OBSTRUCTION THEORY IN THE STABLE RANGE

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Obstruction theory is the branch of algebraic topology that studies the problems of extending or classifying maps, from a complex X to a space Y, one dimension at a time. Implicit in the fundamental work of H. Whitney [9], it was first codified officially by S. Eilenberg [2]. The most general formulation was given by P. Olum [7].

One basic concept in the classical obstruction theory is that of the obstruction set $\mathscr{O}^n(X, Y)$ (see for example p. 181 of [4]). This set consists of the set of all possible obstruction classes, that is, classes in $H^n(X; \Pi_{n-1}(Y))$ that arise as obstructions to the extension of maps from X^{n-1} to Y (here X^i means the i-skeleton of the complex X) to the n-skeleton. In general, the literature contains little information about obstruction sets. It seems that these sets have little interesting structure, without some special assumptions.

The purpose of the present paper, which concerns the theory rather than computations, is to show that the situation is radically different in the stable case. We show that when we stabilize with respect to suspension, the obstruction sets are filtered groups, so that the total obstruction set

$$\mathscr{O}_S^*(X, Y) = \sum_n \mathscr{O}_S^n(X, Y)$$

is a filtered, graded group. It is a subgroup of the total cohomology group

$$\sum_n H^n(X; \Pi_{n-1}^S(Y)),$$

where $\Pi_j^S(Z)$ means the jth-stable homotopy group of Z. Furthermore, if we consider this filtered, graded group for all suspensions of X,

$$\sum_j \mathscr{O}_S^*(\Sigma^j X, Y),$$

where j varies over the integers (including negative integers), and where Σ^j means, of course, the j-fold reduced suspension, we have the structure of a graded, filtered module over G_*, the stable homotopy ring of spheres. This module structure is compatible with the G_*-module structure on

$$\sum_j \{\Sigma^j X, Y\},$$

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where $\{A, B\}$ means the group of stable homotopy classes of maps from A to B. We note that this group also has a filtration, defined in terms of the skeleta of A, that relates to the filtration on the stable obstruction sets.

The methods for introducing all this structure consist in interpreting obstruction theory in terms of the spectral sequence, converging to the groups

$$\sum_j \{\Sigma^j X, Y\},$$

which in turn arises from the filtration of X by skeleta, or equivalently, by filtering Y by successive fibres in a Postnikov system (as originally done by F. P. Peterson [8]). We review the spectral sequence in Section 1. Our basic results on obstructions are in Section 2, and we give the applications in Section 3. The first applications are elementary, dealing with the question when a map $X^{n-1} \to Y$ that does not extend to X^n is decomposable in terms of the G_*-action. We give an exact sequence for obstructions in the top dimension; it implies some numerical estimates. The deeper applications, however, relate to the well-known *generating hypothesis* of P. Freyd [3]. We use our theory to give new consequences of the generating hypothesis, unproved at this time, in terms of the G_*-action on $\sum_j \{\Sigma^j X, Y\}$. The consequences here (Theorem 3 and Corollaries) are all interesting global statements in stable homotopy.

Parallel to this, we have an obstruction theory to homotopy, which in this case can be reduced to obstructions to the formation of null-homotopies. One can also show that if the generating hypothesis is true, the entire obstruction theory in the stable range can be reduced to the study of G_*-modules and their homomorphisms (see [5]). On the other hand, the relative theory does not have these nice algebraic properties, with subgroups becoming cosets. Here we shall leave details to the reader.

We note that much of the additive part of these results follows quickly from the beautiful treatment of A. Dold (see [1]).

1. We shall work in the category of finite CW-complexes that are connected and have base points (omitted from the notation). We write ΣX for the reduced suspension of X, $\{X, Y\}$ for the groups of stable homotopy classes of maps from X to Y. Note that $\{\Sigma^i X, \Sigma^j Y\}$ is defined for all integers i and j.

Definition. $A^{p,q}(X, Y) = \{\Sigma^{-p-q} X^p, Y\}$, $B^{p,q}(X, Y) = \{\Sigma^{-p-q}(X^p/X^{p-1}), Y\}$. Here X^p means the p-skeleton, which is assumed to be the base point if p is negative. Using the maps from the cofibrations

$$X^{p-1} \to X^p \to X^p/X^{p-1},$$

we get an exact couple, denoted by

$$\mathscr{C}(X, Y) = \langle A, B, f, g, h \rangle .$$

The maps have bidegrees (-1, 1), (1, 0), and (0, 0), respectively. The elementary properties of this couple are summarized in the following theorem (see [5] for details).

THEOREM 1. (a) $B^{p,q}(X, Y) = C^p(X; \Pi^S_{-q}(Y))$.

(b) *The homomorphism* d_1 *is the usual coboundary.*

(c) *The construction is functorial in* X *and* Y. *The induced maps on the first derived group*

$$B_2^{p,q} = H^p(X; \Pi_{-q}^S(Y))$$

are the usual induced maps; the coefficient homomorphisms associated with the induced maps on stable homotopy groups express the functoriality in the second variable.

(d) *The entire exact couple has the structure of a couple of graded modules over* G_*. *The action is defined by the action of* $\{\alpha\} \in G_p$ *on* $\{f\} \in \{X, Y\}$ *by the composition*

$$\Sigma^{n+m+p} X = \Sigma^m X \mathbin{\#} S^{n+p} \xrightarrow{i(f \mathbin{\#} \alpha)} \Sigma^m Y \mathbin{\#} S^n = \Sigma^{m+n} Y,$$

where $i = (-1)^{p \cdot m}$.

Associated with the exact couple $\mathscr{C}(X, Y)$, we have a convergent spectral sequence whose first invariant term is

$$E_2^{p,q} = H^p(X; \Pi_{-q}^S(Y));$$

the sequence converges to the graded group associated with $\sum_j \{\Sigma^j X, Y\}$. Because this is zero for sufficiently small j, the entire spectral sequence lies in the upper half-plane and vanishes to the right of a suitable line.

2. We wish to identify the obstruction classes in terms of our spectral sequence.

THEOREM 2. *Suppose* X *and* Y *are* (j - 1)*-connected and*

$$\max(\dim X, \dim Y) < 2j - 2.$$

The class $x \in H^n(X; \Pi_{n-1}^S(Y))$, *with* $x \neq 0$, *is the obstruction to the extension of a map* $f: X^{n-1} \to Y$ *to the* n*-skeleton of* X *if and only if there exists a* $y \in H^{n-k}(X; \Pi_{n-k}^S(Y))$ *such that in our spectral sequence*

(1) $d_j\{y\} = 0 \qquad (j < k)$,

(2) $d_k\{y\} = \{x\} \qquad$ *in* $E_k^{*,*}$.

(Note that $\Pi_{n-k}(Y) = \Pi_{n-k}^S(Y)$.)

Proof. Suppose that the condition is satisfied and y is represented by a map

$$\beta: X^{n-k} \to Y$$

that factors through the map

$$\beta': X^{n-k}/X^{n-k-1} \to Y,$$

with β' representing the cohomology class y. The first condition assures us that β' lifts to $\beta'': X^{n-1} \to Y$, and the second condition says that if we compose on the left with $\Sigma^{-1}(X^n/X^{n-1}) \xrightarrow{\delta} X^{n-1}$, we get a map

$$\beta''': X^n/X^{n-1} \to \Sigma Y$$

that represents x. It is clear that the indeterminacy in $\{x\}$ also has the form of a map $X^{n-1} \to Y$, so that $x \in E_2$ may be represented by such a β'''.

It will obviously suffice to show that the obstruction to extending β'' is β'''. Observe that the suspended attaching map for an n-cell $e^n \in X^n$ is

$$(\Sigma\delta)(1_{S^n}): S^n \to \Sigma X^{n-1};$$

but then the obstruction cocycle, evaluated on e^n, is precisely β'''.

Conversely, let $x \in H^n(X; \Pi^S_{n-1}(Y))$ $(x \neq 0)$ be the obstruction to extending $f: X^{n-1} \to Y$. By previous remarks, x is represented by

$$\Sigma^{-1}(X^n/X^{-n-1}) \xrightarrow{\delta} X^{n-1} \xrightarrow{f} Y.$$

The map f does not extend to X^n, and if we choose k so that $f \mid X^k$ is nontrivial, yet $f \mid X^{k-1}$ is trivial, we see at once that $\{x\}$ is a suitable boundary.

COROLLARY 1. *The* n*th stable obstruction set* $\mathcal{O}^n_S(X, Y)$ *for maps from* X *to* Y *is the kernel of the natural homomorphism*

$$D^n \to E^{n,1-n}_\infty,$$

where

$$D^n = \bigcap_{k \geq 2} \ker(d_k) \subseteq E^{n,1-n}_2 = H^n(X; \Pi^S_{n-1}(Y)).$$

Proof. If $x \in H^n(X; \Pi^S_{n-1}(Y))$ is an obstruction, we see that $\{x\} = d_k(\{y\})$, so that x clearly lives in the desired kernel. The converse also follows at once from Theorem 2.

COROLLARY 2. $\mathcal{O}(X, Y) = \sum_{m,n} \mathcal{O}^n_S(\Sigma^m X, Y)$ *is a graded, filtered* G_*-*module. The filtration arises from the successive kernels*

$$D^n \to E^{n,1-n}_r \qquad (2 \leq r \leq \infty),$$

whereas the G_*-*action arises from the* G_*-*action in the exact couple.*

Proof. Apply Corollary 1 to the various iterated suspensions $\Sigma^m X$.

3. We now give some applications.

PROPOSITION 1. *Let* $f: X^{n-1} \to Y$ *not extend to* X^n, *and suppose the obstruction is* $x \in H^n(X; \Pi^S_{n-1}(Y))$. *Suppose* $f = \alpha \cdot g$, *where* $\alpha \in G_i = \Pi^S_i(S^0)$ *and* $g: \Sigma^{-i}X \to Y$. *Then* x *is an image under the coefficient homomorphism*

$$H^n(X; \Pi^S_{n-i-1}(Y)) \to H^n(X; \Pi^S_{n-1}(Y)),$$

which is given by multiplication by α.

Proof. By our theorem, $\{x\} = d_m\{y\}$, where y is represented by a map

$$X^r/X^{r-1} \to Y \qquad (r = n - m)$$

and $d_m\{y\}$ is given by the correspondence

$$\{X^r/X^{r-1}; Y\} \to \{X^r, Y\} \leftarrow \{X^{r+1}, Y\} \leftarrow \cdots \leftarrow \{X^{n-1}, Y\} \to \{X^n/X^{n-1}; \Sigma Y\}.$$

We choose f in the group which is second from the right so that the class of the image of f on the right is x. All maps are G_*-linear; therefore x is clearly such an image.

PROPOSITION 2. *If* $\dim X = n$, *we have an exact sequence*

$$0 \to \mathcal{O}^n_S(X, Y) \to H^n(X; \Pi^S_{n-1}(Y)) \xrightarrow{p} \ker(\{X, \Sigma Y\} \to \{X^{n-1}, \Sigma Y\}) \to 0,$$

where p *is the projection*

$$[C_n(X; \Pi^S_{n-1}(Y))/\mathrm{Im}(\{X^{n-1}/X^{n-2}; Y\} \xrightarrow{\delta} \{X^n/X^{n-1}, \Sigma Y\})]$$
$$\to [C^n(X; \Pi^S_{n-1}(Y))/\mathrm{Im}(\{\Sigma X^{n-1}, \Sigma Y\} \to \{X^n/X^{n-1}, \Sigma Y\})].$$

The sequences, for suspensions of X, *are natural under the* G_**-action.*

Proof. $\mathcal{O}^n_S(X, Y) = \bigcup_j \mathrm{Im}\, d_j$, because this is the top dimension for X. We then have an obvious exact sequence

$$0 \to \mathcal{O}^n_S(X, Y) \to H^n(X; \Pi^S_{n-1}(Y)) \to E^{n, 1-n}_\infty \to 0.$$

Since $\dim X = n$, the last group is the required kernel. The rest is obvious.

We remark that one can use these techniques to give estimates on orders of elements, and to give a modified version of the sequence, in one dimension lower than that in Proposition 2. Note that Proposition 2 is a generalization of the classical Hopf-Whitney theorem.

Next, we turn to Freyd's generating hypothesis (see [3]). It claims that if X and Y are finite, connected complexes, and $f: X \to Y$ induces 0 on stable homotopy groups, then f is stably null-homotopic. We shall show how, via obstruction theory, this hypothesis implies interesting results about the G_*-action on $\sum_j \{\Sigma^j X, Y\}$.

LEMMA. *Assume the generating hypothesis. Then*

(a) *if* $f: X \to S^n$ *is a stably nontrivial map and* X *is a finite complex, there exist distinct elements* $\alpha_1, \cdots, \alpha_i, \cdots \in G_*$ *such that* $\alpha_i\{f\} \neq 0$ *for each* i;

(b) *if* $0 \neq x \in \Pi^S_*(X)$, *there exist infinitely many* $\beta_i \in G_*$ *such that* $\beta_i \cdot x \neq 0$. *One may assume that each* β_i *is a product of length* i.

Proof. (a) By the generating hypothesis, there exists an $x \in \Pi^S_*(X)$ with $f^S_\#(x) \neq 0$. We use Proposition 9.3 of [3], a consequence of the generating hypothesis, to find elements $\alpha_i \in G_*$ such that $\alpha_i f^S_\#(x) \neq 0$, for each i. It is then easy to verify that $\alpha_i\{f\} \in \{\Sigma^* X, Y\}$ must be nonzero, for each i.

(b) Let $g: S^m \to X$ represent x (stably). Denote Spanier-Whitehead duality by D, and apply (a) to the dual map $Dg: DX \to S^n$. Since the α_i are self-dual, it is clear that the compositions

$$S^k \xrightarrow{\alpha_i} S^m \xrightarrow{g} X$$

are all nontrivial. The generating hypothesis implies (see [3]) that the β_i may be chosen as products, as desired.

THEOREM 3. *We assume the generating hypothesis, and we let* $f\colon X \to Y$ *be a (stably) nontrivial map between finite complexes.*

(a) *There exists an* $\alpha \in \Pi^S_m(X)$ *such that* f *does not extend to a map*

$$X' = \Sigma^j X \bigcup_{\alpha} e^{m+j+1} \to \Sigma^j Y \qquad (m > \dim X).$$

Denote the obstruction by v *(which may be identified with* $f^S_{\#}(\alpha)$*).*

(b) *There exist* $g = \beta \cdot f$ $(\beta \in G_*$ *and* $\{g\} \in \{\Sigma^\ell X, Y\})$ *and infinitely many* $\alpha_i \in G_*$ *such that* $\alpha_i\{g\} \neq 0$ *for all* i. *In fact, if* u *represents* g *in the spectral sequence, say in* $E_k^{*,*}$, *then* $d_k\{u\} = \beta \cdot \{v\}$ *and* $\alpha_i \cdot u \neq 0$ *in* $E_k^{*,*}$ *for all* i *(in other words, the terms* $\alpha_i \cdot u$ *are nonzero for all* i *in the fixed term* $E_k^{*,*}$*).*

(c) *Let* u' *be another class, in our spectral sequence, with* $d_k\{u'\} = \beta \cdot \{v\}$. *Then* u' *is represented by a map* $g'\colon \Sigma^\ell X \to Y$, *where* $\alpha_i \cdot \{g'\} \neq 0$ *for all* i, *and* $g - g'$ *extends to* X'.

Proof. To begin, we observe that the selection of g in part (b) is primarily to show (c), which is a uniqueness assertion for the situation here. The choices of dimension that follow are needed for this.

By the generating hypothesis, there is an $x \in \Pi^S_*(X)$ with $f^S_{\#}(x) \neq 0$. By part (b) of our Lemma, there are $\beta_1, \beta_2, \cdots \in G_*$ such that

$$\beta_j \cdots \beta_i \cdot f^S_{\#}(x) \neq 0 \qquad \text{for all } j.$$

Without loss of generality, we may assume after suspension that

(1) X is (i - 1)-connected,

(2) $i - 1 < \dim X$ and $\dim Y < \left[\frac{5}{4} i\right] - 1$,

(3) there exists a k_1 such that

$$\beta_j \cdots \beta_{k_1} f^S_{\#}(\beta_{k_1 - 1} \cdots \beta_1 x) \neq 0 \qquad \text{for all } j \geq k_1,$$

with the degrees satisfying the condition

$$\left[\frac{7}{4} i\right] < \deg(\beta_{k_1 - 1} \cdots \beta_1 \cdot x) < 2i - 2.$$

Putting $\alpha = \beta_{k_1 - 1} \cdots \beta_1 x$, we see that part (a) is trivially satisfied. Note that

$$H^{m+j+1}(X'; \Pi^S_m(Y)) \approx H^{m+j+1}(X') \otimes \Pi^S_m(Y) \approx \Pi^S_m(Y);$$

this justifies the identification of the obstruction v with $f^S_{\#}(\alpha)$.

From the G_*-linearity of the maps

$$\{X^{n-1}, Y\} \to \{X^n/X^{n-1}; \Sigma Y\},$$

we conclude that $\beta_j \cdots \beta_{k_1} \cdot v \neq 0$ for all $j \geq k_1$. Next, choose $k_0 > k_1$ so that the filtration of $\beta_{k_0} \cdots \beta_{k_1}\{f\}$ achieves its minimum, in other words, the elements vanish in E_r^{**} of lowest r. Write $\beta_{k_0} \cdots \beta_{k_1} = \beta$, and put $g = \beta \cdot f$. If we let the symbols α_i denote the elements $\beta_{k_0+1}, \beta_{k_0+2}, \cdots$, then part (b) is clear.

To prove part (c), observe that for sufficiently early E_r^{**}-terms, in dimensions below dim X', the spectral sequences for maps $\Sigma^j X \to Y$ and for maps $\Sigma^j X' \to Y$ are the same. Our choices of dimension ensure that such an element u' remains to the term E_∞ in the spectral sequence for $\{\Sigma^j X, Y\}$, this representing some map g', as claimed. Because $d_k\{u'\} = \beta\{v\}$ in the sequence involving X', and $\alpha_i \cdot \beta \cdot v \neq 0$ for each i, we see that $\alpha_i\{g'\} \neq 0$ for each i. It is clear that the obstruction to extending $g - g'$ vanishes; this completes the proof.

The following are corollaries to the preceding theorem and lemma. For these corollaries, we shall assume the generating hypothesis.

COROLLARY 1. *If the mapping* $f: X \to Y$ *is stably essential, then there is a stable mapping* $\alpha: S^{m-1} \to X$ (*with* $m > \dim X + 1$) *such that* $f_\#^S(\alpha) \neq 0$. *In fact, there are infinitely many* $\alpha_i \in G_*$ *such that* $\alpha_i\{f\} \neq 0$ *for each* i, *and the obstruction to extending* $\alpha_i f$ *is* $\alpha_i f_\#^S(\alpha)$.

Proof. The existence of such α comes from the lemma, part (b); the rest comes from Theorem 3.

COROLLARY 2. *If* $1_X: X \to X$ *is the identity map of a nontrivial finite complex* X, *then there exists an* $\alpha_i \in G_*$ *for infinitely many* i, *with* $\alpha_i \cdot \{1_X\} \neq 0$.

COROLLARY 3. *If* X *is a nontrivial complex, there are infinitely many stable maps*

$$\Sigma^{n_i} X \xrightarrow{f_i} \Sigma^{m_i} X \qquad (n_i > m_i),$$

such that every finite successive composition is stably essential.

Proof. By the proof of Theorem 3, and by part (b) of the Lemma, we may assume that the elements in Corollary 2 are products of increasing length. The result follows because our actions are module actions, with $(\beta_2 \cdot \beta_1) \cdot f = \beta_2 \cdot (\beta_1 f)$.

COROLLARY 4. *If* $f: X \to Y$ *is stably essential, then there are, after suspension, infinitely many* X' $(X \subset X')$ *such that* f *does not extend to* X'. *One may also assume that* X'/X *is a sphere, in other words, that* X' *is the adjunction obtained by adding one cell.*

Remarks. (a) Corollary 1 to Theorem 3 says that $\sum_j \{\Sigma^j X, Y\}$ has big orbits under G_*-action. One might hope that this module is finitely-generated; but the generating hypothesis implies that if X is a sphere and Y is not stably a wedge of spheres, then the module is infinitely generated. In [5], we show this in various cases without involving the generating hypothesis.

(b) In [6], we give a strengthened version of some of the consequences of the generating hypothesis. Using this, one may show that in Theorem 3 and its corollaries, the elements $\alpha_i \cdot \{g\}$ can be assumed to have divisibility properties and to annihilate prescribed elements (finitely many) in G_*. Here, we omit details because they are technical and would obscure the presentation.

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