A ROTUND REFLEXIVE SPACE HAVING A SUBSPACE OF CODIMENSION TWO WITH A DISCONTINUOUS METRIC PROJECTION

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If E is a strictly convex (rotund) and reflexive Banach space and L is a closed linear subspace of E, then L is a Chebyshev subspace of E; that is, corresponding to each point x in E there exists in L a unique point $P_L(x)$ that is nearest to x. The *metric projection* of E onto L is the mapping P_L. In [2], the authors constructed a strictly convex but nonreflexive Banach space possessing a linear subspace of codimension 2 whose metric projection is discontinuous. They conjectured that if L is a closed subspace of a strictly convex reflexive space, then P_L must be continuous. The conjecture is false even in spaces equivalent to a Hilbert space. An elegant counterexample was constructed by B. Kripke, and independently the present writer constructed a more complicated example, some features of which are more general. In both examples, the subspace L is of infinite codimension, and Ivan Singer, in a private communication, asked whether an example with a subspace L of codimension 2 could be constructed. Here we construct such an example by modifying our original method.

THEOREM. *There exist a strictly convex, reflexive, and separable real Banach space* E *and closed linear subspaces* L *and* M, *with* $L \subseteq M$, *having the properties*

(1) P_L *is discontinuous,*

(2) L *is of codimension* 2 *in* E, *and*

(3) M *is of codimension* 1 *in* E *and is a Hilbert space with respect to the norm of* E.

The construction of E depends upon a lemma asserting the existence of strictly convex norms with prescribed properties.

LEMMA. *Let* F *be a real linear space, and let* p_1 *and* p_2 *be two equivalent norms on* F *with respect to which* F *is separable. If* $p_1(x) \leq p_2(x)$ *for all* $x \in F$ *and the set* $\{x \in F: p_1(x) = p_2(x) = 1\}$ *contains no nondegenerate line segment, then there exists a strictly convex norm* p *on* F *with* $p_1(x) \leq p(x) \leq p_2(x)$ *for all* $x \in F$.

Proof. Throughout the proof, there will be a single topology on F, the norm topology determined by p_1 and p_2.

Suppose that y is a point of the open set $V = \{x \in F: p_1(x) < p_2(x)\}$. The first step in the proof is to show that there exists a norm p_y between p_1 and p_2 that is 'strictly convex near y'. Replacing y by a multiple, we shall suppose that $p_1(y) < 1 < p_2(y)$. Since F is p_2-separable, and by a well known result of J. A. Clarkson [1], there exists a strictly convex norm q on F that is equivalent to p_2. The norm p_y will be obtained as a modification of q.

Let $f \in F^*$ be a continuous linear functional on F that has p_2-norm equal to 1 and attains its norm at y: that is, such that $|f(x)| \leq p_2(x)$ for all $x \in F$, and $f(y) = p_2(y)$. Let

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$$k = \sup\{q(x)/p_2(x): x \in F \setminus \{0\}\} \quad \text{and} \quad \theta = \frac{1}{2k}\left(1 - \frac{1}{p_2(y)}\right).$$

Define q_y on F by

$$q_y(x) = q_y\left(\frac{f(x)}{f(y)}y + \left(x - \frac{f(x)}{f(y)}y\right)\right) = \left(\left(\frac{f(x)}{f(y)}\right)^2 + \theta^2 q\left(x - \frac{f(x)}{f(y)}y\right)^2\right)^{1/2}.$$

Then (i) $q_y(y) = 1$, (ii) q_y is a strictly convex norm on F, and (iii) $q_y(x) \le p_2(x)$ for all $x \in F$. To prove (iii), we use the properties of f to obtain the inequalities

$$q_y(x) \le \left|\frac{f(x)}{f(y)}\right| q_y(y) + q_y\left(x - \frac{f(x)}{f(y)}y\right) \le \left|\frac{f(x)}{f(y)}\right| + \theta\left(q(x) + \left|\frac{f(x)}{f(y)}\right| q(y)\right)$$

$$\le p_2(x)\left(\frac{1}{p_2(y)} + 2\theta k\right) \le p_2(x).$$

Now let $p_y = q_y \vee p_1$. Then $p_1(x) \le p_y(x) \le p_2(x)$ for all $x \in F$, so that p_y is a norm equivalent to p_1 and p_2. Furthermore, $p_y(y) = q_y(y)$. The norms p_1, p_2, p_y are all continuous in the topology of F, and therefore the set

$$V_y = \{x \in F: p_1(x) < p_y(x) < p_2(x)\}$$

is an open subset of F containing y. If $x \in V_y$, then $p_y(x) = q_y(x)$.

The set V is an open subset of the separable metric space F. It thus has a countable base for its topology, and therefore it is a Lindelöf space—that is, every open cover has a countable subcover. Therefore there exists a sequence $(x(n))_{n \ge 1}$ such that $V = \bigcup_{n=1}^{\infty} V_{x(n)}$. Define p by

$$p(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} p_{x(n)}(x).$$

Then p is a norm on F, and it satisfies the conditions of the lemma. Suppose, to the contrary, that p is not strictly convex. Then the set $\{x: p(x) = 1\}$ contains a nondegenerate line segment. The norms p_1, p_2, and p coincide on $F \setminus V$, and therefore, by the condition of the lemma, the line segment must intersect V. It now follows that, for some m, the set $V_{x(m)} \cap \{x: p(x) = 1\}$ contains a nondegenerate line segment, $[y, z]$ say. Then, by the equality of $p_{x(m)}$ and $q_{x(m)}$ on $V_{x(m)}$ and by the strict convexity of $q_{x(m)}$,

$$1 = p\left(\tfrac{1}{2}(y+z)\right) = \frac{1}{2^m} q_{x(m)}\left(\tfrac{1}{2}(y+z)\right) + \sum_{n \ne m} \frac{1}{2^n} p_{x(n)}\left(\tfrac{1}{2}(y+z)\right)$$

$$< \frac{1}{2^m}\left(\tfrac{1}{2} q_{x(m)}(y) + \tfrac{1}{2} q_{x(m)}(z)\right) + \sum_{n \ne m} \frac{1}{2^n}\left(\tfrac{1}{2} p_{x(n)}(y) + \tfrac{1}{2} p_{x(n)}(z)\right) = 1,$$

which is a contradiction. This completes the proof of the lemma.

The construction of an example. Let E be the space ℓ^2 of real sequences,

$$E = \left\{ x = (\xi_k)_{k \geq 1} : \sum_{k=1}^{\infty} \xi_k^2 < \infty \right\},$$

and let $e_1, e_2, \cdots$ be the standard basis. Let $M = \{(\xi_k) \in E: \xi_1 = 0\}$. It will be notationally convenient to write $M = M_2$. For $k = 3, 4, \cdots$, let $M_k = \mathrm{sp}\{e_1, e_k\}$. We shall obtain an example satisfying the statement of the theorem, by giving E a suitable norm q equivalent to the ℓ^2-norm. Throughout the discussion, the topology on E is that of the ℓ^2-norm. In the first place, we shall define the norm q on the subspaces M_k $(k = 2, 3, \cdots)$, and we shall then invoke the lemma to obtain an extension to the whole of E.

For $0 \leq t \leq 1/3$,

$$1 + \frac{1}{2}t \leq (1+t)^{1-t} \leq 1 + t.$$

Therefore we can inductively define positive real sequences $(\delta(k))$, $(p(k))$, $(\alpha(k))$, $(\lambda(k))$ $(k = 3, 4, \cdots)$ by

$$\delta(3) = 1/8, \qquad p(3) = 16,$$

and, for $k = 3, 4, \cdots$,

(1) $$\alpha(k) = p(k)^{-1/p(k)},$$

(2) $$\lambda(k) = \left(1 + \frac{1}{p(k)}\right)^{-1+1/p(k)},$$

(3) $$\delta(k+1) = \frac{1}{4}\delta(k)\, p(k) \left(\frac{1}{\lambda(k)} - 1\right),$$

(4) $$p(k+1) = \delta(k+1)^{-p(k)}.$$

We also put $q(k) = p(k)/(p(k) - 1)$.

It follows that

(5) $$\delta(k+1) \leq \frac{1}{4}\delta(k) \quad \text{and} \quad p(k) \leq p(k+1).$$

Therefore the sequences $(p(k))$, $(\alpha(k))$, and $(\lambda(k))$ are increasing and $(\delta(k))$ is decreasing, with

(6) $$p(k) \to \infty, \quad \alpha(k) \to 1, \quad \delta(k) \to 0, \quad p(k)\,\delta(k) \to \infty \quad \text{as } k \to \infty.$$

Furthermore

(7) $$\frac{1}{2} < \alpha(k), \quad \frac{\delta(k)}{\alpha(k)} \leq \frac{1}{4}, \quad 2 \leq p(k)\,\delta(k), \quad \sum_{k=3}^{\infty} \left(\frac{\delta(k)}{\alpha(k)}\right)^2 \leq \frac{1}{15},$$

and, since $1/p(k) \leq 1/p(3) \leq 1/3$,

(8) $$\lambda(k) \leq \left(1 + \frac{1}{2p(k)}\right)^{-1} < 1.$$

Now for $k = 3, 4, \cdots$ put $x_k = e_1 + \delta(k)e_2$ and $y_k = \delta(k)e_2 + \alpha(k)e_k$, and let L be the closed linear hull of $\{y_k: k = 3, 4, \cdots\}$.

For $k = 2, 3, \cdots$ define seminorms p_k on E by

$$p_2(x) = \left(\sum_{k=2}^{\infty} \xi_k^2\right)^{1/2},$$

$$p_k(x) = (|\xi_1|^{p(k)} + |\xi_k|^{p(k)})^{1/p(k)} \quad (k = 3, 4, \cdots),$$

where $x = (\xi_k) \in E$. Thus the restriction of p_2 to M_2 is the ℓ^2-norm, and the restriction of p_k to M_k $(k = 3, 4, \cdots)$ is the $\ell^{p(k)}$-norm on M_k. These norms are strictly convex, and for all k and m, the seminorms p_k and p_m coincide on $M_k \cap M_m$.

Linear functionals f_k $(k = 3, 4, \cdots)$ and a strictly convex norm q will be defined on E so that they have the following properties: for $k = 2, 3, \cdots$,

$$q(x) = p_k(x) \quad \text{for all } x \in M_k, \tag{9}$$

$$q(x) \geq p_k(x) \quad \text{for all } x \in E, \tag{10}$$

and for $k = 3, 4, \cdots$,

$$q(x) \geq |f_k(x)| \quad \text{for all } x \in E, \tag{11}$$

$$f_k(x) = 0 \quad \text{for all } x \in L, \tag{12}$$

$$f_k(x_k - y_k) = p_k(x_k - y_k). \tag{13}$$

Suppose that such q and f_k have been constructed. Then, writing P_L for the metric projection onto L associated with the norm q, we deduce from (11), (12), and (13) that $P_L(x_k) = y_k$. However, by (10) $(k \geq 3)$ and the definition of p_k we see that $P_L(e_1) = 0$. Therefore, by (6), P_L is discontinuous at e_1.

The $\ell^{p(k)}$-norms are smooth, and therefore the linear functional f_k is determined on M_k by (9), (11), and (13), and then upon E by (12). Let f_k $(k = 3, 4, \cdots)$ be defined by

$$f_k(x) = \lambda(k)\xi_1 + \frac{\lambda(k)}{p(k)\,\delta(k)}\left(\xi_2 - \sum_{m=3}^{\infty} \frac{\delta(m)}{\alpha(m)}\xi_m\right)$$

for $x = (\xi_k) \in E$. It follows from the last inequality of (7) that f_k is a well-defined continuous linear functional on E, and we can verify that (12) and (13) are satisfied.

Next we obtain inequalities for the norms of the restrictions of the functionals f_k $(k = 3, 4, \cdots)$ to the normed linear spaces M_m $(m = 2, 3, \cdots)$.

There is a constant $\theta < 1$ *such that, for all* $x \in M_2$ *and* $k = 3, 4, \cdots$,

$$|f_k(x)| \leq \theta\, p_2(x). \tag{14}$$

If $x \in M_k$, *then*

$$(15) \qquad |f_k(x)| \leq p_k(x).$$

There exist constants $\theta_k < 1$ $(k = 3, 4, \cdots)$, *such that if* $x \in M_m$ *and* $m \neq k$, *then*

$$(16) \qquad |f_k(x)| \leq \theta_k p_m(x).$$

It follows from the Schwarz inequality, (8), and the last two inequalities of (7) that (14) is satisfied with $\theta = \frac{1}{2}\left(\frac{16}{15}\right)^{1/2}$

If $x \in M_m$, then, by Hölder's inequality,

$$|f_k(x)| \leq \theta(k, m)\, p_m(x),$$

where

$$\theta(k, m) = \lambda(k)\left(1 + \left(\frac{\delta(m)}{p(k)\,\delta(k)\,\alpha(m)}\right)^{q(m)}\right)^{1/q(m)}$$

If $m = k$, then $\theta(k, m) = 1$, and we have (15). We shall show that (16) is satisfied with $\theta_k = (1 + 4p(k))(2 + 4p(k))^{-1}$. The cases $m < k$ and $k < m$ must be considered separately. For $m > k \geq 3$ we obtain from the triangle inequality for the $\ell^{q(m)}$-norm, from the first inequality of (7), from the monotonicity of $(\delta(k))$, and from (3) and (8) the inequalities

$$\theta(k, m) \leq \lambda(k)\left(1 + \frac{\delta(m)}{\alpha(m)}\frac{1}{p(k)\,\delta(k)}\right) \leq \lambda(k)\left(1 + \frac{2\delta(k+1)}{p(k)\,\delta(k)}\right) = \frac{1}{2}(1 + \lambda(k)) \leq \theta_k.$$

For $3 \leq m < k$, we use the second inequality of (7), the monotonicity of $(p(k))$, and (8) to deduce that

$$\begin{aligned}\theta(k, m) &\leq \lambda(k)\left(\frac{\theta(k, m)}{\lambda(k)}\right)^{q(m)} \\ &= \lambda(k)\left(1 + \left(\frac{\delta(m)}{\alpha(m)}\right)^{q(m)}\left(\frac{1}{p(k)}\right)^{q(m)}\left(\frac{1}{\delta(k)^{p(m)}}\right)^{q(m)/p(m)}\right) \\ &\leq \lambda(k)\left(1 + \frac{1}{4}\left(\frac{1}{p(k)}\right)^{q(m)}\left(\frac{1}{\delta(k)^{p(k-1)}}\right)^{q(m)/p(m)}\right) = \lambda(k)\left(1 + \frac{1}{4p(k)}\right) \leq \theta_k.\end{aligned}$$

Thus (14), (15), and (16) are satisfied.

Now let q_1 be the norm on E defined by

$$q_1(x) = \sup \{p_k(x) : k = 2, 3, \cdots\} \cup \{|f_k(x)| : k = 3, 4, \cdots\}.$$

The inequalities (14), (15) and (16) ensure that q_1 and p_k coincide on M_k for $k = 2, 3, \cdots$. Later we shall use the fact that for each $x \in E$ the supremum in the definition of q_1 is attained (because $|\xi_1| = \lim_{k\to\infty} |f_k(x)| = \lim_{k\to\infty} p_k(x)$).

Let A be the closed convex hull of $\left\{x \in \bigcup_{k=2}^{\infty} M_k : q_1(x) \leq 1\right\}$, and let q_2 be the norm for which A is the unit ball. Then $q_2(x) \geq q_1(x)$ for all $x \in E$ and $q_1(x) = q_2(x)$ for $x \in \bigcup_{k=2}^{\infty} M_k$. Clearly, q_1 and q_2 are equivalent norms. If it is shown that

$$\{x: q_1(x) = q_2(x)\} = \bigcup_{k=2}^{\infty} M_k,$$

then it will follow by the lemma that there exists an equivalent strictly convex norm q on E such that $q(x) \geq q_1(x)$ for all $x \in E$ (from which it follows that (10) and (11) are satisfied), and such that (9) is satisfied. The proof of the theorem will then be complete.

Suppose that $y = (\eta_k) \in A$ and $q_1(y) = 1$ but that $y \notin \bigcup_{k=2}^{\infty} M_k$. Then $0 < |\eta_1| < 1$.

There exists a functional f *such that* $f(y) = 1$,

$$|f(x)| \leq q_1(x) \quad \textit{for all } x \in E,$$

and such that for some k *and some* $\theta' < 1$

$$|f(x)| \leq \theta' p_m(x) \quad \textit{for all } x \in M_m \textit{ if } m \neq k.$$

Since the supremum in the definition of q_1 is attained, there must exist a k for which either $|f_k(y)| = 1$ or $p_k(y) = 1$. If $|f_k(y)| = 1$ for some k, then we can take $f = f_k$ and $\theta' = \theta(k)$. If $p_2(y) = 1$, take $k = 2$, define f by

$$f(x) = \sum_{m=2}^{\infty} \eta_m x_m,$$

and take $\theta' = (1 - \eta_1^2)^{1/2}$. If $p_k(y) = 1$ for some $k \geq 3$, then with this k define f by

$$f(x) = \operatorname{sgn} \eta_1 \, |\eta_1|^{p(k)-1} \xi_1 + \operatorname{sgn} \eta_k \, |\eta_k|^{p(k)-1} \xi_k$$

and take $\theta' = (1 - |\eta_1|^{p(k)})^{1/q(k)}$.

We can now obtain a contradiction. For each $\varepsilon > 0$ we can find a finite convex combination $\sum_{m \geq 2} \beta_m z_m$ with $z_m \in M_m$, $q_1(z_m) \leq 1$ for $m \geq 2$, and $q_1\left(y - \sum \beta_m z_m\right) < \varepsilon$. Then

$$1 = |f(y)| \leq \left|f\left(y - \sum \beta_m z_m\right)\right| + \left|f\left(\sum \beta_m z_m\right)\right|$$

$$\leq \varepsilon + \sum_{m \neq k} \beta_m \theta' + \beta_k = \beta_k(1 - \theta') + \theta' + \varepsilon,$$

so that $\beta_k \geq 1 - \dfrac{\varepsilon}{1 - \theta'}$ and

$$q_1(y - \beta_k z_k) = q_1\left(\left(y - \sum \beta_m z_m\right) + \sum_{m \neq k} \beta_m z_m\right) \leq \varepsilon + \sum_{m \neq k} \beta_m$$

$$= \varepsilon + (1 - \beta_k) \leq \varepsilon \, \frac{2 - \theta'}{1 - \theta'}.$$

It now follows that y is in the closure of M_k, so that it is in M_k; this is a contradiction.

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