RIESZ POTENTIALS, k, p-CAPACITY, AND p-MODULES

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1. INTRODUCTION

Let R^m denote m-dimensional Euclidean space with points $x = (x_1, x_2, \cdots, x_m)$ and Euclidean norm $|x|$. For $p \geq 1$, we denote by $\|f\|_p$ the L^p-norm of f taken over the whole space R^m. Let $s = (s_1, s_2, \cdots, s_m)$ be a multi-index with length $|s| = \sum s_i$, and let $D^s f$ be the corresponding derivative of f of order $|s|$. As usual, C_0^∞ is the class of all infinitely differentiable functions with compact support. Finally, k is a positive integer, and F is a compact subset of R^m.

A measure of the size of a set F is given by the k, p-capacity of F, which we define as follows.

Definition 1. The k, p-*capacity of* F is

$$\Gamma_{k,p}(F) = \inf_f \sum_{|s| \leq k} \|D^s f\|_p^p,$$

where the infimum is taken over all $f \in C_0^\infty$ with $f \geq 1$ on F.

We get the same class of null-sets if in the definition we require all the functions f to have support in some fixed neighbourhood O of F. In fact, if $\phi \in C_0^\infty$ has support in O and $\phi = 1$ on F, then $f\phi$ has support in O, $f\phi \geq 1$ on F, and

$$\sum_{|s| \leq k} \|D^s(f\phi)\|_p \leq \text{const.} \sum_{|s| \leq k} \|D^s f\|_p,$$

where the constant does not depend on f.

We also get the same class of null-sets if in the sum in the definition we take $|s| = k$ instead of $|s| \leq k$ (if $kp \geq m$, we must then assume that the support of f is a subset of a fixed sphere). This may be proved by means of inequalities of Sobolev type.

For k = 1, the notion of k, p-capacity was used by Serrin [4] in the investigation of removable singularities of partial differential equations. It has also been used in the theory of quasiconformal mappings in R^m (Gehring [3]).

By the *Riesz potential of order* α $(0 < \alpha < m)$ *of the function* f (or the α-potential of f) we shall mean the function U_α^f defined by

$$U_\alpha^f(x) = \int \frac{f(y)\,dy}{|x - y|^{m-\alpha}}.$$

The purpose of this paper is to prove the following theorem.

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THEOREM 1. *Suppose that* F *is a compact subset of* R^m, $p \geq 1$, *and* $1 \leq k < m$. *A necessary condition that*

$$\Gamma_{k,p}(F) = 0$$

is that there exists a nonnegative function $f \in L^p(R^m)$ *with compact support such that*

$$U_k^f(x) = \int \frac{f(y)\,dy}{|x-y|^{m-k}} = \infty \quad \textit{for every } x \in F .$$

For $p > 1$, *the condition is also sufficient; for* $p = 1$, *it is sufficient under the additional assumption that* $f \log^+ f \in L^1(R^m)$.

The condition $f \log^+ f \in L^1(R^m)$ for $p = 1$ may not be omitted. We prove this in Section 4, where we also comment on the cases $kp > m$ and $k = m$.

Before proceeding to the proof of Theorem 1, we shall give an account of the connection between this theorem and earlier results. For this purpose, we introduce the p-module of certain systems of k-dimensional Lipschitz surfaces in R^m. A *k-dimensional Lipschitz surface* S *in* R^m is a nonempty Borel subset of R^m that is locally the image of some open subset of R^k, under a one-to-one transformation having the Lipschitz property in both directions. Let $d\sigma$ denote the surface measure of S (which can be defined for a k-dimensional Lipschitz surface; see Fuglede [2, p. 184]).

Definition 2 (Fuglede [2, p. 187]). Let E be a nonempty subset of R^m, and let k be an integer $(1 \leq k < m)$. Let $\mathfrak{S}^k(E)$ be the system of all k-dimensional Lipschitz surfaces that intersect E. The p-*module* of $\mathfrak{S}^k(E)$ is

$$M_p(\mathfrak{S}^k(E)) = \inf_f \|f\|_p^p ,$$

where the infimum is taken over all Lebesgue-measurable, nonnegative functions f such that

$$\int_S f\,d\sigma \geq 1 \qquad \text{for every } S \in \mathfrak{S}^k(E)$$

($d\sigma$ denotes surface measure).

Fuglede has proved the following theorem (stated for $kp \leq m$).

THEOREM 2 (Fuglede [2, p. 191]). *Theorem* 1 *remains valid if* $\Gamma_{k,p}(F)$ *is replaced by* $M_p(\mathfrak{S}^k(F))$ (*even if* F *is an arbitrary set*).

Remark. Theorem 2 remains true even if only very regular surfaces are considered, instead of Lipschitz surfaces.

Fuglede [2, p. 199] and I [6] have given connections between the condition that a certain potential-theoretic α-capacity of F equals zero and the conditions $M_p(\mathfrak{S}^k(F)) = 0$ and $\Gamma_{k,p}(F) = 0$, respectively. However, because of an ε-gap, these results do not reveal the exact connection between the conditions $M_p(\mathfrak{S}^k(F)) = 0$ and $\Gamma_{k,p}(F) = 0$. Recently, Ziemer proved a theorem (stated for $p < m$) that implies the following result.

THEOREM 3 (Ziemer [7, p. 50]). *Suppose* F *is a compact subset of* R^m, $p \geq 1$, *and* $k = 1 < m$. *Then*

$$\Gamma_{k,p}(F) = 0 \Longleftrightarrow M_p(\mathfrak{S}^k(F)) = 0\,. \tag{1}$$

It is not obvious that Ziemer's proof can be generalized to the case $k > 1$. For $k = 1$, Theorem 1 is a consequence of Theorems 2 and 3. On the other hand, for $p > 1$, Theorem 1 together with Theorem 2 yields a new proof of Theorem 3 and generalizes it to the case $k > 1$. Theorems 1 and 2 imply that (1) *holds if* F *is compact,* $p > 1$, *and* $1 \leq k < m$. The method by which we shall prove Theorem 1 shows clearly the connection between α-potentials and the condition $\Gamma_{k,p}(F) = 0$.

2. PROOF OF THE NECESSITY IN THEOREM 1

Suppose F is a compact set, $p \geq 1$, and $1 \leq k < m$. Suppose that $\Gamma_{k,p}(F) = 0$. By the definition of $\Gamma_{k,p}(F)$, there exist functions $g_n \in C_0^\infty$ ($g_n \geq 1$ on F) such that

$$\sum_{|s|=k} \|D^s g_n\|_p < 2^{-n} \qquad (n = 1, 2, \cdots)\,. \tag{2}$$

We may also assume that the support of g_n is a subset of a fixed bounded set (see the remarks after Definition 1).

We shall use the following representation formulas (see Wallin [5, p. 71]), where the a_s are constants and the sums extend over a number of multiindices s with length $|s| = k$: If either $m - 2k > 0$ or else m is odd and $m - 2k < 0$, then

$$g_n(x) = \sum a_s \int D^s(|x - y|^{2k-m}) D^s g_n(y)\,dy\,; \tag{3}$$

if $m - 2k \leq 0$ and m is even, then

$$g_n(x) = \sum a_s \int D^s(|x - y|^{2k-m} \log|x - y|) D^s g_n(y)\,dy\,. \tag{4}$$

Observe that the equations (3) and (4) hold for all x, since both members are continuous functions. Now, for $|s| = k$,

$$D^s(|x - y|^{2k-m}) \leq \frac{\text{const.}}{|x - y|^{m-k}}\,.$$

Furthermore, for $m - 2k \leq 0$ (m even) and $m - k > 0$, it is easy to see that

$$D^s(|x - y|^{2k-m} \log|x - y|) \leq \frac{\text{const.}}{|x - y|^{m-k}}\,.$$

If we put

$$f_n(y) = \sum_{|s|=k} b_s\, |D^s g_n(y)|\,,$$

then these estimates and the formulas (3) and (4) give (with appropriate constants $b_s \geq 0$) the bound

$$|g_n(x)| \leq \int \frac{f_n(y)\,dy}{|x - y|^{m-k}} = U_k^{f_n}(x)\,. \tag{5}$$

Clearly, $f_n \geq 0$, the supports satisfy the inclusion relation $\operatorname{supp} f_n \subseteq \operatorname{supp} g_n$, and (by virtue of (2)),

$$\|f_n\|_p < \text{const.}\, 2^{-n}.$$

From (5) and the fact that $g_n \geq 1$ on F, we infer that $U_k^{f_n} \geq 1$ on F. Now we put

$$f = \sum_1^\infty f_n .$$

Then $f \geq 0$ and $f \in L^p(R^m)$, since

$$\|f\|_p \leq \sum_1^\infty \|f_n\|_p < \text{const.} \sum_1^\infty 2^{-n} < \infty .$$

The function f has bounded support, since $\operatorname{supp} f_n \subseteq \operatorname{supp} g_n$ and $\operatorname{supp} g_n$ is uniformly bounded in n. Finally, $U_k^f = \infty$ on F, because for each N

$$U_k^f(x) \geq \sum_{n=1}^N U_k^{f_n}(x) \geq N \qquad \text{when } x \in F .$$

Hence the function f has all the properties required for f in Theorem 1.

3. PROOF OF THE SUFFICIENCY IN THEOREM 1

We shall use the theory of singular integrals from Calderon and Zygmund [1] and Fuglede [2] (see in particular [2, pp. 193-198]). For $\varepsilon > 0$ and corresponding to any function f, we put

$$\phi_\varepsilon(x) = \frac{1}{(|x|^2 + \varepsilon^2)^{(m-k)/2}},$$

$$\phi(x) = \frac{1}{|x|^{m-k}},$$

and

$$u_\varepsilon(x) = \int \phi_\varepsilon(x-y)\, f(y)\, dy = (\phi_\varepsilon * f)(x),$$

$$u(x) = (\phi * f)(x) = U_k^f(x) .$$

Then $u_\varepsilon \in C^\infty$ and $D^s u_\varepsilon(x) = (D^s \phi_\varepsilon * f)(x)$ for every s. Now

$$|D^s \phi_\varepsilon(x)| < \frac{\text{const.}}{|x|^{m-k+|s|}},$$

and the right-hand side is locally integrable when $|s| < k$. By Lebesgue's dominated-convergence theorem, this means that, for $|s| < k$, $D^s \phi_\varepsilon \to D^s \phi$ in the mean of order 1 over every bounded set, as $\varepsilon \to 0$. Using the inequality

$\|g*f\|_p \leq \|g\|_1 \|f\|_p$, with g equal to $D^s \phi_\varepsilon - D^s \phi$ in a certain neighbourhood of the origin and 0 elsewhere, we obtain the following proposition.

If $f \in L^p$, f has bounded support, and $|s| < k$, then $D^s u_\varepsilon \to (D^s \phi) * f$ in the mean of order p over every bounded subset of R^m, as $\varepsilon \to 0$.

For $|s| = k$, we shall use the following lemma from the theory of singular integrals.

LEMMA (Calderon and Zygmund [1]; Fuglede [2, p. 195]). *Let* $D^s u_\varepsilon$ *be an arbitrary derivative of order* $|s| = k$ *of the function* $u_\varepsilon = \phi_\varepsilon * f$, *where* ϕ_ε *is defined by* (6) *and* $1 \leq k < m$.

a) *If* $1 < p < \infty$ *and* $f \in L^p(R^m)$, *then* $D^s u_\varepsilon$ *converges in the mean of order* p *over* R^m, *as* $\varepsilon \to 0$.

b) *If* f *is Lebesgue-measurable*, $f \log^+ |f| \in L^1(R^m)$, *and* f *has compact support, then* $D^s u_\varepsilon$ *converges in the mean of order* 1 *over every subset of* R^m *of finite Lebesgue measure, as* $\varepsilon \to 0$.

Now suppose that F is a compact subset of R^m, $p \geq 1$, and $1 \leq k < m$. Suppose that there exists a nonnegative function $f \in L^p(R^m)$ with compact support such that the potential U_k^f is infinite on F. If $p = 1$, we also assume that $f \log^+ f \in L^1(R^m)$. Consider $u_\varepsilon = \phi_\varepsilon * f$ with this function f, where ϕ_ε is defined by (6). Clearly, $u_\varepsilon(x) \nearrow u(x) = U_k^f(x)$ as $\varepsilon \searrow 0$. Since F is compact and $U_k^f(x) = \infty$ on F, we can, for every positive integer n, choose an $\varepsilon_n > 0$ such that $u_{\varepsilon_n}(x) > n$ for $x \in F$. Put

$$f_n = \frac{u_{\varepsilon_n} \cdot \psi}{n},$$

where $\psi \in C_0^\infty$ is 1 on F. Then $f_n \in C_0^\infty$ and $f_n \geq 1$ on F. We wish to prove that

$$\sum_{|s| \leq k} \|D^s f_n\|_p \to 0 \quad \text{as } n \to \infty. \tag{7}$$

In fact, by Definition 1, (7) implies that $\Gamma_{k,p}(F) = 0$.

By Leibnitz's rule,

$$D^s f_n = \frac{1}{n} \sum_{\alpha \leq s} c_{s,\alpha} D^{s-\alpha} u_{\varepsilon_n} \cdot D^\alpha \psi, \tag{8}$$

where the $c_{s,\alpha}$ are constants. To prove (7), we shall estimate $\|D^{s-\alpha} u_{\varepsilon_n} D^\alpha \psi\|_p$ for different values of α and s, with $|s| \leq k$.

Convergence in the mean of order p implies uniform boundedness in the L^p-norm. We may therefore use the lemma and the remarks preceeding it to conclude that if $|s| \leq k$, then the L^p-norm of $D^{s-\alpha} u_{\varepsilon_n}$ over any fixed compact subset of R^m is bounded in n. Hence, for all α, the quantity $\|D^{s-\alpha} u_{\varepsilon_n} \cdot D^\alpha \psi\|_p$ is bounded in n when $|s| \leq k$. This and (8) give (7), and the proof of Theorem 1 is complete.

4. REMARKS

For $p = 1$, the condition $f \log^+ f \in L^1(R^m)$ occurs in one half of Theorem 1. In the following proposition we shall show that the theorem is not true if we omit this condition.

PROPOSITION 1. *Consider* R^m *for* $m = 2$, $k = 1$, $p = 1$, *and*

$$F = \{x \in R^2 \mid |x| = 1\}.$$

Then $\Gamma_{1,1}(F) > 0$, *and there exists a nonnegative function* $f \in L^1(R^m)$ *with compact support such that*

$$U_1^f(x) = \int \frac{f(y)\,dy}{|x - y|} = \infty \quad \textit{for } x \in F.$$

Proof. Suppose $f \in C_0^\infty$ and $f \geq 1$ on F. If (r, θ) are polar coordinates in R^2, then for all θ

$$\int_0^\infty \left|\frac{\partial f(r, \theta)}{\partial r}\right| r\,dr \geq \int_1^\infty \left|\frac{\partial f(r, \theta)}{\partial r}\right| dr \geq 1,$$

because the total variation of f is at least 1 on the half-line determined by θ and $1 \leq r < \infty$. By integrating with respect to θ, we conclude that $\Gamma_{1,1}(F) > 0$.

Now we turn to the construction of f. Let μ be a measure supported by F such that μ is uniformly distributed on F and $\mu(R^m) > 0$. It is easy to see that

$$U_1^\mu(x) = \int \frac{1}{|x - y|}\,d\mu(y) = \infty \quad \text{for } x \in F.$$

Take $\mu = \mu_i$ so that $\sum_1^\infty \mu_i(R^m) < \infty$. Put $f_i = \psi_i * \mu_i$, where $\psi_i \geq 0$, $\psi_i \in C_0^\infty$, ψ_i is supported by a fixed bounded set, and $\|\psi_i\|_1 = 1$. We can also choose each ψ_i so that $U_1^{f_i} \geq 1$ on F, since

$$U_1^{f_i} = \psi_i * U_1^{\mu_i} \quad \text{and} \quad U_1^{\mu_i}(x) \to \infty \quad \text{as } x \to x_0 \in F.$$

If we now put $f = \sum_1^\infty f_i$, then the function f fulfills the conditions in the proposition. In fact, $U_1^f = \infty$ on F, since $U_1^{f_i} \geq 1$ on F for each i, and

$$\|f\|_1 \leq \sum_1^\infty \|f_i\|_1 \leq \sum_1^\infty \|\psi_i\|_1\,\mu_i(R^m) < \infty.$$

We shall now comment on the cases $kp > m$ and $k = m$.

PROPOSITION 2. *Suppose* $p \geq 1$ *and* $m > k \geq 1$. *Then* $\Gamma_{k,p}(F) > 0$ *for every nonempty compact set* F *if and only if* $kp > m$.

Proof. This may be proved by means of Theorem 1, for example. Assume that $kp > m$, and let $f \in L^p(R^m)$ be a function with compact support. An application of Hölder's inequality shows that

$$|U_k^f(x)| \leq \|f\|_p \left(\int_{\mathrm{supp}\, f} \frac{dy}{|x-y|^{(m-k)p/(p-1)}}\right)^{(p-1)/p} < \infty$$

for all x, since $(m - k)p/(p - 1) < m$. Hence U_k^f is finite everywhere, and Theorem 1 gives one half of our assertion.

If $kp \leq m$ and $1 \leq k < m$, we can make U_k^f infinite at some point, for instance at 0, by using a nonnegative function f with compact support such that $f \in L^p(R^m)$ if $p > 1$ and $f \log^+ f \in L^1(R^m)$ if $p = 1$. In fact, $U_k^f(0) = \infty$ if we choose

$$f(x) = \begin{cases} |x|^{-k} \,|\log |x||^{-1} & (|x| \leq 1/2), \\ 0 & (|x| > 1/2). \end{cases}$$

By Theorem 1, this means that $\Gamma_{k,p}(F) = 0$ when F consists of a single point.

Remark. The result corresponding to Proposition 2 for the p-module was proved in a different way by Fuglede [2, p. 190].

PROPOSITION 3. *If* $p \geq 1$ *and* $k = m$, *then* $\Gamma_{k,p}(F) > 0$ *for all nonempty sets* F.

Proof. For every $f \in C_0^\infty$,

$$f(x) = \int_{y \leq x} \frac{\partial^m f(y)\, dy}{\partial y_1 \partial y_2 \cdots \partial y_m}.$$

This is obtained by repeated integration in the right-hand member of the equation. If there exists at least one point x at which $f(x) \geq 1$, the equation above gives

$$\left\| \frac{\partial^m f}{\partial y_1 \partial y_2 \cdots \partial y_m} \right\|_1 \geq 1,$$

and from this the proposition follows.

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