ON A CLASS OF SCHLICHT FUNCTIONS

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Let S be the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

that are analytic and schlicht in $|z| < 1$. In 1934, O. Dvořák [1] made the interesting observation that if $f \in S$ and

$$\Re \sqrt{\frac{f(z)}{z}} > \frac{1}{2} \quad (|z| < 1), \tag{1}$$

then $|a_n| \leq n$ $(n = 2, 3, \cdots)$. The proof simply uses the fact that

$$\sqrt{\frac{f(z)}{z}} - \frac{1}{2} = \frac{1}{2} + c_1 z + c_2 z^2 + \cdots$$

has positive real part, so that $|c_n| \leq 1$ $(n = 1, 2, \cdots)$. Dvořák [1] further showed that every function $f \in S$ satisfies (1) in the disk $|z| < \rho$, where

$$\rho \log \frac{1+\rho}{1-\rho} = 2\,.$$

A calculation shows that $0.833 < \rho < 0.834$. Recently, Dvořák [2] claimed to show that (1) holds in a disk $|z| < r_0$, where $0.90 < r_0 < 0.91$. He later [3] claimed an improvement to $0.98 < r_0 < 0.99$.

Unfortunately, however, these last two estimates are incorrect. In the present note, we show that the best possible radius is $R = 0.835 \cdots$. In other words, for every $f \in S$, the inequality (1) holds for every z in $|z| < R$; but for each z in $|z| > R$, there is some $f \in S$ for which (1) fails to hold. Our procedure allows the computation of R to any desired accuracy. It is curious that although Dvořák derived the constant ρ by what appears to be crude estimation, the sharp constant R is only slightly larger.

For $f \in S$, G. M. Goluzin [5] used Loewner's differential equation to establish the sharp estimate

$$\left|\arg \frac{f(z)}{z}\right| \leq \log \frac{1+r}{1-r} \quad (r = |z| < 1)\,.$$

Thus

$$\Re \sqrt{\frac{f(z)}{z}} > 0 \quad \text{for all } f \in S$$

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if and only if

$$\log \frac{1+r}{1-r} < \pi ;$$

that is, if and only if

$$|z| = r < \tanh \pi/2 = 0.917 \cdots .$$

Hence

$$(2) \qquad 0.83 < \rho \leq R < \tanh \pi/2 < 0.92 .$$

H. Grunsky [6] and Goluzin [4] also obtained the deeper result that for each fixed z, the region of values of $\log \frac{f(z)}{z}$ for all $f \in S$ is exactly the circular disk

$$|w + \log(1 - r^2)| \leq \log \frac{1+r}{1-r} \qquad (r = |z|) .$$

Thus the region of values of $\log \sqrt{\frac{f(z)}{z}}$ is the circular disk $|w - a| \leq b$, where

$$a = -\frac{1}{2} \log(1 - r^2), \qquad b = \frac{1}{2} \log \frac{1+r}{1-r} .$$

In view of (2), we need consider only the interval $\rho \leq r < \tanh \pi/2$, where $1 < b < \pi/2$. It follows from the considerations above that the set of all values of $\sqrt{\frac{f(z)}{z}}$, as f ranges over S, is

$$E_r = \{e^w : |w - a| \leq b\} \qquad (r = |z|) .$$

The radius R is then the largest value of r for which the set E_r lies in the half-plane $\Re \zeta \geq 1/2$.

For each r, we wish to compute the minimum of $\Re \zeta$ for all $\zeta \in E_r$. This is equivalent to finding the minimum of

$$F(\theta) = \{\exp(a + b \cos \theta)\} \cos(b \sin \theta) \qquad (-\pi < \theta \leq \pi) .$$

But differentiation gives the equation

$$F'(\theta) = -b \{\exp(a + b \cos \theta)\} \sin(\theta + b \sin \theta),$$

which shows that $F'(\theta) = 0$ if and only if

$$\sin(\theta + b \sin \theta) = 0 \qquad (-\pi < \theta \leq \pi) .$$

This implies that

$$(3) \qquad \theta + b \sin \theta = n\pi \qquad (n = 0, \pm 1, \pm 2, \cdots) .$$

But since $-\pi < \theta \leq \pi$ and $1 < b < \pi/2$, the only possibilities are $n = 0$, $n = 1$, and $n = -1$.

Case I: $n = 0$. Here the only solution is $\theta = 0$, where F attains its maximum.

Case II: $n = 1$. Here equation (3) has the solution $\theta = \pi$, but

$$F(\pi) = e^{a-b} = (1+r)^{-1} > 1/2$$

for all r $(0 \le r < 1)$. Equation (3) also has the solution $\theta = \theta_0$ $(\pi/2 < \theta_0 < \pi)$; this gives the point where the sinusoid $y = \sin\theta$ intersects the line $y = (\pi - \theta)/b$. There is exactly one intersection in this interval, because $(\pi - \theta)/b > \sin\theta$ at $\theta = \pi/2$, the two curves intersect at $\theta = \pi$, and the line has slope $-1/b > -1$.

Case III: $n = -1$. Here equation (3) has only the solution $\theta = -\theta_0$; but $F(-\theta_0) = F(\theta_0)$.

Thus, for each value of r $(\rho \le r < \tanh \pi/2)$, one may compute the minimum

$$m(r) = \min_{\zeta \in E_r} \Re \zeta$$

by determining the unique solution $\theta_0 = \theta_0(r)$ of the equation

$$\theta + b\sin\theta = \pi \qquad (\pi/2 < \theta < \pi).$$

If $F(\theta_0(r)) \le (1+r)^{-1}$, then $m(r) = F(\theta_0(r))$. Using detailed tables of logarithms and trigonometric functions, we found that

$$m(0.835) = 0.5011 \cdots > 1/2, \qquad m(0.836) = 0.4992 \cdots < 1/2.$$

Thus $0.835 < R < 0.836$.

Dvořák [1] also remarked that for odd univalent functions $f \in S$ such that

$$\Re \frac{f(z)}{z} > \frac{1}{2} \tag{4}$$

in $|z| < 1$, the coefficients satisfy the inequality $|a_n| \le 1$ $(n = 3, 5, \cdots)$. He noted that (4) holds in the disk of radius

$$\sqrt{\rho} = 0.912 \cdots;$$

and he claimed corresponding improvements in [2] and [3]. However, the preceding discussion shows after further calculation that the sharp radius is

$$\sqrt{R} = 0.914 \cdots.$$

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