PENETRATION INDICES OF ARC-AND-BALL PAIRS AND UNCOUNTABLY MANY QUASI-TRANSLATIONS OF THE 3-SPHERE

Shin'ichi Kinoshita and Carl D. Sikkema

INTRODUCTION

This paper is a continuation of [4], and we assume that readers are familiar with that paper. We shall prove, among other things, that there exist uncountably many mutually inequivalent quasi-translations of the 3-sphere. From this it follows that *there exist uncountably many open, orientable* 3-*manifolds whose fundamental group is infinite cyclic and whose universal covering space is* 3-*space* E^3.

In addition to the statement above, which can be proved rather simply, we study some related topics. Not only do they have independent interest, but they are helpful for understanding the background of the proofs of the statement above and the whole structure of the study developed in [4] and this paper.

The statement above was announced in [4]. There we also announced that the composition of two open arc-and-ball pairs (see below) is not commutative. However, we have found a gap in the proof.

In Section I we study an *open arc-and-ball pair* (a, B), where a is an arc in a 3-ball B such that only the endpoint p(a) of a is on the boundary ∂B of B and a is locally tame in B except at p(a). (This is what in [4] we called an arc-and-ball pair.) We introduce the penetration index P(a, B) for (a, B) and prove that

$$P((a_1, B_1) \# (a_2, B_2)) = P(a_1, B_1) \cdot P(a_2, B_2).$$

In Section II we study a *closed arc-and-ball pair* [a, B], where a is an arc in a 3-ball B such that only the initial point q(a) and the endpoint p(a) are on ∂B, and such that a is locally tame in B except possibly at p(a). One of the significant differences between the two concepts (a, B) and [a, B] is that the infinite composition $\#_{n=-\infty}^{+\infty} [a_n, B_n]$ can be defined, but $\#_{n=-\infty}^{+\infty} (a_n, B_n)$ can not. We shall apply this infinite composition in Section III.

In [4], we associate with each (a, B) a quasi-translation h(a, B) of the 3-sphere. In Section III, we consider open arc-and-ball pairs that are constructed from Wilder arcs and their associated quasi-translations. We study the mutual inequivalence of some of these quasi-translations through the concept of positively characteristic translation curves, and we close the discussion with the classification of Wilder arcs by R. H. Fox and O. G. Harrold [3].

Received April 5, 1972.
The authors of this paper are partially supported by NSF Grant GP-19964.

Michigan Math. J. 20 (1973).

I. THE PENETRATION INDEX OF AN OPEN ARC-AND-BALL PAIR

1. Let (a, B) be an open arc-and-ball pair, and let p(a) be the endpoint of a that lies on ∂B. Then an odd natural number n is called the *penetration index* of (a, B), denoted by P(a, B), if it satisfies the following two conditions:

(1) For each standard neighborhood U of p(a) in B, there exists a standard neighborhood V of p(a) in B such that $V \subset \operatorname{Int} U$ and

$$c(\text{Bdry } V \cap a) \leq n,$$

where c(X) denotes the cardinality of the set X. (For the definition of a standard neighborhood, see [4].)

(2) There exists a standard neighborhood U of p(a) in B such that

$$c(\text{Bdry } V \cap a) \geq n$$

for every standard neighborhood V contained in U. If no such number n exists, we put $P(a, B) = \infty$.

If (a, B) satisfies condition (1), then $P(a, B) \leq n$. If (a, B) satisfies condition (2), then $P(a, B) \geq n$.

The penetration index for an arc in a 3-sphere was first introduced by W. R. Alford and B. J. Ball [1].

Let (a, B) be an open arc-and-ball pair, and suppose that B is tamely imbedded in S^3. Let $P(a, S^3)$ be the penetration index of a at p(a) in S^3. Then $P(a, S^3) \leq P(a, B)$.

2. Let (a_1, B_1) and (a_2, B_2) be open arc-and-ball pairs. In $(B_2)^\circ$, take a small tame 3-ball B' such that $a_2 \cap B' = q(a_2)$ and $a_2 \cup B'$ is locally tame at $q(a_2)$, where $q(a_2)$ is the initial point of a_2. Let g be an orientation-preserving homeomorphism of B_1 onto B' such that $g(p(a_1)) = q(a_2)$, where $p(a_1)$ is the endpoint of a_1. Consider the decomposition d: $B_2 \to d(B_2)$, where the only nondegenerate element of d is the arc a_2. Then $d(B_2)$ is a 3-ball and $(d(a_1), d(B_2))$ is an open arc-and-ball pair, which is defined as

$$(a_1, B_1) \# (a_2, B_2)$$

in [4].

Let V_1 and V_2 be standard neighborhoods of $p(a_1)$ and $p(a_2)$ in (a_1, B_1) and (a_2, B_2), respectively, such that Bdry V_2 separates B' and $p(a_2)$ in B_2. The purpose of this section is to construct a subset V of B_2 as follows: The number of points of $a_2 \cap \text{Bdry } V_2$ is an odd natural number $2n + 1$. The intersection $a_2 \cap V_2$ consists of $n + 1$ arcs $c_1, \cdots, c_n$, and c, oriented subarcs arranged in the natural order on a_2. Hence, the endpoint of c is $p(a_2)$. Let $T_1, \cdots, T_n$ be mutually disjoint 3-balls such that, for each i $(1 \leq i \leq n)$,

(i) $c_i \subset T_i \subset V_2$,

(ii) $c_i^\circ \subset T_i^\circ \subset V_2^\circ$,

(iii) $T_i \cap \partial V_2$ consists of two disjoint disks D_i and D_i',

(iv) $D_i \cup D_i' \subset (\text{Bdry } V_2)^\circ$,

(v) $c_i \cap D_i$ is the initial point of c_i, and $c_i \cap D_i'$ is the endpoint of c_i, and

(vi) the pair $[c_i, T_i]$ represents a trivial knot.

In other words, let T_i be a thickening of c_i in V_2. (See Figure 1.)

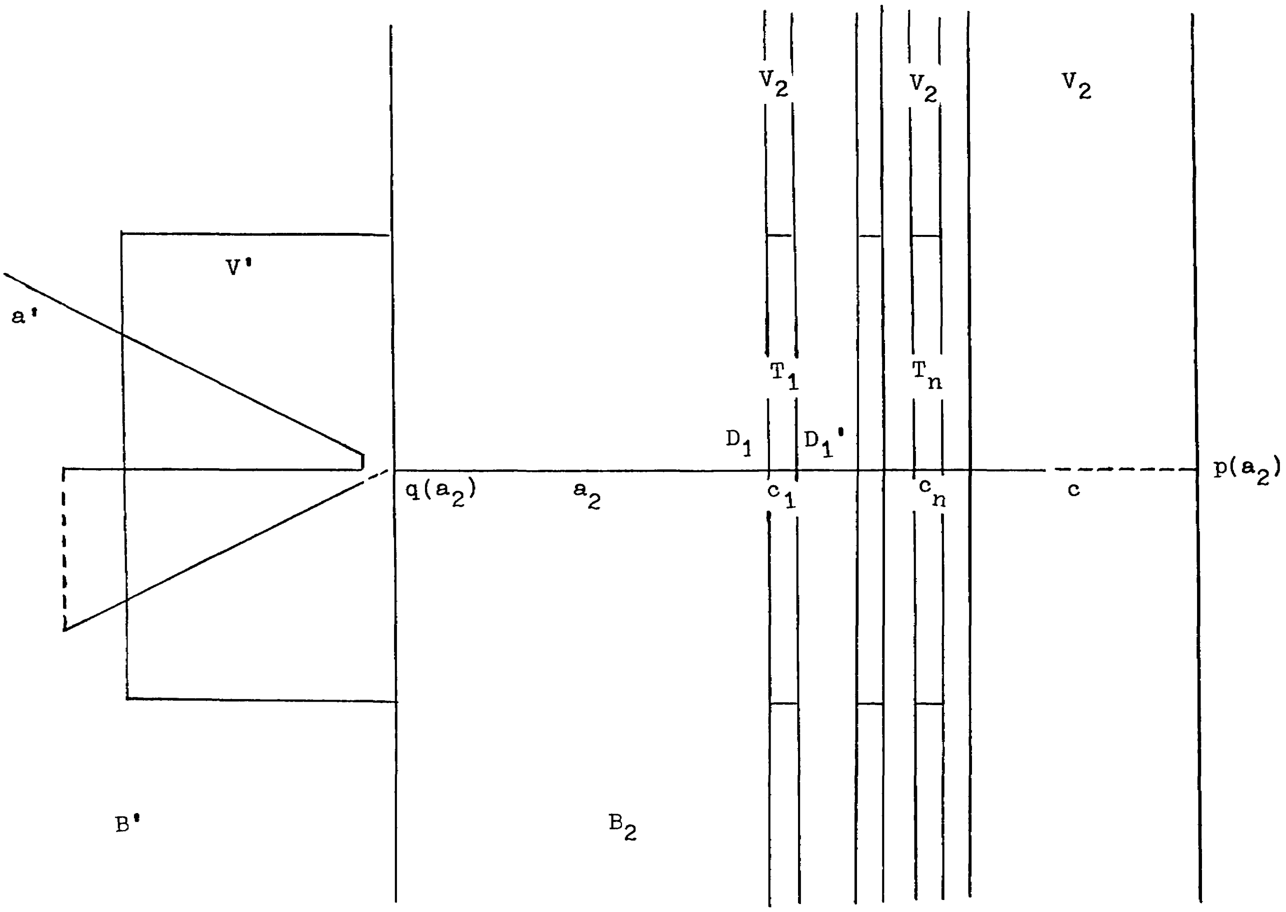

Figure 1.

Now $g(V_1) = V'$ is a standard neighborhood of $g(a_1) = a'$ in $g(B_1) = B'$. Let D be a disk, and let I be the closed interval [0, 1]. Consider a homeomorphism g_0 of $D \times I$ into V' such that

(i) $g_0(D \times 1) = \text{Bdry } V'$ in B',

(ii) $g_0(\partial D \times I) \subset \partial B'$, and

(iii) $g_0(D \times I)$ is so close to $g_0(D \times 1)$ that the arc a' pierces $g_0(D \times t)$ at each point of $a' \cap g_0(D \times t)$ for $0 \leq t \leq 1$.

Let $E_i = D \times [2i - 1/2n, i/n]$ for $i = 1, \cdots, n$, and $E_i' = g_0(E_i)$. Further, let $E_0' = \text{Cl}(V' - g_0(D \times I))$ in B'.

First consider $H_0 = \text{Cl}\left(V_2 - \bigcup_{i=1}^n T_i\right)$, a 3-ball with n holes. Connect H_0 and E_0' by a narrow tube F_0 along a_2. Then F_0 is a natural thickening of the subarc of a_2 from $q(a_2)$ to the initial point of the arc c. Let $H_1 = H_0 \cup F_0 \cup E_0'$; this is a 3-ball with n holes. (See Figure 2.)

Next, connect H_1 and E_1' by a solid torus F_1, as in Figure 2. The solid torus F_1 is a homeomorphic image of the product of an annulus A and the interval I, and we shall denote the homeomorphism by g_1. We suppose that

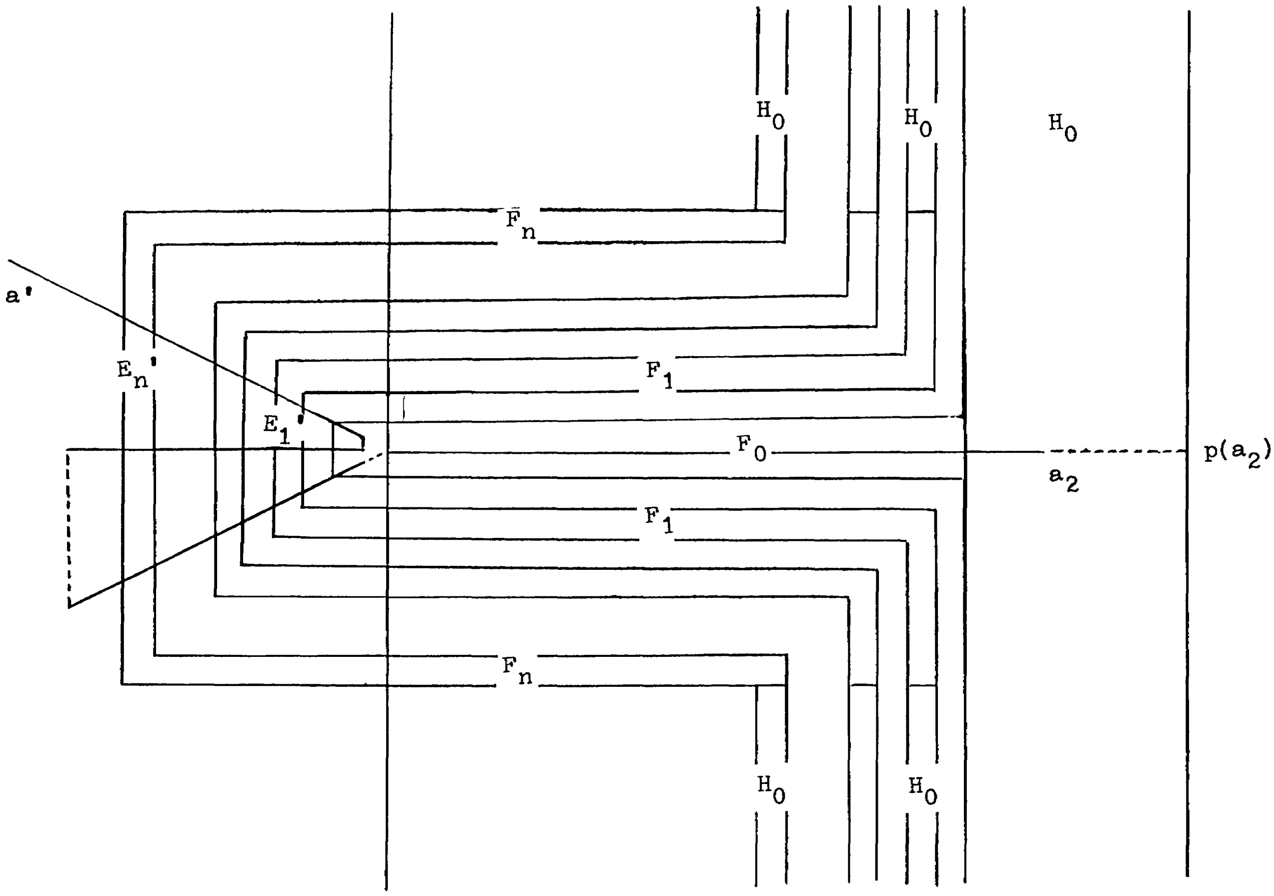

Figure 2.

(i) $g_1(A \times 0) = F_1 \cap H_1 = \mathrm{Cl}(\mathrm{Bdry}\ T_n - D_n \cup D_n')$ and

(ii) $g_1(A \times 1) = F_1 \cap E_1' = g_0(\partial D \times [1/2n, 1/n])$.

Roughly speaking, F_1 is almost a thin thickening of a cylinder along the part of F_0 that is a thickening of the subarc of a_2 from $q(a_2)$ to the point $c_n \cap D_n'$. Let

$$H_2 = H_1 \cup F_1 \cup E_1';$$

this is a 3-ball with $n - 1$ holes.

We continue this process so that H_i and E_i' are connected by an appropriate solid torus F_i along the part of F_{i-1} corresponding to the subarc of a_2 from $q(a_2)$ to the point $c_{n-i+1} \cap D_{n-i+1}'$ (see Figure 2). Let

$$H_{i+1} = H_i \cup F_i \cup E_i';$$

this is a 3-ball with $n - i$ holes. At the nth step, we have the 3-ball

$$(V_1, V_2) = H_n \cup F_n \cup E_n'.$$

The following proposition is clear from the construction of the set (V_1, V_2).

PROPOSITION 1. (1) *The set* $d(V_1, V_2)$ *is a standard neighborhood of* $d(a_2)$ *in* $d(B_2)$.

(2) *We have the relation*

$$c(dg(a_1) \cap \text{Bdry } d(V_1, V_2)) = c(a_1 \cap \text{Bdry } V_1) \cdot c(a_2 \cap \text{Bdry } V_2).$$

(3) *The set of all* (V_1, V_2) *forms a basis for neighborhoods of* $d(a_2)$ *in* $d(B_2)$, *where* V_1 *and* V_2 *run over all standard neighborhoods of* $p(a_1)$ *in* B_1 *and* $p(a_2)$ *in* B_2, *respectively, such that* Bdry V_2 *separates* B' *and* $p(a_2)$ *in* B_2.

It follows from Proposition 1 that

$$(*) \qquad P((a_1, B_1) \# (a_2, B_2)) \leq P(a_1, B_1) \cdot P(a_2, B_2),$$

which includes the case $P(a_1, B_1) = \infty$ or $P(a_2, B_2) = \infty$.

3. Now let U_1 be a standard neighborhood of $p(a_1)$ in B_1 such that for every standard neighborhood V_1 contained in U_1, we have the inequality

$$c(\text{Bdry } V_1 \cap a_1) \geq c(\text{Bdry } U_1 \cap a_1),$$

and let U_2 be a standard neighborhood of $p(a_2)$ in B_2 such that for every standard neighborhood V_2 contained in U_2, we have the inequality

$$c(\text{Bdry } V_2 \cap a_2) \geq c(\text{Bdry } U_2 \cap a_2).$$

We assume that Bdry U_2 separates B' and $p(a_2)$ in B_2. Let $U = (U_1, U_2)$.

PROPOSITION 2. *For every standard neighborhood* V *of* $d(a_2)$ *in* $d(B_2)$ *that is contained in* $d(U)$,

$$c(\text{Bdry } V \cap dg(a_1)) \geq c(\text{Bdry } d(U) \cap dg(a_1)).$$

Proof. By the cutting-and-pasting method, we can deform $d^{-1}(V)$ into a 3-cell W in $d^{-1}(U)$ such that

(i) $d(W)$ is a standard neighborhood of $d(a_2)$ in $d(B_2)$,

(ii) Bdry $W \cap g(\partial B_1)$ consists of a finite number of mutually disjoint simple closed curves, none of which bounds a disk on

$$g(\partial B_1) \cap d^{-1}(U) - q(a_2),$$

and

(iii) $c(\text{Bdry } W \cap g(a_1)) \leq c(\text{Bdry } d^{-1}(V) \cap g(a_1))$.

Now consider the set Bdry $W \cap g(\partial B_1)$, which consists of a finite number of simple closed curves $\{c_i\}$. If c_i is one of the innermost of these curves on Bdry W, then c_i bounds a 2-cell D_i on Bdry W such that $D_i \subset g(U_1)$. Hence

$$c(\text{Bdry } g(U_1) \cap g(a_1)) \leq c(D_i \cap g(a_1)).$$

Now let c_j be one of the outermost curves on Bdry W. Then c_j bounds a 2-cell D_j on Bdry W, which contains a c_i, one of the innermost curves on Bdry W. Possibly, $c_i = c_j$. Replace D_j by a 2-cell D_j' that is parallel to $g(\text{Bdry } U_1)$. We suppose that $D_j' \subset d^{-1}(U)$. Then

$$c(\text{Bdry } U_1 \cap a_1) = c(D_j' \cap g(a_1)) \leq c(D_i \cap g(a_1)) \leq c(D_j \cap g(a_1)).$$

We make this replacement for all of the outermost curves on Bdry W. Then Bdry W is modified into a 2-cell D contained in $d^{-1}(U)$. The 2-cell D bounds a 3-cell W' that is contained in $d^{-1}(U)$, and $d(W')$ is a standard neighborhood of $d(a_2)$ in $d(B_2)$. Clearly,

$$c(\text{Bdry } W' \cap g(a_1)) \leq c(\text{Bdry } W \cap g(a_1)).$$

Further, $c(\text{Bdry } W' \cap g(a_1))$ is equal to the product of $c(\text{Bdry } U_1 \cap a_1)$ and the number of outermost curves on Bdry W.

Now consider the construction of $U = (U_1, U_2)$. This can be thought of as a deformation of U_2 onto U. Then the inverse is a deformation of U onto U_2. This inverse deforms W' into U_2. Let W" be the image of W'. Because $g(B_1) \cap \text{Bdry } W'$ consists of a finite number of 2-cells, W" is a standard neighborhood of $p(a_2)$ in B_2, and $W'' \subset U_2$. Hence

$$c(\text{Bdry } W'' \cap a_2) \geq c(\text{Bdry } U_2 \cap a_2).$$

On the other hand, $c(\text{Bdry } W'' \cap a_2)$ is equal to the number of outermost curves on Bdry W. Hence,

$$c(\text{Bdry } U_1 \cap a_1) \cdot c(\text{Bdry } U_2 \cap a_2) \leq c(\text{Bdry } W' \cap g(a_1)) \leq c(\text{Bdry } W \cap g(a_1))$$
$$\leq c(\text{Bdry } V \cap dg(a_1)),$$

and the proof is complete.

It follows from Proposition 2 that

$$(**) \qquad P((a_1, B_1) \# (a_2, B_2)) \geq P(a_1, B_1) \cdot P(a_2, B_2);$$

this includes the case where $P(a_1, B_1) = \infty$ or $P(a_2, B_2) = \infty$.

By (*) and (**), we have the following theorem.

THEOREM 1. $P((a_1, B_1) \# (a_2, B_2)) = P(a_1, B_1) \cdot P(a_2, B_2)$.

Remark. In [4], we have proved that

$$(a_1, B_1) \# (a_2, B_2) = (e, B)$$

if and only if

$$(a_1, B_1) = (a_2, B_2) = (e, B),$$

where (e, B) is the trivial pair (see Theorem 1 in [4]). Applying Theorem 1, we have the following alternate proof of this statement: Assume that

$$(a_1, B_1) \# (a_2, B_2) = (e, B).$$

Then

$$P(a_1, B_1) \cdot P(a_2, B_2) = 1,$$

by Theorem 1. Hence

$$P(a_1, B_1) = P(a_2, B_2) = 1,$$

which means that both (a_1, B_1) and (a_2, B_2) are open arc-and-ball pairs constructed from Wilder arcs. Now we can easily see that

$$(a_1, B_1) = (a_2, B_2) = (e, B),$$

by applying the classification theorem on Wilder arcs by Fox and Harrold [3]. The converse is trivial.

II. CLOSED ARC-AND-BALL PAIRS

4. Let a be an arc in a 3-ball B such that only the initial point q(a) and the endpoint p(a) of the arc a are on ∂B and a is locally tame in B except possibly at the endpoint p(a). We call such an arc-and-ball pair [a, B] a *closed arc-and-ball pair*.

The definition of the *equivalence* of two closed arc-and-ball pairs is similar to that of open arc-and-ball pairs (see [4]).

Let $[a_1, B_1]$ and $[a_2, B_2]$ be two closed arc-and-ball pairs. Then the *composition* of $[a_1, B_1]$ and $[a_2, B_2]$, which will be denoted by $[a_1, B_1] \# [a_2, B_2]$, is defined as follows: Suppose that

$$B_1 = \{(x, y, z) \mid 0 \leq x \leq 1, \ -1 \leq y \leq 1, \ -1 \leq z \leq 1\},$$

$$B_2 = \{(x, y, z) \mid 1 \leq x \leq 2, \ -1 \leq y \leq 1, \ -1 \leq z \leq 1\}.$$

Further, suppose that $q(a_1) = (0, 0, 0)$, $p(a_1) = q(a_2) = (1, 0, 0)$, and $p(a_2) = (2, 0, 0)$. Consider the decomposition

$$d: B_1 \cup B_2 \to d(B_1 \cup B_2),$$

where the only nondegenerate element of d is the arc a_2. Then $d(B_1 \cup B_2)$ is a 3-ball and $[d(a_1), d(B_1 \cup B_2)]$ is a closed arc-and-ball pair. We define

$$[a_1, B_1] \# [a_2, B_2] = [d(a_1), d(B_1 \cup B_2)].$$

Note that this generalizes the composition (product) of two knots.

It can easily be seen that the family of all equivalence classes of [a, B] with operation # forms a semigroup with the identity [e, B], where e represents a trivial knot in B. Further, if $[a_1, B_1] \# [a_2, B_2] = [e, B]$, then

$$[a_1, B_1] = [a_2, B_2] = [e, B].$$

The definition of the *penetration index* P[a, B] of a closed arc-and-ball pair [a, B] is similar to that of an open arc-and-ball pair.

With each closed arc-and-ball pair [a, B] we associate an open arc-and-ball pair $\phi[a, B]$ as follows: Let b be a subarc of a whose initial point is in the interior of a and whose endpoint is p(a). Let $\phi[a, B] = (b, B)$. This induces a many-to-one correspondence of equivalence classes. However, it is easy to see that $P[a, B] = P(\phi[a, B])$, because the penetration index depends only on the position of the arc a near p(a) in B. Hence we have the formula

$$(***) \qquad P([a_1, B_1] \# [a_2, B_2]) = P[a_1, B_1] \cdot P[a_2, B_2].$$

5. For every integer n, let $[a_n, B_n]$ be a closed arc-and-ball pair. Then the infinite product $\#_{n=-\infty}^{+\infty} [a_n, B_n]$ is defined as follows: Let

$$C = \{(x, y, z) |\ x^2 + y^2 + z^2 \leq 1\},$$

$$C_n = C \cap \left\{(x, y, z) \middle|\ 1 - \frac{1}{n} \leq x \leq 1 - \frac{1}{n+1}\right\} \text{ if } n \geq 2,$$

$$C_1 = C \cap \left\{(x, y, z) \middle|\ \frac{1}{4} \leq x \leq \frac{1}{2}\right\},$$

$$C_0 = C \cap \left\{(x, y, z) \middle|\ -\frac{1}{4} \leq x \leq \frac{1}{4}\right\},$$

$$C_{-1} = C \cap \left\{(x, y, z) \middle|\ -\frac{1}{2} \leq x \leq -\frac{1}{4}\right\}, \quad \text{and}$$

$$C_{-n} = C \cap \left\{(x, y, z) \middle|\ -1 + \frac{1}{n+1} \leq x \leq -1 + \frac{1}{n}\right\} \text{ if } n \geq 2.$$

Further, let

$$L = \{(x, y, z) |\ y = 0,\ z = 0\} \cup (\infty),$$

as in [4]. Now, for each integer n, let g_n be an orientation-preserving homeomorphism of B_n onto C_n such that

$$g_n(q(a_n)) = C_{n-1} \cap C_n \cap L, \qquad g_n(p(a_n)) = C_n \cap C_{n+1} \cap L.$$

Then

$$\alpha = \bigcup_{n=-\infty}^{\infty} g_n(a_n) \cup (-1, 0, 0) \cup (1, 0, 0)$$

is an arc. Let

$$B = \{(x, y, z) |\ -2 \leq x \leq 1,\ -2 \leq y \leq 2,\ -2 \leq z \leq 2\},$$

$$a = \{(x, y, z) |\ -2 \leq x \leq -1,\ y = 0,\ z = 0\}.$$

Consider the decomposition d: $B \to d(B)$, where the only nondegenerate element of d is the arc α. Since every subarc of the arc α is locally peripherally unknotted at its initial point, d(B) is a 3-ball and [d(a), d(B)] is a closed arc-and-ball pair (see [2]). We define

$$\mathop{\#}_{n=-\infty}^{\infty} [a_n, B_n] = [d(a), d(B)].$$

Then we have the following generalization of (***).

$$(\substack{**\\**}) \qquad P\left(\mathop{\#}_{n=-\infty}^{\infty} [a_n, B_n]\right) = \prod_{n=-\infty}^{\infty} P[a_n, B_n].$$

Proof. Note that the proof is divided into two cases: (1) $P[a_n, B_n] = 1$ for almost all n, and (2) $P[a_n, B_n] \geq 3$ for infinitely many n. For each of these cases, the proof is easy.

Remark. We point out that the infinite product of open arc-and-ball pairs does not seem to be well-defined.

III. UNCOUNTABLY MANY QUASI-TRANSLATIONS

6. Let N be the set of all natural numbers. Let $\{k_n \mid n \in N\}$ be a family of mutually inequivalent prime knots in S^3, and suppose that a closed arc-and-ball pair $[a_n, B_n]$ represents k_n for each $n \in N$. Let $\{x_\lambda\}$ be the set of all correspondences of N into the set of two points $\{0, 1\}$. The set $\{x_\lambda\}$ has the cardinality of the continuum.

For each x_λ and each integer n, let

$$[a_n', B_n'] = \begin{cases} [a_n, B_n] & \text{if } x_\lambda(n) = 1, \\ [e, B] & \text{otherwise.} \end{cases}$$

Then let

$$[a_\lambda', B_\lambda'] = \mathop{\#}_{n=-\infty}^{\infty} [a_n', B_n'].$$

Now we let

$$[a_n'', B_n''] = \begin{cases} [a_\lambda', B_\lambda'] & \text{if } n \in N, \\ [e, B] & \text{otherwise.} \end{cases}$$

Then we let

$$(a_\lambda, B_\lambda) = \phi\left(\mathop{\#}_{n=-\infty}^{\infty} [a_n'', B_n''] \right).$$

Hence (a_λ, B_λ) is an open arc-and-ball pair constructed from a Wilder arc that contains infinitely many copies of k_n if and only if $x_\lambda(n) = 1$. It was proved by Fox and Harrold [3] that (a_λ, B_λ) is topologically equivalent to (a_μ, B_μ) if and only if $x_\lambda = x_\mu$.

7. With each open arc-and-ball pair (a_λ, B_λ) we now associate the quasi-translation $h(a_\lambda, B_\lambda)$ of S^3 defined in Section 6 of [4].

THEOREM 2. *Two quasi-translation* $h(a_\lambda, B_\lambda)$ *and* $h(a_\mu, B_\mu)$ *are topologically equivalent if and only if* $x_\lambda = x_\mu$.

COROLLARY. *There exist uncountably many mutually inequivalent quasi-translations of* S^3.

Proof. First we note that in the construction of $h(a_\lambda, B_\lambda)$ we also construct a closed arc-and-ball pair $[g(a), A_0]$ such that $\phi[g(a), A_0] = (a_\lambda, B_\lambda)$. Without loss of generality, we assume that

$$[g(a), A_0] = \mathop{\#}_{n=-\infty}^{\infty} [a_n'', B_n''],$$

and for simplicity, we let $[a_\lambda, B_\lambda] = [g(a), A_0]$.

Next we note that the positively characteristic translation curve $c_\lambda = d(J)$ for $h(a_\lambda, B_\lambda)$ is defined in the proof of Theorem 3 in [4]. With the curve c_λ we associate an open arc-and-ball pair $(c_{\lambda+}, B_{c_\lambda})$ (see [4, Section 8]), and now it is clear that $(c_{\lambda+}, B_{c_\lambda})$ is topologically equivalent to

$$\phi\left(\mathop{\#}_{n=-\infty}^{\infty} [a_n''', B_n''']\right),$$

where $[a_n''', B_n'''] = [a_\lambda, B_\lambda]$ for every integer n. Note that the curve c_λ is uniquely determined by $[a_\lambda, B_\lambda]$, but not by (a_λ, B_λ). As we noted in the last section of [4], or more generally by $(\substack{**\\**})$, the open arc-and-ball pair $(c_{\lambda+}, B_{c_\lambda})$ is constructed from a Wilder arc that contains the knot k_n infinitely many times if and only if $x_\lambda(n) = 1$.

Now suppose that $x_\lambda \neq x_\mu$. Then there exists a natural number n such that $x_\lambda(n) \neq x_\mu(n)$. We assume that $x_\lambda(n) = 0$ and $x_\mu(n) = 1$. Suppose on the contrary that $h(a_\lambda, B_\lambda)$ is topologically equivalent to $h(a_\mu; B_\mu)$. Let f be an orientation-preserving autohomeomorphism of S^3 that gives rise to the equivalence of $h(a_\lambda, B_\lambda)$ and $h(a_\mu, B_\mu)$. As we noted before, to the quasi-translation $h(a_\lambda, B_\lambda)$ there corresponds a positively characteristic translation curve c_λ. Its image $f(c_\lambda)$ is a positively characteristic translation curve of $h(a_\mu, B_\mu)$. Hence, by Theorem 4 in [4], there exist two open arc-and-ball pairs (a_1^*, B_1^*) and (a_2^*, B_2^*) such that

$$(f(c_\lambda)_+, B_{f(c_\lambda)}) = (a_1^*, B_1^*) \# (a_\mu, B_\mu) \# (a_2^*, B_2^*) .$$

Since $P(c_{\lambda+}, B_{c_\lambda}) = P(f(c_\lambda)_+, B_{f(c_\lambda)}) = 1$, we see by Theorem 1 that

$$P(a_1^*, B_1^*) = P(a_2^*, B_2^*) = 1 .$$

Hence (a_1^*, B_1^*) and (a_2^*, B_2^*) are constructed from Wilder arcs. Therefore, $(f(c_\lambda)_+, B_{f(c_\lambda)})$ is constructed from a Wilder arc that contains the knot k_n infinitely many times. But $(f(c_\lambda)_+, B_{f(c_\lambda)})$ is topologically equivalent to $(c_{\lambda+}, B_{c_\lambda})$, which is constructed from a Wilder arc that does not contain k_n. This contradicts the classification theorem for Wilder arcs by Fox and Harrold [3]. Hence $h(a_\lambda, B_\lambda)$ is not topologically equivalent to $h(a_\mu, B_\mu)$.

If $x_\lambda = x_\mu$, then $(a_\lambda, B_\lambda) = (a_\mu, B_\mu)$ and $h(a_\lambda, B_\lambda) = h(a_\mu, B_\mu)$. Thus the proof is complete.

REFERENCES

1. W. R. Alford and B. J. Ball, *Some almost polyhedral wild arcs.* Duke Math. J. 30 (1963), 33-38.

2. P. H. Doyle, *A sufficient condition that an arc in* S^n *be cellular.* Pacific J. Math. 14 (1964), 501-503.

3. R. H. Fox and O. G. Harrold, Jr., *The Wilder arcs*. Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), pp. 184-187. Prentice-Hall, Englewood Cliffs, N.J., 1962.

4. S. Kinoshita and C. D. Sikkema, *On arc and ball pairs and quasi-translations of the* 3-*sphere*. Michigan Math. J. 17 (1970), 165-173.

The Florida State University
Tallahassee, Florida 32306