COUNTEREXAMPLES FOR THE SPACE OF MINIMAL SOLUTIONS OF THE EQUATION $\Delta u = Pu$ ON A RIEMANN SURFACE

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1. Let $P \geq 0$ $(P \not\equiv 0)$ be a C^1-density on an open Riemann surface R. The space of nonnegative C^2-solutions on R of the elliptic equation $\Delta u = Pu$ is denoted by PN(R). A function $u \in PN(R)$ is said to be PN-minimal if for every $v \in PN(R)$ with $0 \leq v \leq u$, there exists a constant c_v such that $v = c_v u$ on R.

It was established in a former work by the second author [6] that *if the space* PB(R) *of bounded* C^2*-solutions of* $\Delta u = Pu$ *is of dimension at least* 2, *then every* PN-*minimal function on* R *has zero infimum*. The purpose of this note is to demonstrate that the conclusion is *no longer* valid in the remaining two cases: dim PB(R) = 0 or 1.

2. First consider the case dim PB(R) = 0. Take R to be the complex plane. It is well-known that dim PB(R) = 0 since R is parabolic (see H. L. Royden [4]). For a constant $M \geq 2$ and the density

$$P(z) = M^2 |z|^{M-2}(1 + |z|^M)^{-1} \tag{1}$$

on R, it is not difficult to see that the function $v(z) = |z|^M + 1$ belongs to the class PN(R).

We claim that v(z) is PN-minimal on R. A bit more strongly, it is true that PN(R) is generated by v(z). For a function $u \in PN(R)$, set $\phi = u \cdot v^{-1}$. We need to show that ϕ is a constant. In view of the conditions $\Delta u = Pu$ and $\Delta v = Pv$, the function ϕ must satisfy the partial differential equation

$$\Delta\phi + \frac{2M|z|^{M-2}}{|z|^M + 1}\left(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}\right) = 0$$

on R. It follows from Liouville's theorem (see for example M. Prother and G. Weinberger [3, p. 120]) that every nonnegative solution of this equation is a constant. Thus the plane R with the density (1) carries a PN-minimal function $v(z) = |z|^M + 1$ (≥ 1), although dim PB(R) = 0.

3. Turning now to the case dim PB(R) = 1, we base our argument on the remarkable examples of Y. Tôki [7], [8] (see also L. Sario [5]) of a hyperbolic Riemann surface carrying no nonconstant positive harmonic functions. We take such a surface $R \in O_{HB} - O_G$ and construct a C^1-density $Q \geq 0$ $(Q \not\equiv 0)$ on R such that $\int_R Q(z)\,dx\,dy < \infty$. Thus HB(R) and QB(R) are isomorphic (Royden [4], also M. Nakai [1]), which implies that dim QB(R) = 1. Moreover, dim HBD(R) = 1 implies

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dim QBD(R) = 1 (Royden [4], also Nakai [2]). Here H and D stand for harmonicity and Dirichlet-finiteness, respectively.

Choose a function $u \in QBD(R)$ $(u > 0)$, and set $v = e^u$. A simple calculation shows that $v \in PB(R)$, where

$$P = Qu + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \geq 0 \qquad (P \not\equiv 0)$$

is a C^1-density on R. Since $0 < u \leq M$ for some constant $M < \infty$ and the Dirichlet integral D(u) of u is finite, it follows that

$$\int_R P\,dx\,dy = \int_R \left[Qu + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dx\,dy \leq M \int_R Q\,dx\,dy + D(u) < \infty .$$

As a consequence, PB(R) is isomorphic to HB(R), and hence dim PB(R) = 1. Therefore $v \in PB(R)$ is PN-minimal on R, and $v \geq 1$, as desired.

We conjecture that as in the case dim PB(R) = 0, there exists a pair (R, P) for which dim PB(R) = 1 and every PN-minimal function on R has a positive infimum. But we are unable to prove this.

4. In summary, we have established the following result.

THEOREM. *There exist a Riemann surface* R_i *and a density* P_i *such that* $\dim P_iN(R_i) = i$ $(i = 0, 1)$ *and* R_i *carries a* P_iN-*minimal function with positive infimum.*

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