COMPACT INTERTWINING OPERATORS

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Let T_1 and T_2 be bounded operators acting on Hilbert spaces H_1 and H_2, respectively. A bounded operator X from H_1 to H_2 that satisfies the condition $XT_1 = T_2X$ is called an *intertwining operator* for T_1 and T_2. We denote the class of such operators by $\mathscr{I}(T_1, T_2)$. Clearly, with the usual operator norm, $\mathscr{I}(T_1, T_2)$ is a Banach space.

It is of interest to look for compact operators in $\mathscr{I}(T_1, T_2)$. That nonzero compact operators need not exist in $\mathscr{I}(T_1, T_2)$, even when $\mathscr{I}(T_1, T_2)$ is relatively large, is clear if for example we take T_1 and T_2 to be the simple unilateral shift; it is well known that there exist no compact analytic Toeplitz operators different from zero. P. Muhly [7] in effect found conditions that guarantee the existence of nonzero compact operators in $\mathscr{I}(T_1, T_2)$.

An example of the opposite extreme occurs when both T_1 and T_2 are the identity operator on a Hilbert space. In this case, there are so many compact operators in $\mathscr{I}(T_1, T_2)$ that if we let $\mathscr{C}(T_1, T_2)$ denote the subspace of $\mathscr{I}(T_1, T_2)$ consisting of the compact operators, then $\mathscr{C}(T_1, T_2)$ has $\mathscr{I}(T_1, T_2)$ as its second dual (R. Schatten [11]). Another example of this phenomenon is the case where T_1 is a unilateral shift of arbitrary multiplicity on a separable, complex Hilbert space, while $T_2 = T_1^*$. That $\mathscr{I}(T_1, T_2)$ is here the second dual of $\mathscr{C}(T_1, T_2)$ was shown by R. N. Hevener [5] in case T_1 is the simple unilateral shift, and by Page [9] in case T_1 is a shift of higher multiplicity. Hevener's result is a consequence of the work of Z. Nehari [8] on bounded Hankel operators and of P. Hartman [3] on compact Hankel operators.

This paper concerns the second of the two extreme possibilities, that is, the case where

$$\mathscr{C}(T_1, T_2)^{**} \simeq \mathscr{I}(T_1, T_2)\,.$$

We prove that this biduality relation holds if T_1 and T_2 are compressions of the simple unilateral shift to co-invariant subspaces. (See concluding comments.)

Let $\mathscr{T}$ and m denote the unit circle in the complex plane and normalized Lebesgue measure on $[0, 2\pi)$. The spaces $L^p(dm)$ $(1 \leq p \leq \infty)$ will be the standard spaces of appropriate complex functions on $\mathscr{T}$. Usually, $L^p(dm)$ and its Hardy subspace $H^p(dm)$ will be denoted simply by L^p and H^p. By H_0^1 we mean the subspace of functions f in H^1 for which $\int_0^{2\pi} f(e^{it})\,dm(t) = 0$. Following tradition, we use χ to denote the identity function on $\mathscr{T}$, so that the simple unilateral shift on H^2 is given by U: $f \to \chi f$. For a more detailed discussion, see [4] and [6].

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Let ψ_1 and ψ_2 be nonconstant inner functions, and set $K_i = H^2 \ominus \psi_i H^2$ $(i = 1, 2)$. Let P_i be the orthogonal projection of H^2 onto K_i and set $S_i = P_i U \mid K_i$. If by δ we denote the greatest common divisor (g.c.d.) of ψ_1 and ψ_2, then we can formulate a slight generalization of a result of D. Sarason [10, Section 2] as follows (the symbol $\simeq$ is to be interpreted throughout as meaning "isometrically isomorphic as Banach spaces"):

LEMMA 1. $\mathscr{I}(S_1, S_2) \simeq H^\infty/\delta H^\infty$.

Proof. By the lifting theorem of Sz.-Nagy and Foiaş (see [12] and [2]), $X \in \mathscr{I}(S_1, S_2)$ if and only if $X = P_2 Y \mid K_1$, where

i) $YU = UY$,

ii) $P_2 Y(I - P_1) = 0$, and

iii) $\|Y\| = \|X\|$.

Since the commutant of U is H^∞, it follows from (i) that $Y: f \to \phi f$ for some $\phi \in H^\infty$ with the property $\|\phi\|_\infty = \|Y\|$. Therefore $\phi\psi_1 H^2 \subseteq \psi_2 H^2$, by (ii), so that $\phi\psi_1 = \psi_2 \lambda$ for some $\lambda \in H^\infty$. If we set $\gamma_1 = \bar{\delta}\psi_1$ and $\gamma_2 = \bar{\delta}\psi_2$, then the g.c.d. of γ_1 and γ_2 is a constant of modulus 1. Using this and the relation $\phi\gamma_1 = \gamma_2\lambda$, we can conclude that $\phi \in \gamma_2 H^\infty$. Conversely, a retracing of the above argument shows that if $\phi \in \gamma_2 H^\infty$, then $Y(\psi_1 H^2) \subseteq \psi_2 H^2$. Therefore, if

$$\mathscr{A} = \{Y \in \mathscr{I}(U, U): Y(\psi_1 H^2) \subseteq \psi_2 H^2\},$$

then $\mathscr{A} \simeq \gamma_2 H^\infty$.

The linear mapping $Y \to P_2 Y \mid K_1$ from $\mathscr{A}$ to $\mathscr{I}(S_1, S_2)$ is onto by the lifting theorem. Its kernel is $\{Y \in \mathscr{A}: YK_1 \subseteq \psi_2 H^2\}$, which is easily seen to be isometrically isomorphic to $\psi_2 H^\infty$ under the natural isomorphism of the commutant of U onto H^∞. Hence $\gamma_2 H^\infty/\psi_2 H^\infty \simeq \mathscr{I}(S_1, S_2)$. Finally, the mapping that sends a coset $\phi + \delta H^\infty$ in $H^\infty/\delta H^\infty$ onto the coset $\gamma_2\phi + \psi_2 H^\infty$ in $\gamma_2 H^\infty/\psi_2 H^\infty$ is an isometric isomorphism. Therefore $H^\infty/\delta H^\infty \simeq \mathscr{I}(S_1, S_2)$, and the proof is complete.

Let C denote the complex-valued continuous functions on the unit circle. Define $T_\phi: K_1 \to K_2$ to be the operator $T_\phi f = P_2(\phi f)$ for $\phi \in \gamma_2 H^\infty$, and let A denote $H^\infty \cap C$.

To follow the thrust of Lemma 1, we shall need Sarason's Theorem 2 of [10] stated in an intertwining rather than commutant framework. The original proof goes through with only minor modification, and therefore we omit it.

LEMMA 2. *If* $\phi \in \gamma_2 H^\infty$, *then* T_ϕ *is compact if and only if* $\bar{\psi}_2 \phi \in H^\infty + C$.

LEMMA 3. $C(S_1, S_2) \simeq [\bar{\delta} H^\infty \cap C]/A$.

Proof. By Lemma 2, T_ϕ is compact if and only if $\bar{\psi}_2 \phi \in H^\infty + C$, or equivalently, if and only if $\phi \in \psi_2(H^\infty + C)$. Hence

$$[\psi_2(H^\infty + C) \cap \gamma_2 H^\infty]/\psi_2 H^\infty \simeq C(S_1, S_2).$$

In addition,

$$[\bar{\delta} H^\infty \cap (H^\infty + C)]/H^\infty \simeq [\psi_2(H^\infty + C) \cap \gamma_2 H^\infty]/\psi_2 H^\infty.$$

Let F denote the mapping of $[\bar{\delta} H^\infty \cap C]/A$ to $[\bar{\delta} H^\infty \cap (H^\infty + C)]/H^\infty$ given by $F(\phi + A) = \phi + H^\infty$.

Sarason [10] and other authors have observed that the theorem of F. and M. Riesz may be used to show that H_0^1 is the dual of C/A. Furthermore, L^∞/H^∞ is the dual of H_0^1, and the canonical injection of C/A into its second dual L^∞/H^∞ is the mapping $\phi + A \to \phi + H^\infty$. (See the proof of Theorem 2 in [10].) Clearly, F above is isometric, since it is the restriction to $[\bar{\delta} H^\infty \cap C]/A$ of the canonical injection of C/A into its second dual. We need only show that F is onto. If $y \in \bar{\delta} H^\infty \cap (H^\infty + C)$, then there exist η and λ in H^∞ and $c \in C$ such that $y = \bar{\delta}\eta = \lambda + c$. Moreover, $c \in \bar{\delta} H^\infty \cap C$, and this implies that $F(c + A) = c + H^\infty = \bar{\delta}\eta + H^\infty = y + H^\infty$. Thus F is onto; hence,

$$[\bar{\delta} H^\infty \cap C]/A \simeq [\bar{\delta} H^\infty \cap (H^\infty + C)]/H^\infty \simeq C(S_1, S_2),$$

and the proof is complete.

Suppose that, considered as a function on the open unit disc $\mathscr{D}$, α is an inner function, and let $\beta\sigma = \alpha$ be its unique factorization into a Blaschke product β and a singular inner function σ. Then

$$\beta(z) = kz^n \prod_{j \geq 1} \frac{\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z} \qquad (|z| < 1) \tag{1}$$

and

$$\sigma(z) = \exp\left\{ - \int_0^{2\pi} h(t, z)\, ds(t) \right\} \qquad (|z| < 1); \tag{2}$$

here k is a constant ($|k| = 1$), n is a nonnegative integer, $\{z_j\}$ is the sequence of zeros of α in $0 < |z| < 1$, $h(t, z) = (e^{it} + z)(e^{it} - z)^{-1}$, and s is a singular, finite, positive, regular Borel measure on $[0, 2\pi]$.

The support of β is the intersection of the closure of $\{z_j\}_{j \geq 1}$ with $\mathscr{T}$, the support of σ is the closed support of the measure on $\mathscr{T}$ induced by s, and the support of α is the union of the supports of β and σ.

Remark. An inner function is continuous on the complement of its support.

Corresponding to each inner function α, we denote by Q_α the class of inner functions μ that divide α and for which the support of $\alpha\bar{\mu}$ has Lebesgue measure 0.

LEMMA 4. *Let* α *be an inner function. The* g.c.d. *of all functions in* Q_α *is a constant.*

Proof. Let $\alpha = \beta\sigma$ where β and σ have the forms (1) and (2). Then

$$Q_\alpha = \{\mu_1 \mu_2 \colon \mu_1 \in Q_\beta \text{ and } \mu_2 \in Q_\sigma\}.$$

If α_0 is the g.c.d. of all functions in Q_α, then $\alpha_0 = \beta_0 \sigma_0$, where β_0 and σ_0 are the g.c.d.'s of all the functions in Q_β and Q_σ, respectively. Note that β_0 is a Blaschke product and that σ_0 is singular. We shall show that both are constant inner functions.

Assume that β_0 is not constant, and write $\beta_0(z) = kz^m \prod_{j \geq 1} \frac{\bar{w}_j}{|w_j|} \frac{w_j - z}{1 - \bar{w}_j z}$.

If the set $\{w_j\}$ is empty, then $\beta_0(z) = kz^m$, where $m > 0$. Let

$$\beta_1(z) = k \prod_{j \geq 1} \frac{\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z}.$$

Clearly, β_1 divides β, and $\beta\bar{\beta}_1 = z^n$ has empty support. Thus, $\beta_1 \in Q_\beta$. However, β_0 does not divide β_1, contrary to the definition of β_0.

If $\{w_j\}$ is not empty, set $\beta_1(z) = kz^n \prod_{z_j \neq w_1} \frac{\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z}$. As before, $\beta_1 \in Q_\beta$ but β_0 does not divide β_1. Hence β_0 is constant.

Now assume that the factor σ is not constant. If s_0 is the associated singular measure of σ_0, then $s_0 \leq s$, since σ_0 divides σ. By the definition of σ_0, it follows that $s_0 \leq u$ whenever u is a positive Borel measure with $u \leq s$ and $s - u$ is supported on a closed set of measure 0.

Since s is singular, there exists a set $\mathcal{E} \subset [0, 2\pi)$ of Lebesgue measure 0 such that $s(\mathcal{E}) = s([0, 2\pi)) > 0$. Corresponding to each $\varepsilon > 0$, there exists a closed set $\mathcal{K} \subseteq \mathcal{E}$ such that $s(\mathcal{K}) > s(\mathcal{E}) - \varepsilon$. We define a measure u by

$$u(\mathcal{F}) = s(\mathcal{F}) - s(\mathcal{F} \cap \mathcal{K}).$$

Clearly, $u \leq s$ and $(s - u)(\mathcal{F}) = s(\mathcal{F} \cap \mathcal{K})$, so that $s - u$ is supported in $\mathcal{K}$. Hence $s_0 \leq u$, but $u(\mathcal{E}) = s(\mathcal{E}) - s(\mathcal{K}) < \varepsilon$; therefore, s_0 is the zero measure. Thus σ_0 is constant, and together with the argument above, this implies that α_0 is constant.

LEMMA 5. *$H^\infty \cap \delta C$ is weak-star-dense in H^∞.*

Proof. $H^\infty \cap \delta C$ is invariant under multiplication by χ, as is its weak-star closure. Thus the weak-star closure of $H^\infty \cap \delta C$ has the form ξH^∞ for some inner function ξ [4, p. 25]. Suppose $\alpha \in Q_\delta$; then $\delta\bar{\alpha}$ has closed support $\mathcal{J}$ of Lebesgue measure zero. We can choose a continuous outer function η that vanishes on $\mathcal{J}$ [6, p. 80]. Since $\delta\bar{\alpha}$ is continuous on $\mathcal{T} - \mathcal{J}$, it follows that $\bar{\delta}\alpha\eta$ is continuous on $\mathcal{T}$. Therefore, $\alpha\eta \in H^\infty \cap \delta C \subseteq \xi H^\infty$. Thus $\alpha\eta = \xi\mu$ for some $\mu \in H^\infty$. Since η is an outer function, ξ divides α. Thus, by Lemma 4, ξ is a constant inner function. The proof of the lemma is complete.

Definition. If B is a Banach space and D is a subspace of B, then $D^\perp$, the *annihilator* in B^* of D, is

$$D^\perp = \{b \in B^*: b(d) = 0 \text{ for all } d \in D\}.$$

THEOREM. $C(S_1, S_2)^{**} \simeq \mathcal{I}(S_1, S_2)$.

Proof. By virtue of Lemmas 1 and 2, it suffices to identify $H^\infty/\delta H^\infty$ with the bidual of $[\bar{\delta}H^\infty \cap C]/A$. We indicated earlier that $(C/A)^* \simeq H_0^1$. We claim now that δH_0^1 is the annihilator in H_0^1 of $[\bar{\delta}H^\infty \cap C]/A$ in C/A. To see this, note that $\delta H_0^1 \subseteq \{[\bar{\delta}H^\infty \cap C]/A\}^\perp$. On the other hand, if $g \in H_0^1$ is in $\{[\bar{\delta}H^\infty \cap C]/A\}^\perp$, then $\int g\bar{\delta}\lambda\, dm = 0$ for all $\lambda \in H^\infty \cap \delta C$. Let $\phi \in H^\infty$; by Lemma 5, some net $\{\lambda_j\}$ in $H^\infty \cap \delta C$ converges to ϕ in the weak-star topology. Consequently,

$$\int (g\bar{\delta})\phi\, dm = \lim \int (g\bar{\delta})\lambda_j\, dm = 0$$

and hence $g\bar{\delta} \in H_0^1$. Therefore, $\delta H_0^1 = \{[\bar{\delta}H^\infty \cap C]/A\}^\perp$.

It is easy to verify that H^∞ in L^∞ is the annihilator of H_0^1 in L^1. It follows then from a standard dual-space theorem [1, Section 1, Theorem 3] that $H^\infty \simeq [L^1/H_0^1]^*$. Also, δH^∞ in H^∞ is the annihilator of $\bar{\delta}H_0^1/H_0^1$ in L^1/H_0^1. Putting the pieces together, and using repeatedly the theorem cited above, we conclude that

$$\{[\bar{\delta}H^\infty \cap C]/A\}^{**} \simeq [H_0^1/\delta H_0^1]^* \simeq [\bar{\delta}H_0^1/H_0^1]^* \simeq H^\infty/\delta H^\infty.$$

Therefore, $C(S_1, S_2)^{**} \simeq \mathscr{I}(S_1, S_2)$.

Comments. We would like to know what happens when U is replaced by a shift of infinite multiplicity and the inner functions ψ_1 and ψ_2 are replaced by operator-valued inner functions. In this general setting, S_1 and S_2 represent the most general strict contractions. Since the relation $C(S_1, S_2)^{**} \simeq \mathscr{I}(S_1, S_2)$ does not always hold, any generalization of the theorem of this paper involves complications that are not obvious to us. A step in the direction of this study is provided by the existence of an analogue of Lemma 2 for operator-valued inner functions [7], at least when $\psi_1 = \psi_2$.

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