This means that if $\mathbb{Z}_p$ acts freely on X, there is a map $c_X\colon X/\mathbb{Z}_p \to B\mathbb{Z}_p$, depending on the action but otherwise unique up to homotopy, such that if $f\colon X \to Y$ is an equivariant map between spaces with free $\mathbb{Z}_p$-actions and $\hat{f}\colon X/\mathbb{Z}_p \to Y/\mathbb{Z}_p$ is the induced map, then the diagram

$$\begin{array}{ccc} X/\mathbb{Z}_p & \xrightarrow{\hat{f}} & Y/\mathbb{Z}_p \\ & {\scriptstyle c_X}\searrow \quad \swarrow{\scriptstyle c_Y} & \\ & B\mathbb{Z}_p & \end{array}$$

is homotopy-commutative. We also recall that the cohomology of $B\mathbb{Z}_p$ is known to be the tensor product of a polynomial algebra on a two-dimensional class with an exterior algebra on a one-dimensional class in the case p is odd, and a polynomial algebra on a one-dimensional class in the case $p = 2$. In either case, there is one generator g_n in each dimension n.

Definition. Let $\mathbb{Z}_p$ act freely on X, and let $c_X\colon X/\mathbb{Z}_p \to B\mathbb{Z}_p$ be the classifying map for the covering $X \to X/\mathbb{Z}_p$. Let z be an element of $H_n(X/\mathbb{Z}_p)$. Then the *index* $\nu(z)$ is the element of $\mathbb{Z}_p$ defined by $[c_X^*(g_n)](z)$.

PROPOSITION 1. *If* $f\colon X \to Y$ *is an equivariant map, then* $\nu(\hat{f}_*(z)) = \nu(z)$ *for all* $z \in H_n(X/\mathbb{Z}_p)$.

Proof. $\nu(\hat{f}_*(z)) = [c_Y^*(g_n)](\hat{f}_*(z)) = [\hat{f}^* c_Y^*(g_n)](z) = [c_X^*(g_n)](z) = \nu(z)$.

PROPOSITION 2. *Let* $z \in H_n(X/\mathbb{Z}_p)$. *Then* $\nu(z) \neq 0$ *if and only if for each space* Y *supporting a free* $\mathbb{Z}_p$*-action and each equivariant map* $f\colon X \to Y$ *the condition* $\hat{f}_*(z) \neq 0$ *is satisfied in* $H_n(Y/\mathbb{Z}_p)$.

Proof. If $f\colon X \to Y$ is an equivariant map and $\nu(z) \neq 0$, then $\nu(\hat{f}_*(z)) = \nu(z) \neq 0$, so that $\hat{f}_*(z) \neq 0$. To prove the converse, apply the hypothesis to the map $X \to E\mathbb{Z}_p$ that induces the classifying map. Then, if $(c_X)_*(z) \neq 0$, we have the relations $\nu(z) = [c_X^*(g_n)](z) = g_n[(c_X)_*(z)] \neq 0$.

Definition. Let X support a free $\mathbb{Z}_p$-action. Then the *index* $\nu(X)$ is the largest dimension n for which there exists an element z in $H_n(X/\mathbb{Z}_p)$ such that $\nu(z) \neq 0$.

PROPOSITION 3. *The index* $\nu(X)$ *is the largest* n *such that* $c_X^*\colon H\ (B\mathbb{Z}_p) \to H^n(X/\mathbb{Z}_p)$ *is nonzero.*

Proof. Let ν_1 be the index in the definition, and let ν_2 be the index in the statement of the proposition. If $\nu_1(X) = n$, then there exists an element z in $H_n(X/\mathbb{Z}_p)$ such that $\nu(z) \neq 0$, which implies c_X^* is nonzero. Hence $\nu_2(X) \geq \nu_1(X)$. If $\nu_2(X) = n$, then $c_X^*: H^n(B\mathbb{Z}_p) \to H^n(X/\mathbb{Z}_p)$ is nonzero; in particular, $c_X^*(g_n) \neq 0$. Therefore there is a z in $H_n(X/\mathbb{Z}_p)$ such that $[c_X^*(g_n)](z) \neq 0$, which implies that $\nu_1(X) \geq \nu_2(x)$.

To compute the index of certain spaces, we recall that if G acts freely on a space X, there exists a cohomology spectral sequence whose E_2-term is given by $E_2^{s,t} = H^s(G; H^t(X))$ and which converges to $H^*(X/G)$.

THEOREM 3. *If* $H^i(X) = 0$ *for* $0 < i < m$, *then* $\nu(X) \geq m$.

Proof. Consider the map of covering spaces

$$\begin{array}{ccc} X & \longrightarrow & E\mathbb{Z}_p \\ \downarrow & & \downarrow \\ X\ \mathbb{Z}_p & \xrightarrow{c_X} & B\mathbb{Z}_p \end{array}$$

Since the spectral sequence described above is natural and $E\mathbb{Z}_p$ is contractible, c_X^* must be an isomorphism in dimensions less than or equal to m. The result now follows from Proposition 3.

COROLLARY 1. *If* S^m *is a homology* m-*sphere, then* $\nu(S^m) = m$.

3. A BOUND FOR THE COHOMOLOGY OF A CERTAIN SPACE

Let

$$F'(R^n, p) = \{(x_1, x_2, \cdots, x_p) \in (R^n)^p \mid x_i \neq x_{i+1} \text{ for } i = 1, \cdots, p-1 \text{ and } x_1 \neq x_p\}.$$

There is a free $\mathbb{Z}_p$-action on $F'(R^n, p)$, defined by cyclic permutation of coordinates. In this section we find a bound for the index of $F'(R^n, p)$.

Let

$$K = \{(x_1, \cdots, x_p) \in (R^n)^p \mid x_1 = x_2, \text{ or } x_2 = x_3, \cdots, \text{ or } x_{p-1} = x_p, \text{ or } x_p = x_1\} \cap S^{np-1}.$$

Then $F'(R^n, p)$ equivariantly deforms onto $S^{np-1} - K$. We shall find a lower bound for $H_*(K)$ and then apply Alexander duality in S^{np-1}.

Definition. Let $J = (i(1), \cdots, i(r))$ be an ordered set of positive integers whose sum is p. Define the length $\ell(J)$ to be r in this case. Let

$$X(i(1), \cdots, i(r)) = \{(x_1, \cdots, x_p) \in (R^n)^p \mid x_1 = x_2 = \cdots = x_{i(1)};\ x_{i(1)+1} = \cdots = x_{i(1)+i(2)};\ x_{i(1)+\cdots+i(r-1)+1} = \cdots = x_{i(1)+\cdots+i(r)}\} \cap S^{np-1}.$$

Then $X(i(1), \cdots, i(r))$ is a sphere of dimension $rn - 1$. If I is an ordered set of positive integers with $\ell(I) = j < k$, not necessarily having sum p, we define

$X(I, k, m)$ to be equal to $\bigcup X(i(1), \cdots, i(r))$, where the union is taken over all sequences $(i(1), \cdots, i(r))$ such that

$$\sum_{s=1}^{r} i(s) = p, \quad (i(1), \cdots, i(j)) = I, \quad i(j+1) = m, \quad j < r \leq k.$$

We allow I to be empty, in which case $\ell(I) = 0$.

LEMMA 1. $\left[\bigcup_{q=1}^{m} X(I, k, q)\right] \cap X(i, k, m+1) = \bigcup_{s \geq 1} X(I, k-1, m+s)$.

Proof. Let $I = (i(1), \cdots, i(j))$. If $(x_1, \cdots, x_p)$ is the left-hand side, then for some $q \leq m$,

$$x_1 = \cdots = x_{i(1)}; \cdots; x_{i(1)+\cdots+i(j)+1} = \cdots = x_{i(1)+\cdots+i(j)+q};$$

and

$$x_{i(1)+\cdots+i(j)+1} = \cdots = x_{i(1)+\cdots+i(j)+m+1}.$$

The fact that $q \leq m$ causes at least two equations to overlap, and therefore k, the total number of equations, can be reduced by at least one. Hence

$$(x_1, \cdots, x_p) \in \bigcup_{s \geq 1} X(I, k-1, m+s).$$

(Of course, the union is actually finite.) If the point $(x_1, \cdots, x_p)$ is in the right-hand side, then its coordinates satisfy the equation

$$x_{i(1)+\cdots+i(j)+1} = \cdots = x_{i(1)+\cdots+i(j)+m+s}.$$

If we rewrite this equation as two equations, one of which is

$$x_{i(1)+\cdots+i(j)+1} = \cdots = x_{i(1)+\cdots+(j)+m},$$

it can be seen that $(x_1, \cdots, x_p) \in X(I, k, m)$. If $s > 1$, splitting this same equation shows that $(x_1, \cdots, x_p) \in X(I, k, m+1)$; if $s = 1$, we split one of the other equations.

LEMMA 2. *For any* m *and* w, $H_i\left(\bigcup_{s=0}^{w} X(I, k, m+s)\right) = 0$ *if* $0 < i < n+k-2$.

Proof. The proof is by induction on k. Since $X(0, 1, p)$ is an $(n-1)$-sphere, the lemma is true when $k = 1$. Next we use downward induction on $\ell(I)$ to prove the lemma for $w = 0$ and $k > 1$. The largest $\ell(I)$ can be is $k - 1$; in this case, $X(I, k, m)$ is an $(nk-1)$-sphere, and the lemma is true if $w = 0$. Now suppose I is fixed and that the lemma is true for all $\bigcup_{s=0}^{w'} X(J, k, m+s)$ with $\ell(J) > \ell(I)$. Observe that if $I = (i(1), \cdots, i(j))$, then $X(I, k, m) = \bigcup_{q=1}^{r} X(J, k, q)$, where $J = (i(1), \cdots, i(j), m)$ and r is some number less than or equal to

$$p - (i_1 + \cdots + i_j + m).$$

Therefore we apply the Mayer-Vietoris sequence, together with induction on r and Lemma 1. Part of this sequence is

$$\cdots \to H_i\left(\bigcup_{q=1}^{r-1} X(J, k, q)\right) \oplus H_i X(J, k, r) \to H_i X(I, k, m)$$

$$\to H_{i-1}\left(\bigcup_{t\geq 0} X(J, k-1, r+t)\right) \to \cdots .$$

Now the left side is zero for $0 < i < n + k - 2$, by the inductions on r and $\ell(I)$; and the right side is zero for $0 < i < n + k - 2$, by the induction on k. Therefore $H_i X(I, k, m) = 0$ for $0 < i < n + k - 2$. This is the statement of the lemma if $w = 0$. To complete the induction step we again apply the Mayer-Vietoris sequence, this time to the union $\bigcup_{s=0}^{w} X(I, k, m+s)$. Part of this sequence is

$$\cdots \to H_i\left(\bigcup_{s=0}^{w-1} X(I, k, m+s)\right) \oplus H_i X(I, k, m+w) \to H_i\left(\bigcup_{s=0}^{w} X(I, k, m+s)\right)$$

$$\to H_{i-1}\left(\bigcup_{t\geq 0} X, k-1, m+w+t\right) \to \cdots .$$

The left side is zero for $i < i < n + k - 2$, by induction on w, and the right side is zero by induction on k. This completes the proof.

Let

$$Z = \{(x_1, \cdots, x_p) \in (R^n)^p \mid x_1 = x_p\} \cap S^{np-1},$$

and let $Y(I, k, m) = X(I, k+1, m) \cap Z$. Then, like $X(I, k, m)$, $Y(I, k, m)$ is a union of spheres of dimension $rn - 1$, for $j < r \leq k$; and indeed Lemmas 1 and 2 are true if $X(I, k, m)$ is replaced by $Y(I, k, m)$.

LEMMA 3. $H_i(K) = 0$ *for* $0 < i < n + p - 3$.

Proof. First observe that if we let $X = \bigcup_{s=1}^{2} X(0, p-1, s)$, then $K = X \cup Z$. By Lemma 2, $H_i(X) = 0$ for $0 < i < n + p - 3$, and $H_i(Z) = 0$ in the same range, since Z is a $((p-1)n-1)$-sphere. Furthermore, $X \cap Z = \bigcup_{s=1} Y(0, p-2, s)$, and therefore by Lemma 2 and the remark following it, $H_i(X \cap Z) = 0$ for $0 < i < n + p - 4$. An application of the Mayer-Vietoris sequence completes the proof.

THEOREM 4. $H^i F'(R^n, p) = 0$ *if* $i > (n-1)(p-1)$.

Proof. By the remarks at the beginning of this section,

$$H^i F'(R^n, p) \cong H^i(S^{np-1} - K) \cong H_{np-1-i}(S^{np-1}, K).$$

Now we apply the homology exact sequence of the pair (S^{np-1}, K), together with Lemma 3, to obtain the desired result.

COROLLARY 2. *Let* $\mathbb{Z}_p$ *act on* $F'(R^n, p)$ *by cyclic permutation of coordinates. Then* $H_i(F'(R^n, p)/\mathbb{Z}_p) = 0$ *for* $i > (n-1)(p-1)$.

Proof. One can apply Lemma 3 of [2], among other arguments.

4. PROOFS OF THEOREMS 1 AND 2

As in [3], we may assume that f is PL and nondegenerate, since it can be approximated by such functions. If $T: S^m \to S^m$ is a free PL $\mathbb{Z}_p$-action on S^m, there are triangulations L and K of S^m such that the maps $L \xrightarrow{f} K \xrightarrow{T} K$ are simplicial, and we may assume that the triangulations are so fine that if

$$f(x_0) = Tf(x_1) = \cdots = T^{p-1} f(x_{p-1}),$$

then $st(x_i; L) \cap st(x_j; L) = \emptyset$ for $i \neq j$. Suppose further that each simplex of K and L is oriented so that $\sum_{\sigma \in L} 1 \cdot \sigma$ and $\sum_{\rho \in K} 1 \cdot \rho$ each represent the fundamental cycle of S^m if the sums are taken over all m-simplices. These will be called the *positive* orientations of these simplices. Let

$$A = \{(x_0, \cdots, x_{p-1}) \in S^m \times \cdots \times S^m \mid f(x_0) = Tf(x_1) = \cdots = T^{p-1}(x_{p-1})\}.$$

We shall show that A has a triangulation that can be oriented so that A is an invariant cycle of index m.

If $h: L \to K$ is simplical and σ and ρ are oriented simplices of L and K of the same dimension, then we let the statement $h(\sigma) = \rho$ signify that h maps σ onto ρ in an orientation-preserving manner. Now suppose that $\sigma^0, \cdots, \sigma^{p-1}$ are r-simplices of L such that $f(\sigma^0) = Tf(\sigma^1) = \cdots = T^{p-1} f(\sigma^{p-1})$, oriented so that $f(\sigma^0)$ is a positively oriented simplex of K. Suppose $\sigma^i = \langle a_0^i, \cdots, a_r^i \rangle$ for each i, with

$$f(a_j^0) = Tf(a_j^1) = \cdots = T^{p-1} f(a_1^{p-1})$$

for each j, noting that $\langle a_0^i, \cdots, a_r^i \rangle$ may or may not be the positive orientation for σ^i. Let $C(\sigma^0, \cdots, \sigma^{p-1})$ denote the simplex

$$\langle (a_0^0, \cdots, a_0^{p-1}), (a_1^0, \cdots, a_1^{p-1}), \cdots, (a_r^0, \cdots, a_r^{p-1}) \rangle$$

of $S^m \times \cdots \times S^m$. Then the collection J of all such simplices is a triangulation for A. We orient each $C(\sigma^0, \cdots, \sigma^{p-1})$ by prefixing it by $(-1)^j$, where j is the number of simplices in the set $\sigma^0, \cdots, \sigma^{p-1}$ for which $\langle a_0^i, \cdots, a_n^i \rangle$ is the negative orientation for σ^i.

LEMMA 4. *Let* C *be the* m-*chain consisting of the sum of all the* m-*simplices of* J, *oriented as above. Then* C *is a cycle.*

Proof. Let $C(\tau^0, \cdots, \tau^{p-1})$ be an $(m-1)$-simplex of J. We shall show that if it occurs in ∂C, then it occurs once with negative orientation for each time it occurs with a positive orientation. Let σ^j and η^j be the m-simplices that lie on either side of τ^j, for each j. Since τ^j inherits opposite orientations from σ^j and η^j, we may assume that

$$\tau^j = \langle a_0^j, \cdots, a_{m-1}^j \rangle, \qquad \sigma^j = \langle b^j, a_0^j, a_1^j, \cdots, a_{m-1}^j \rangle,$$

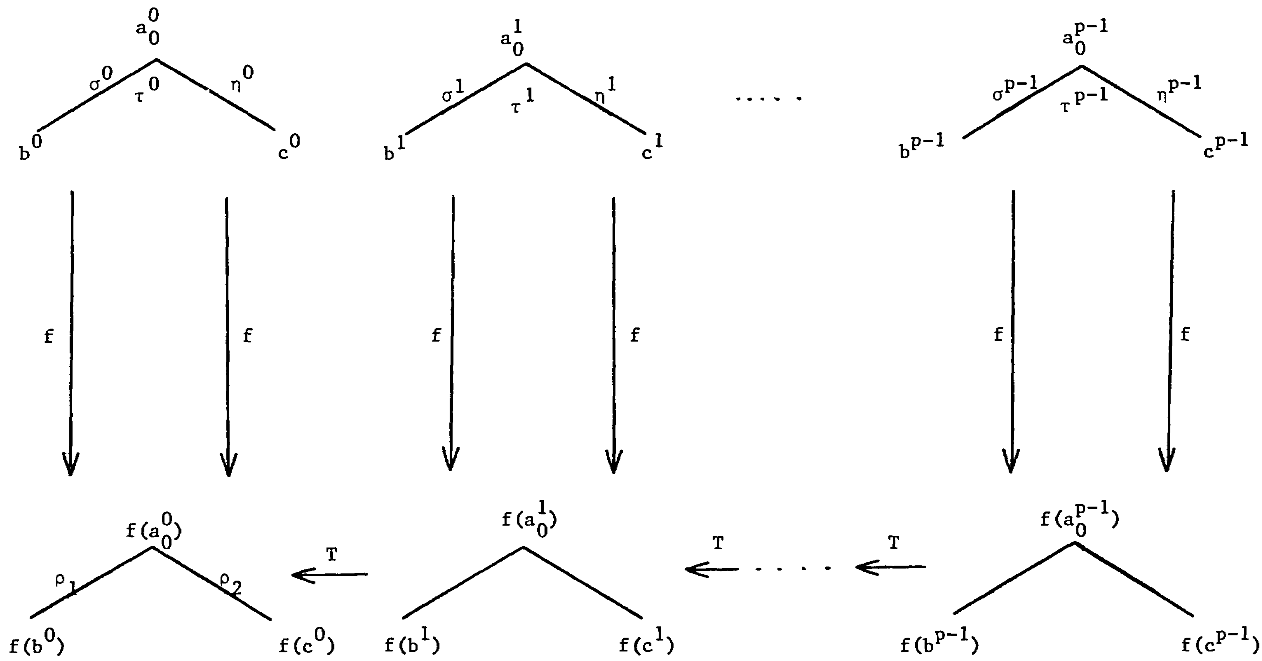
$$\eta_j = \langle a_0^j, c^j, a_1^j, \cdots, a_{m-1}^j \rangle,$$

and that these are the positive orientations for σ^j and η^j under $T^j f$ for all j. There are essentially three cases.

Case 1. The simplices $f(\sigma^0), \cdots, f(\sigma^{p-1}), f(\eta^0), \cdots, f(\eta^{p-1})$ are all distinct, and

$$f(\sigma^0) = Tf(\sigma^1) = \cdots = T^{p-1} f(\sigma^{p-1}) = \rho_1, \quad f(\eta^0) = Tf(\eta^1) = \cdots = T^{p-1} f(\eta^{p-1}) = \rho_2,$$

where ρ_1 and ρ_2 are positively oriented simplices of K. The following diagram illustrates this situation in the case m = 1. Note that $\tau^i = \langle a_0^i \rangle$.



Here C contains

$$C(\sigma^0, \cdots, \sigma^{p-1}) = \langle (b^0, \cdots, b^{p-1}), (a_0^0, \cdots, a_0^{p-1}), \cdots, (a_{m-1}^0, \cdots, a_{m-1}^{p-1}) \rangle$$

and

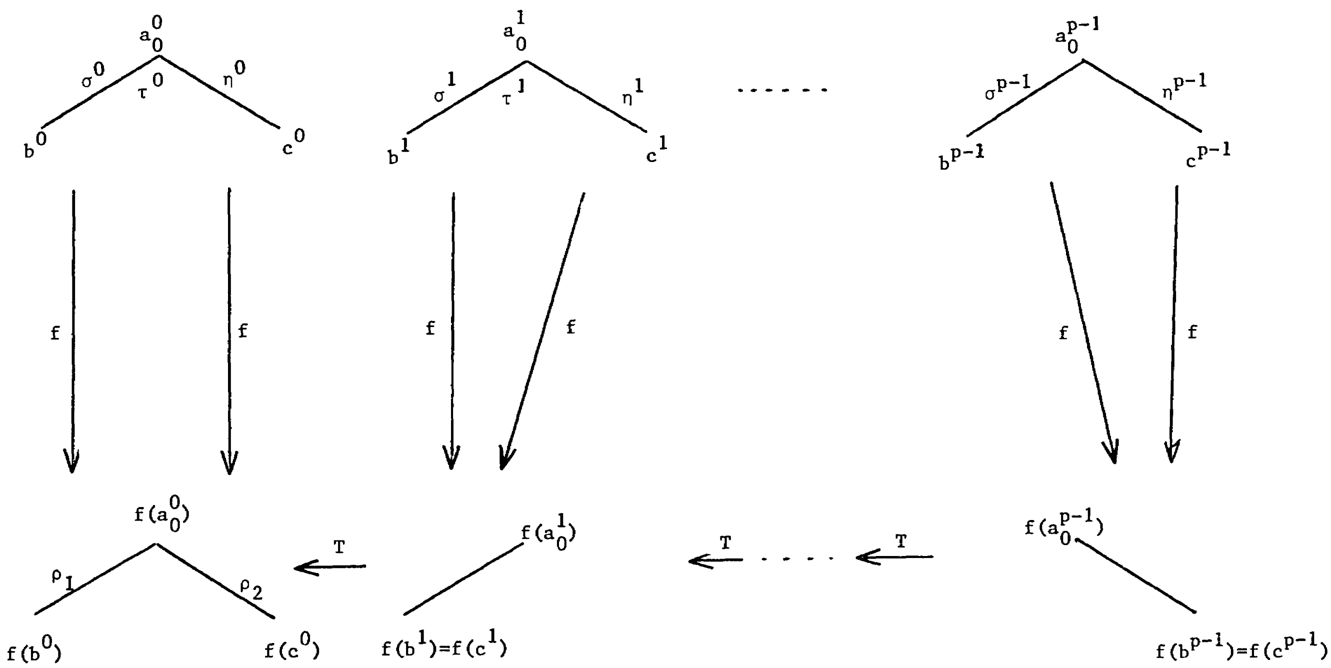
$$C(\eta^0, \cdots, \eta^{p-1}) = \langle (a_0^0, \cdots, a_0^{p-1}), (c^0, \cdots, c^{p-1}), (a_1^0, \cdots, a_1^{p-1}), \cdots, (a_{m-1}^0, \cdots, a_{m-1}^{p-1}) \rangle .$$

Therefore, in ∂C, $C(\tau^0, \cdots, \tau^{p-1}) = \langle (a_0^0, \cdots, a_0^{p-1}), \cdots, (a_{m-1}^0, \cdots, a_{m-1}^{p-1}) \rangle$ occurs once with each orientation and hence cancels out.

Case 2. The simplices $f(\sigma^0), \cdots, f(\sigma^{p-1}), f(\eta^0), \cdots, f(\eta^{p-1})$ are not all distinct, but it is not possible to choose $\xi^j = \sigma^j$ or $\xi^j = \eta^j$ for each $j = 0, \cdots, p-1$ such that

$$f(\xi^0) = Tf(\xi^1) = \cdots = T^{p-1} f(\xi^{p-1}).$$

In this diagram $f(\sigma^1) = f(\eta^1)$ and $f(\sigma^{p-1}) = f(\eta^{p-1})$, but $T^{p-1} f(\xi^{p-1}) \neq Tf(\xi^1)$ regardless of the choices of ξ^{p-1} and ξ^1.



Here $C(\tau^0, \cdots, \tau^{p-1})$ does not occur at all in ∂C.

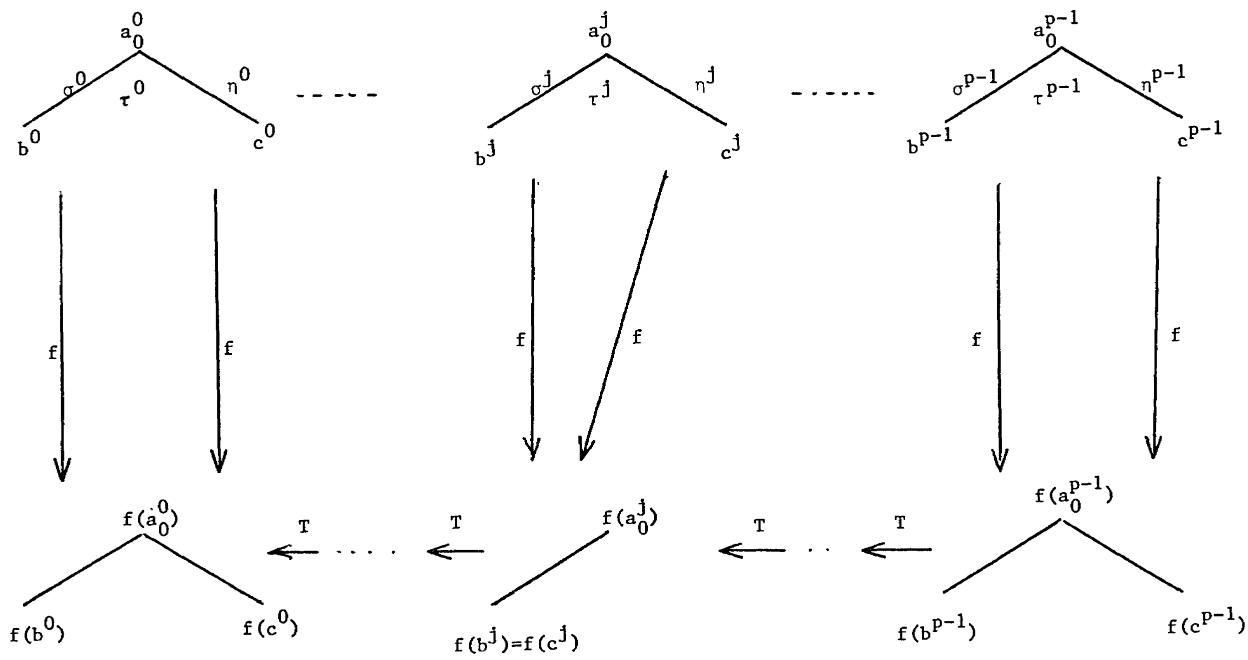
Case 3. There is at least one j such that

(a) $$f(\sigma^0) = Tf(\sigma^1) = \cdots = T^j f(\sigma^j) = T^j f(-\eta^j) = \cdots = T^{p-1} f(\sigma^{p-1})$$

or

(b) $$f(\eta^0) = Tf(\eta^1) = \cdots = T^j f(\eta^j) = T^j f(-\sigma^j) = \cdots = T^{p-1} f(\sigma^{p-1}) .$$

(Note that the relation $T^j f(\sigma^j) = T^j f(\eta^j)$ is impossible.) It suffices to assume that (a) occurs and that it occurs for only one value j.



Then the simplices

$$C(\sigma^0, \cdots, \sigma^j, \cdots, \sigma^{p-1}) = \left\langle (b^0, \cdots, b^j, \cdots, b^{p-1}), (a_0^0, a_0^1, \cdots, a_0^{p-1}), \cdots, (a_{m-1}^0, \cdots, a_{m-1}^{p-1}) \right\rangle$$

and

$$C(\sigma^0, \cdots, -\eta^j, \cdots, \sigma^{p-1}) = -\left\langle (b^0, \cdots, c^j, \cdots, b^{p-1}), (a_0^0, a_0^1, \cdots, a_0^{p-1}), \cdots, (a_{m-1}^0, \cdots, a_{m-1}^{p-1}) \right\rangle$$

occur in C. (The minus sign occurs because $\langle c^j, a_0^j, \cdots, a_{p-1}^j \rangle$ is the negative orientation for η^j.) In the boundary of each of these simplices, the simplex

$$C(\tau^0, \cdots, \tau^{p-1}) = \left\langle (a_0^0, \cdots, a_0^{p-1}), \cdots, (a_{m-1}^0, \cdots, a_{m-1}^{p-1}) \right\rangle$$

occurs with the same orientation but cancels out because of the sign difference. This concludes the proof that C is a cycle.

Consider the $\mathbb{Z}_p$-action on $S^m \times \cdots \times S^m$ generated by the homeomorphism h, where h is defined by the equation $h(x_0, \cdots, x_{p-1}) = (x_{p-1}, x_0, \cdots, x_{p-2})$. This action is free when restricted to the invariant subspace A, and the cycle C is invariant under this action. Therefore C represents a cycle C in $A/\mathbb{Z}_p$. We shall now show that C has index m.

LEMMA 5. *Let* K *and* L *be homology* m*-spheres, and let* $f: K \to L$ *be a nondegenerate simplicial map of degree* d. *Then, if* $f_\#: C(K) \to C(L)$ *is the induced chain map and* τ *is an* m*-simplex of* L, *there exist* d *simplices that are mapped onto* τ *in the sense that* $f_\#^{-1}(\tau)$ *is a chain whose coefficients add up to* d.

Proof. Let $\sum 1 \cdot \sigma_i$ and $\sum 1 \cdot \tau_i$ be the fundamental cycles of K and L, respectively. Then $f_\#\left(\sum 1 \cdot \sigma_i\right)$ is homologous to $d \sum 1 \cdot \tau_i$; but since there are no $(m+1)$-chains, we see that $f_\#\left(\sum 1 \cdot \sigma_i\right) = d \sum 1 \cdot \tau_i$. Therefore

$$\sum_{f_\#(\sigma_i) = \pm\tau_{i_0}} f_\#(\sigma^i) = d\tau_{i_0},$$

and the result follows.

THEOREM 5. C *is a cycle of index* m.

Proof. Define a simplicial map ϕ of J onto L by the equation

$$\phi(C(\sigma^0, \cdots, \sigma^{p-1})) = f(\sigma^0).$$

Then ϕ is equivariant with respect to the actions generated by h on A and T on S^m. It is easy to see that since $f^{-1}(\sigma^0)$ contains d simplices in the sense of Lemma 5, the simplex $f(\sigma^0)$ will occur d^p times in $\sigma_\#(C)$. Since we are assuming that d is not a multiple of p, this says that $\sigma_\#(C)$ is a nonzero multiple of the fundamental cycle of S^m. The theorem now follows from Proposition 1 and Corollary 1.

Proof of Theorem 1. Suppose the conclusion is false. Then $g(x) \neq g(y)$, for every pair (x, y) of points in S^m such that $f(x) = Tf(y)$. Thus, if $(x_0, \cdots, x_{p-1}) \in A$, we see that $g(x_0) \neq g(x_1)$ since $f(x_0) = Tf(x_1)$, and $g(x_1) \neq g(x_2)$ since $Tf(x_1) = T^2 f(x_2)$ implies $f(x_1) = Tf(x_2)$, and so forth. Therefore the map

$\psi: A \to (R^n)^p$ defined by $\psi(x_0, \cdots, x_{p-1}) = (g(x_0), \cdots, g(x_{p-1}))$ is actually a map into $F'(R^n, p)$, equivariant with respect to the actions that cyclically permute the coordinates. Therefore $\psi: A/\mathbb{Z}_p \to F'(R^n, p)/\mathbb{Z}_p$ is defined, and $\psi_*([C])$ has index m, which contradicts Corollary 2 if $m > (n - 1)(p - 1)$.

Proof of Theorem 2. The argument is the same as above, except that the space $F'(R^n, p)$ is replaced by the space $(R^n)^p - \Delta_{R^n}$, which has the homotopy type of $S^{n(p-1)-1}$.

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Northern Illinois University
DeKalb, Illinois 60115