ON SUBNORMAL SUBGROUPS OF FUNDAMENTAL GROUPS OF CERTAIN 3-MANIFOLDS

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Let M be a P^2-irreducible 3-manifold. In [2], it is shown that if F is a 2-sided, closed, incompressible surface in M such that $i_*\pi_1(F)$ is normal in $\pi_1(M)$, then M is either a fibre bundle over S^1 with fibre F, or a line bundle over a closed surface G (and F is then parallel to ∂M), or a union of two such line bundles. In particular, if $\partial M \neq \emptyset$ and $i_*\pi_1(F)$ is normal in $\pi_1(M)$, then $i_*\pi_1(F)$ is of index 2 or 1 in $\pi_1(M)$. In this paper, we show that the same result holds if we replace "normal" by "subnormal." Hence $i_*\pi_1(F)$ is subnormal in $\pi_1(M)$ if and only if it is normal in $\pi_1(M)$. An analogous result holds for noncontractible, simple closed curves in 2-manifolds. By way of an application, we classify the sufficiently large 3-manifolds that have fundamental groups each of whose subgroups is subnormal.

We work throughout in the piecewise linear category. "A surface $F \subset M$" always means a 2-sided embedded surface in M. We say that F is *incompressible* in M if genus$(F) \geq 1$ and $\ker(i_*\pi_1(F) \to \pi_1(M)) = 0$, where i: $F \to M$ denotes inclusion. A 3-manifold is called P^2-*irreducible* if M is irreducible and contains no (2-sided) projective planes.

A subgroup S of a group G is called *subnormal* (in G) if there exists a finite sequence of subgroups $S_1, \cdots, S_n$ of G such that $S \lhd S_1 \lhd \cdots S_n \lhd G$.

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1. SUBNORMAL SUBGROUPS OF $\pi_1(M)$

If F is a surface in M, let $i_*: \pi_1(F) \to \pi_1(M)$ denote the homomorphism induced by inclusion.

THEOREM 1. *Let* M *be a compact,* P^2-*irreducible* 3-*manifold, and suppose* F *is a* 2-*sided, closed, incompressible surface in* M *such that* $i_*\pi_1(F)$ *is subnormal in* $\pi_1(M)$. *Then one of the following holds:*

(a) M *is a fibre bundle over* S^1 *with fiber* F.

(b) $M \cong F \times I$.

(c) M *is a twisted line bundle over a closed surface* G, *and* F *is parallel to* ∂M.

(d) F *separates* M *into two twisted line bundles of type* (c).

LEMMA 1. *If* S *is subnormal in* G *and* $U \subseteq G$ *is a subgroup containing* S, *then* S *is subnormal in* U.

Proof. We have subgroups $S_1, \cdots, S_n$ of G such that $S \lhd S_1 \lhd \cdots S_n \lhd G$. Forming intersections with U, we obtain the sequence $S \lhd S_1 \cap U \lhd \cdots \lhd S_n \cap U \lhd U$, and this proves the lemma.

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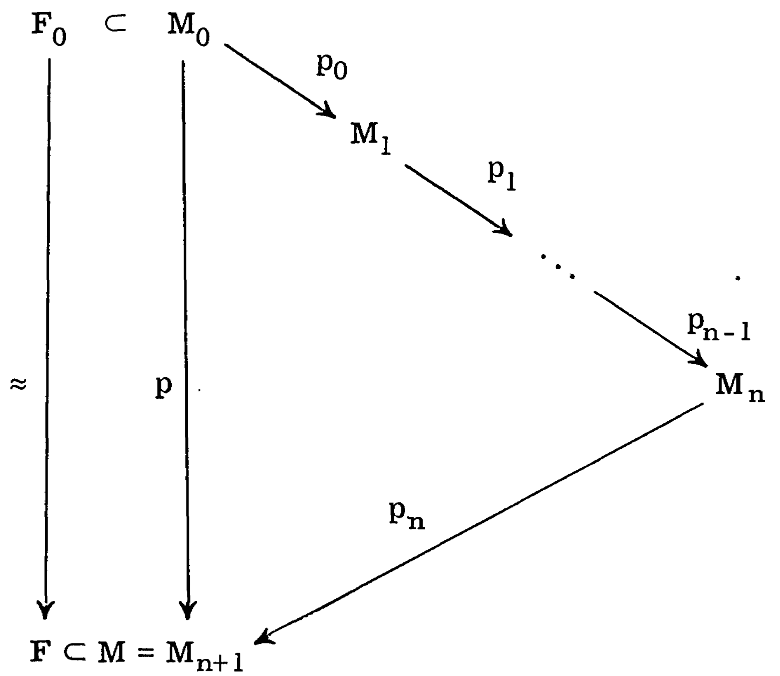
Proof of Theorem 1. We consider two cases:

Case 1. F is a boundary component of M (hence $\partial M \neq \emptyset$). We have subgroups $S_1, \cdots, S_n$ of $\pi_1(M)$ such that

$$i_*\pi_1(F) = S_0 \lhd S_1 \lhd \cdots \lhd S_n \lhd S_{n+1} = \pi_1(M).$$

Consider the covering spaces of M associated with these subgroups: Let $M_{n+1} = M$, and let $M_i \xrightarrow{p_i} M_{i+1}$ be the covering of M_{i+1} associated with $S_i \lhd S_{i+1}$ $(i = 0, \cdots, n)$. We consider the diagram



where $p = p_n \cdot \cdots \cdot p_0$. Clearly, each covering p_i: $M_i \to M_{i+1}$ $(i = 0, \cdots, n)$ is regular. Let $F_0 \in p^{-1}F$ be a copy over $F \subset \partial M$ for which $p \mid F_0$ is a homeomorphism. If there exists a component $F' \in \partial M_0$ (F' compact, $F' \neq F_0$), then it follows from [10, Lemma 5.1] and [1, Proposition 5] that

$$M_0 \cong F_0 \times I \cong F \times I.$$

Since $p \mid F_0$ (where F_0 is short for $F_0 \times 0$ or $F_0 \times 1$) is one-to-one, it follows that $M_0 \xrightarrow{p} M$ is either 1- or 2-sheeted and therefore M is a line bundle over a closed surface, by [10, Lemma 4.1] and [1, Proposition 4], and we have case (b) or (c) of the theorem.

We therefore assume that F_0 is the only closed component of ∂M_0; in particular, F_0 is then the only compact component of $p^{-1}F$. For $i = 1, \cdots, n$, there exists a closed surface $F_i \subset \partial M_i$ $(F_i \in (p_n \cdots p_i)^{-1} F)$ that is covered by F_0. Let $F_{n+1} = F$. Since p_0: $M_0 \to M_1$ is a regular covering and $F_0 \in p_0^{-1} F_1$ is compact, it follows that all components of $p_0^{-1} F_1$ are closed surfaces. Therefore $p_0^{-1} F_1 = F_0$ (otherwise, there would be another compact component in $p^{-1} F$). Since $p \mid F_0$ is a homeomorphism, it follows that $p_0 \mid F_0$ is a homeomorphism and hence that p_0: $M_0 \to M_1$ is a homeomorphism. By the same argument, it follows inductively that p_i: $M_i \to M_{i+1}$ is a homeomorphism and thus that p: $M_0 \to M$ is a homeomorphism. Therefore $\partial M_0 = F_0$, and it follows from [2, Proposition 1] that M_0 is a line bundle over F. But then $\pi_1(F_0)$ would be a proper subgroup (of index 2) of $\pi_1(M_0)$. Thus the case considered cannot occur, since $\pi_1(F_0) = \pi_1(M_0)$.

Case 2. $F \subset \mathring{M}$. Let U(F) be a regular neighbourhood of F in M. If F does not separate M, let $M' = \overline{M - U(F)}$; if F separates M, let M' be a component of $\overline{M - U(F)}$. Then F is an incompressible boundary component of M', and therefore the inclusion j: $M' \to M$ induces a monomorphism $j_*: \pi_1(M') \to \pi_1(M)$. It follows from Lemma 1 that $i_*\pi_1(F)$ is subnormal in $j_*\pi_1(M')$. Therefore we may apply Case 1 to M' and conclude that M' is a line bundle over F. If F is nonseparating in M, then M' has (at least) two boundary components; therefore $M' = F \times I$, and we have Case (a) of the theorem. If F separates M into M_1 and M_2, then (as in [2]) we have Case (b) or (d).

2. SUBNORMAL SUBGROUPS OF $\pi_1(F)$

THEOREM 2. *Let* F *be a surface, and let* k *be a* 2*-sided, noncontractible, simple closed curve in* F. *If* $i_*\pi_1(k)$ *is subnormal in* $\pi_1(F)$ (i: $k \to F$ *denotes inclusion*), *then one of the following holds:*

(a) F *is a torus or Klein bottle,*

(b) F *is an annulus,*

(c) F *is a Moebius strip.*

The proof of this is analogous to that of Theorem 1. We replace M by F, "incompressible surface" by "noncontractible curve," and so forth.

Definition. An n*-group* G is a group each of whose subgroups is subnormal in G.

COROLLARY. *If the fundamental group of a surface* F (*which need not be compact*) *is an* n*-group, then it is* $\mathbb{Z}_2$, $\mathbb{Z}$, *or* $\mathbb{Z} \oplus \mathbb{Z}$ (*or the trivial group*).

Proof. If F is noncompact, then $\pi_1(F)$ is free. If F is compact, then by the previous theorem, $\pi_1(F)$ is the trivial group, or $\mathbb{Z}_2$, or $\mathbb{Z}$, or $\mathbb{Z} \oplus \mathbb{Z}$, or the fundamental group of a Klein bottle K with $\pi_1(K) = |c, d: cdc^{-1} = d^{-1}|$. The latter group is not an n-group. We may either prove this directly, by showing that the normalizer of the cyclic group $\mathbb{Z}(c)$ generated by c equals $\mathbb{Z}(c)$, or we may show that $\mathbb{Z}(c)$ is not subnormal in $\pi_1(K)$, by looking at the covering space K_0 associated to $\mathbb{Z}(c)$. The surface K_0 is an open Moebius strip, and the covering $K_0 \to K$ cannot be factored by a finite sequence of regular coverings.

3. n-GROUPS AND 3-MANIFOLDS

Definition. A P^2-irreducible compact 3-manifold is called *strongly sufficiently large* if it contains a 2-sided, closed, incompressible surface.

Examples. If M is orientable and closed, then M is strongly sufficiently large if and only if it is sufficiently large (in the sense of F. Waldhausen [10]). If M is nonorientable, closed, and P^2-irreducible, then M is strongly sufficiently large. If $\partial M \neq \emptyset$ and M is boundary-irreducible (that is, if every boundary component is incompressible), then M is strongly sufficiently large. In particular, if M is the closure of the complement of a regular neighborhood of a nontrivial knot in S^3, then M is strongly sufficiently large.

Following A. G. Kurosh [3, p. 220 ff.], we call a group G an N-*group* if every proper subgroup is distinct from its normalizer. From the definition it follows that

(i) every subgroup and every factor group of an N-group is itself an N-group,

(ii) if a normal subgroup H of an N-group G has a nontrivial center and G/H is cyclic, then G itself has a nontrivial center (see [3, p. 224]).

Clearly, every n-group (as defined in Section 2) is an N-group.

THEOREM 3. (a) *If the fundamental group of a strongly sufficiently large* P^2-*irreducible* 3-*manifold* M *is an* n-*group, then* $\pi_1(M)$ *is either* $\mathbb{Z} \times \mathbb{Z}$ *or an extension*

$$1 \to \mathbb{Z}(a) \times \mathbb{Z}(b) \to \pi_1(M) \to \mathbb{Z}(c) \to 1,$$

where the matrix of the automorphism defined by c *has the form* $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ (y *an integer*).

(b) *Conversely, all such extensions are* n-*groups, and each occurs as the fundamental group of exactly one* P^2-*irreducible* 3-*manifold.*

Proof. M contains a closed, incompressible surface F, and $i_*\pi_1(F)$ is subnormal in $\pi_1(M)$. In cases (b) and (c) of Theorem 1, $\pi_1(M)$ is the fundamental group of a closed surface, and therefore, by the Corollary, $\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since every subgroup of an n-group is an n-group (Lemma 1), it follows again from the corollary that in case (d) of Theorem 1 we have the isomorphism $\pi_1(M) \cong A \underset{C}{*} B$, where A, B, C are free abelian groups of rank 2 and C is a normal subgroup of A and B of index 2. But then $\pi_1(M)$ is not an N-group, by virtue of the following result, which becomes evident if we write the elements of G in normal form.

LEMMA 2. *If* $G = A \underset{C}{*} B$ *is a nontrivial free product with amalgamated subgroup, then* A *is equal to its normalizer in* G.

In case (a) of Theorem 1, $\pi_1(M)$ is an extension of $\pi_1(F)$ by $\mathbb{Z}$, and again the corollary implies that $\pi_1(F) \cong \mathbb{Z} \oplus \mathbb{Z}$. Now $M = F \times I/(x, 0) \sim (\phi x, 1)$, where $\phi\colon F \to F$ is a homeomorphism and F is a torus. The group $\pi_1(M)$ has a presentation

$$\{u, v, t\colon\ uvu^{-1}v^{-1} = 1,\ \ t^{-1}ut = \phi_*(u),\ \ t^{-1}vt = \phi_*(v)\},$$

where ϕ_* is the automorphism induced by ϕ.

By (ii), $\pi_1(M)$ has nontrivial center. If the center is not contained in $\pi_1(F)$, let $t^k z$ be an element of the center ($k > 0$, $z \in \pi_1(F)$). Then

$$x = (z^{-1}t^{-k})x(t^k z) = z^{-1}\phi_*^k(x)z = \phi_*^k(x), \qquad \text{if } x \in \pi_1(F).$$

Thus a certain power of ϕ_* is the identity, and by [5] we may assume that ϕ is of finite order. Hence M is a Seifert fibre space whose orbit surface ("Zerlegungsfläche") is a torus or a Klein bottle (compare [9, p. 515]). This is also true if the center of $\pi_1(M)$ is contained in $\pi_1(F)$, and it may be proved as in [9, p. 516]. (Note: if M is orientable, we may apply Satz 4.1 of [9] in both cases.)

Thus $\pi_1(M)$ has a presentation (see [6])

$$\{q_1, \cdots, q_n, r, s, h:$$
$$q_i h q_i^{-1} = h, \quad rhr^{-1} = h^{\varepsilon} \ (\varepsilon = \pm 1), \quad shs^{-1} = h^{\varepsilon},$$
$$q_i^{\alpha_i} h^{\beta_i} = 1 \ [(\alpha_i, \beta_i) = 1, \ \alpha_i \geq 2], \text{ and either}$$
$$(*)\ q_1 \cdots q_n\, rsr^{-1}s^{-1} = h^b \text{ (b an integer) or}$$
$$(**)\ q_1 \cdots q_n r^2 s^2 = h^b\}.$$

If the orbit surface is a Klein bottle, that is, if we have the presentation with $q_1 \cdots q_n r^2 s^2 = h^b$, then $\pi_1(M)$ is not an N-group. For if H is the smallest normal subgroup generated by $\{q_1, \cdots, q_n, h\}$, then $\pi_1(M)/H$ is isomorphic to the fundamental group of a Klein bottle, and this is not an N-group (see the proof of Corollary 4). Thus $\pi_1(M)$ has a presentation with relation (*).

First we claim that if $\pi_1(M)$ is an N-group, then $n = 0$. For if $n \geq 1$, let H be the smallest normal subgroup containing $q_2, \cdots, q_n, h$. Then

$$\pi_1(M)/H = \{q_1, r, s: q_1^{\alpha_1} = 1, \ q_1 rsr^{-1}s^{-1} = 1; \ \alpha_1 \geq 2\} = \{r, s: (rsr^{-1}s^{-1})^{\alpha_1}\}.$$

But if $\alpha_1 \geq 2$, this group is not an N-group. (It follows from the theory of W. Magnus on groups with one defining relator [4, p. 252] that the subgroup generated by $\{s, rsr^{-1}, r^2sr^{-2}\}$ has a presentation

$$|s_0, s_1: (s_1 s_0^{-1})^{\alpha_1} = 1| \underset{\mathbb{Z}(s_1)}{*} |s_1, s_2: (s_2 s_1^{-1})^{\alpha_1} = 1|$$

and therefore is not an N-group, by Lemma 6.)

Thus, since $\alpha_i \geq 2$ for all i, we conclude that $n = 0$ and (see [6]) $\pi_1(M)$ has a presentation

$$\{r, s, h: rhr^{-1} = h^{\varepsilon}, \quad shs^{-1} = h^{\varepsilon} \ (\varepsilon = \pm 1), \quad rsr^{-1}s^{-1} = h^b$$
$$\text{(b an integer, and if } \varepsilon = -1, \text{ then } b = 0 \text{ or } b = 1)\}.$$

Now the subgroup H generated by r and h is normal, and $G/H = Z(s)$. By Stalling's Theorem [7], M is therefore a fiber bundle with $\pi_1(\text{fiber}) = H$. Since the fiber is incompressible and $\pi_1(M)$ is an n-group, there is no other possibility for H than to be a free abelian group of rank 2. In particular, r and h commute, and it follows that $\varepsilon = +1$ (since $\pi_1(M)$ has no element of finite order).

Thus the only remaining groups are

$$\pi_1(M) = \{r, s, h: rhr^{-1} = h, \quad shs^{-1} = h, \quad rsr^{-1}s^{-1} = h^b\}$$
$$= \{a, b, t: aba^{-1}b^{-1} = 1, \quad t^{-1}bt = b, \quad t^{-1}at = ab^y \text{ (y an integer)}\},$$

which proves part (a) of the theorem.

To prove part (b), let G be an extension $1 \to \mathbb{Z}(a) \times \mathbb{Z}(b) \to G \to \mathbb{Z}(c) \to 1$, where the automorphism ϕ^* defined by c is given by $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ (y an integer). It is easy to see that G is an n-group. Let T be the torus, let ϕ: T → T be a homeomorphism that induces ϕ^*, and let M = T × I/(x, 0) ~ (ϕx, 1). The 3-manifold M is P^2-irreducible, and $\pi_1(M) \approx G$. If N is a P^2-irreducible 3-manifold with $\pi_1(N) \approx G$, then N ≈ M, by Waldhausen's Theorem [10, Corollary 6.5] and [1, Corollary].

Now a nilpotent group is an n-group, and the group listed in Theorem 3 are nilpotent. Thus Theorem 3 remains true if we replace "n-group" by "nilpotent." Therefore, in cases where M is P^2-irreducible and strongly sufficiently large, we get the same groups as Charles Thomas [8]. Of course, with our methods we cannot obtain the complete list of nilpotent groups, as Thomas does for closed manifolds. However, our manifolds need not be closed. But if $\pi_1(M) \not\approx \mathbb{Z} \times \mathbb{Z}$ (that is, if M is not a line bundle over a torus) and if M is strongly sufficiently large and P^2-irreducible, it follows from Theorem 3 that if $\pi_1(M)$ is an n-group, then M is closed. In particular, there exists no strongly sufficiently large P^2-irreducible 3-manifold with boundary, other than a line bundle over a torus, with nilpotent fundamental group. More generally, we can make the following assertion.

PROPOSITION. *Suppose that* M *is a compact* 3-*manifold containing no* (2-sided) *projective planes, and that* $\pi_1(M)$ *is nilpotent.*

(a) *If* ∂M *consists only of* 2-*spheres, then* $\pi_1(M)$ *is one of the groups listed by C. Thomas* [8].

(b) *If* ∂M *contains a surface of genus at least* 1, *then* $\pi_1(M) \approx \mathbb{Z} \times \mathbb{Z}$ *or* $\mathbb{Z}$.

Proof. (a) If we fill in the 2-spheres with 3-balls, we get a closed 3-manifold M^* such that $\pi_1(M^*) \cong \pi_1(M)$, and we can apply C. Thomas's results.

(b) Let $F \subset \partial M$ have genus at least 1. We may assume that ∂M contains no 2-spheres (otherwise, we again fill these in with 3-balls).

If F is not incompressible, then by a standard argument ([2, proof of Proposition 1]) either $\pi_1(M)$ is a nontrivial free product of two groups or $\pi_1(M) \approx \mathbb{Z}$. In the first case, $\pi_1(M)$ is not an n-group. Thus we assume F to be incompressible.

Now M has a decomposition $M \approx M_1 \# \cdots \# M_n$ into handles and irreducible manifolds. Since $\pi_1(M) \cong \pi_1(M_1) * \cdots * \pi_1(M_n)$ and $\pi_1(M)$ is an n-group, it follows that $\pi_1(M_i) = 1$, for $i = 2, \cdots, n$, say. Thus $\pi_1(M) = \pi_1(M_1)$, where M_1 is either irreducible or is a handle. In the first case, M_1 is strongly sufficiently large (since $F \subset \partial M_1$) and P^2-irreducible, and thus, by Theorem 3, $\pi_1(M) \approx \mathbb{Z} \times \mathbb{Z}$. In the second case, $\pi_1(M) \approx \mathbb{Z}$.

Remark. If M contains 2-sided projective planes and $\pi_1(M)$ is nilpotent, then, looking at the 2-fold orientable cover M' and applying the Proposition to M', we see immediately that $\pi_1(M)$ is an extension of the groups in the Proposition by $\mathbb{Z}_2$. In particular, if ∂M contains an orientable surface of genus at least 1, or a nonorientable surface of genus at least 2, then $\pi_1(M)$ is an extension of $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ by $\mathbb{Z}_2$.

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